

Research Article

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Multiplicity of positive solutions for a degenerate nonlocal problem with p -Laplacian

<https://doi.org/10.1515/anona-2021-0200>

Received March 26, 2021; accepted June 28, 2021.

Abstract: We consider a nonlinear boundary value problem with degenerate nonlocal term depending on the L^q -norm of the solution and the p -Laplace operator. We prove the multiplicity of positive solutions for the problem, where the number of solutions doubles the number of “positive bumps” of the degenerate term. The solutions are also ordered according to their L^q -norms.

Keywords: Nonlocal problems, p -Laplacian, sign-changing coefficient, multiple fixed points

MSC: 35J20, 35J25, 35Q74

1 Introduction

We study the following class of degenerate nonlocal boundary value problems with p -Laplacian

$$\begin{cases} -a \left(\int_{\Omega} u^q dx \right) \Delta_p u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (P)$$

where Ω is a bounded smooth domain in \mathbb{R}^N , $N \geq 1$, $q \geq 1$, $1 < p < +\infty$; $a \in C([0, \infty))$ and $f \in C([0, t_*])$, with t_* large enough, are functions satisfying some conditions which we will introduce later. The above problem generalizes the one considered in Gasiński-Santos Júnior [9] to the p -Laplacian case as well as with degenerate term a possibly sign changing.

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Motivations for the nonlocal problems like (P) come from the biological models of the population diffusion where the velocity of the dispersion, i.e.,

$$v = -a \left(\int_{\Omega} u^q dx \right) \nabla u$$

depends on the whole population (compare with Chipot-Rodrigues [6]). In this case $u(x)$ denotes the population density at x and Ω is a previous container of bacteria.

Another situation where a nonlocal model of this form is used (although in a much simpler setting, with $p = q = 2$ and $N = 1$) is a model for transversal vibrations of elastic strings where the displacements are not necessarily small (see Carrier [2]).

The main feature considered here is the degeneracy of the reaction term, namely, the function a appearing in front of the operator may have change sign several times. More precisely, we suppose that a has a multiplicity of “positive bumps” and possibly “other bumps” between them which can be negative (and as we will see later, they will not “generate” any solutions), namely:

(H₀) $a : [0, +\infty) \rightarrow \mathbb{R}$ is a continuous function and there exist positive numbers $0 =: t_0 \leq t_1 < t_2 \leq t_3 < t_4 \leq \dots \leq t_{2K-1} < t_{2K}$ ($K \geq 1$) such that: $a > 0$ in (t_{2k-1}, t_{2k}) and $a(t_{2k-1}) = a(t_{2k}) = 0$ for all $k \in \{1, \dots, K\}$.

In the past the nonlocal problems with degenerate nonlocal term were considered by Ambrosetti-Arcoya [1] (degeneracy appeared at zero and at infinity) and Santos Júnior-Siciliano [15] (where the problem was variational and in obtaining multiplicity of solutions the so called area condition was exploited). Moreover, we have two papers of Gasínski-Santos Júnior [9, 10], where a similar problem was considered with the Laplacian as a main operator on the left hand side. The authors proved existence, multiplicity as well as nonexistence results for the degenerate nonlocal term depending on the L^q -norm of the solution. For recent developments in problems with nonstandard growth and nonuniform ellipticity with an extensive focus on regularity theory, we mention [12] and the therein references.

Our paper here is the continuation of these works in the direction of a more general operator, namely the p -Laplacian. As far as we know, this is the first time where a p -Laplacian problem ($1 < p < +\infty$) is investigated under the degenerate condition described in (H_0) . Although, on the right hand side, nonlinearity has a particular growth near zero, as in the case $p = 2$, see [9]. More precisely, we assume:

(H₁) There exists $t_* > 0$ such that $f(t) > 0$ in $(0, t_*)$, $f(t_*) = 0$, $f \in C([0, t_*])$ and the map $(0, t_*) \ni t \mapsto f(t)/t^{p-1}$ is strictly decreasing.

To obtain our goal we need to put in good order different properties of the minus p -Laplacian, as the (S_+) property and monotonicity, see [13], that are combined in a suitable way with the Diaz-Saa’s formula [7], truncation techniques, maximum principle and regularity theory, see for instance [3–5] and the therein references. Owing to these refined tools we are able to realize, also for $1 < p < +\infty$, the general strategy developed in [9]. In particular, we investigate the uniqueness, regularity and positivity of the solution u_α of the auxiliary problem

$$\begin{cases} -a(\alpha) \Delta_p u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (P_{k,\alpha,f})$$

with $\alpha \in (t_{2k-1}, t_{2k})$, $k = 1, 2, \dots, K$.

Let us introduce some notation. Along the paper, λ_1 is the first eigenvalue of the minus p -Laplacian operator with zero Dirichlet boundary condition, φ_1 is the positive eigenfunction associated to λ_1 normalized in $W_0^{1,p}(\Omega)$ -norm and e_1 is the positive eigenfunction associated to λ_1 normalized in $L^\infty(\Omega)$ -norm.

The relation between the domains of a and f are stated in the following assumption:

(H₂) $t_{2K} < t_*^q \int_{\Omega} e_1^q dx$.

Finally, to prove the multiplicity result for problem (P), namely the existence of $2K$ positive solutions corresponding to the “positive bumps” of the function a , we will also need the following two assumptions:

- (H_3) $\max_{t \in [t_{2k-1}, t_{2k}]} a(t) < \gamma/\lambda_1$, for all $k \in \{1, \dots, K\}$, where $\gamma = \lim_{t \rightarrow 0^+} f(t)/t^{p-1}$;
- (H_4) $\max_{t \in [t_{2k-1}, t_{2k}]} [t^p a(t)] > \theta$, for all $k \in \{1, \dots, K\}$, where $\theta = |\Omega|^p t_*^{p(q-1)} \lambda_1^{-1} \max_{t \in [0, t_*]} [tf(t)]$.

Remark 1.1. (a) All the hypotheses on $a(t)$, namely (H_0) , (H_3) and (H_4) deal only with “even” intervals $[t_{2k-1}, t_{2k}]$ (for $k \in \{1, \dots, K\}$). The function $a(t)$ can be arbitrary in “odd” intervals (can be negative, zero or positive without satisfying bound conditions like (H_3) or (H_4)) and these intervals can be even degenerated to a point. In other words $a(t)$ can be any continuous function defined on a positive interval and we can split its domain into “even” intervals where the above assumptions are satisfied and “odd” intervals (possibly degenerated to a single point), where the above assumptions need not be satisfied. The number of solutions obtained in the paper will double the number of “even” intervals.

(b) Condition (H_1) implies that $\gamma = \lim_{t \rightarrow 0^+} f(t)/t^{p-1}$ is well defined and it can be a positive number (see also assumption (H_3)) or $+\infty$. In particular, if $f(0) > 0$ then $\gamma = +\infty$.

(c) Condition (H_3) is trivially satisfied if $f(0) > 0$ (see Remark 1.1**(b)**). If $\gamma < +\infty$, (H_3) basically means that the peaks of $a(t)$ in even intervals are controlled from above by variation of f at 0. On the other hand, hypothesis (H_4) means that the peaks of $t^p a(t)$ in “even” intervals are controlled from below by the maximum value of $tf(t)$ in $[0, t_*]$ (see Fig. 1).

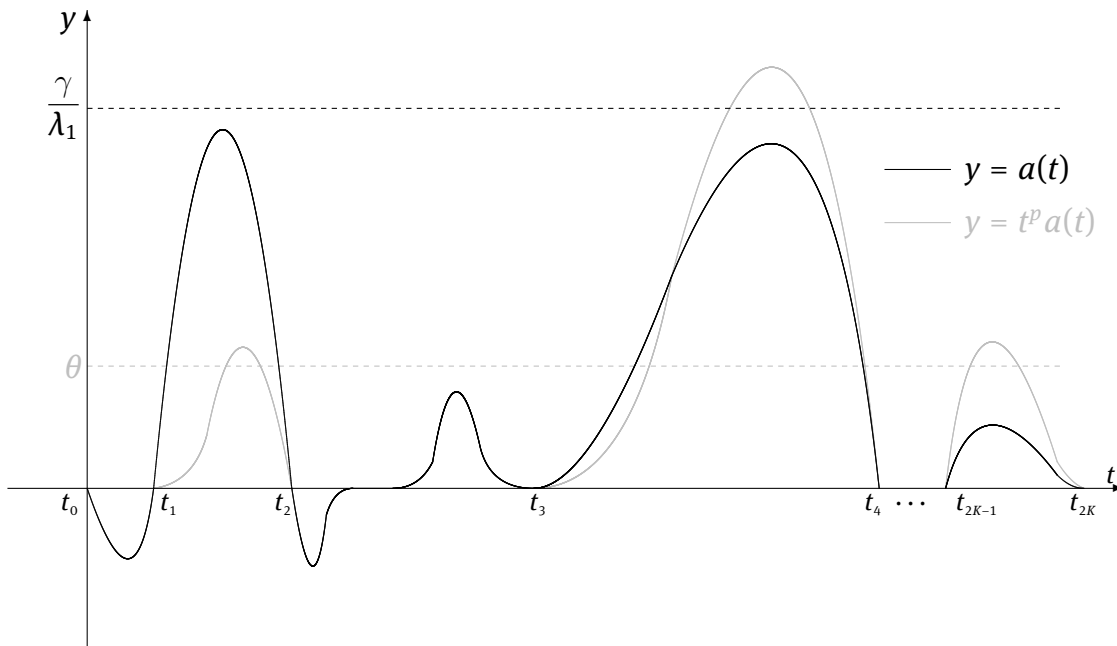


Fig. 1: Geometry of $a(t)$ and $t^p a(t)$ satisfying (H_1) , (H_3) and (H_4) .

(d) Finally, we wish to explicitly observe that $1 < p < +\infty$ and $q \geq 1$ are not required to satisfy any particular further condition. As well as, we emphasize that the nonlinear term f has a behaviour that is prescribed only on $[0, t^*]$, so that, nor asymptotic conditions at infinity, neither critical/subcritical growth are involved.

2 Multiplicity of solutions

Hereinafter, $\|\cdot\|$ denotes the $W_0^{1,p}(\Omega)$ -norm and $|\cdot|_r$ the $L^r(\Omega)$ -norm with $r \geq 1$. In this Section, we will produce regular positive solutions of problem (P). In particular, they will belong to $\text{int}(C_+)$, where

$$C_+ = \{u \in C_0^1(\bar{\Omega}) : u(x) \geq 0, \forall x \in \Omega\} \quad \text{and} \quad C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u(x) = 0, \forall x \in \partial\Omega\}.$$

For the sake of completeness, recall that

$$\text{int}(C_+) = \left\{ u \in C_0^1(\bar{\Omega}) : u > 0, \forall x \in \Omega, \text{ and } \frac{\partial u}{\partial \nu}(x) < 0 \forall x \in \partial\Omega \right\},$$

where $\nu = \nu(x)$ denotes the outer unit normal for all $x \in \partial\Omega$.

The main result of the present note can be stated as follows.

Theorem 2.1. *If hypotheses (H_0) , (H_1) , (H_2) , (H_3) , and (H_4) hold, then problem (P) has at least $2K$ positive solutions belonging to $\text{int}(C_+)$, with ordered L^q -norms, namely*

$$t_{2k-1} < \int_{\Omega} u_{k,1}^q dx < \int_{\Omega} u_{k,2}^q dx < t_{2k} \quad \text{for all } k \in \{1, \dots, K\}.$$

In the proof of Theorem 2.1 we will consider a suitable auxiliary problem involving the following truncation function

$$f_*(t) = \begin{cases} f(0) & \text{if } t \leq 0, \\ f(t) & \text{if } 0 \leq t \leq t_*, \\ 0 & \text{if } t_* \leq t. \end{cases} \tag{2.1}$$

Fix $k \in \{1, \dots, K\}$ and $\alpha \in (t_{2k-1}, t_{2k})$, here is the announced auxiliary problem that will play a crucial role

$$\begin{cases} -a(\alpha) \Delta_p u = f_*(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_{k,\alpha}}$$

We start solving problem $(P_{k,\alpha})$ by the variational approach. Observe that, after removing the nonlocal term of (P) we obtained a variational problem, so that we can look for solutions of $(P_{k,\alpha})$ searching functions $u \in W_0^{1,p}(\Omega)$ such that

$$-a(\alpha) \int_{\Omega} |\nabla u|^{p-2} |\nabla u| |\nabla v| dx - \int_{\Omega} f_*(u) v dx = 0 \quad \forall v \in W_0^{1,p}(\Omega). \tag{2.2}$$

Proposition 2.2. *If hypotheses (H_0) , (H_1) and (H_3) hold, then for each $k \in \{1, \dots, K\}$ and $\alpha \in (t_{2k-1}, t_{2k})$ fixed, problem $(P_{k,\alpha})$ has a unique solution $u_\alpha \in \text{int}(C_+)$ such that $0 < u_\alpha \leq t_*$ in Ω . In particular, u_α solves the problem*

$$\begin{cases} -a(\alpha) \Delta_p u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_{k,\alpha,f}}$$

Proof. Fix $k \in \{1, \dots, K\}$ and $\alpha \in (t_{2k-1}, t_{2k})$. Let us define the energy functional $I_{k,\alpha} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ corresponding to problem $(P_{k,\alpha})$, namely

$$I_{k,\alpha}(u) = \frac{1}{p} a(\alpha) \|u\|^p - \int_{\Omega} F_*(u) dx, \tag{2.3}$$

where $F_*(t) = \int_0^t f_*(s) ds$. Since f_* is bounded and continuous, it is clear that $I_{k,\alpha}$ is coercive and weakly lower semicontinuous. Therefore $I_{k,\alpha}$ has a minimizer u_α which is a solution of $(P_{k,\alpha})$. Since $\varphi_1 \in \text{int}(C_+)$ (recall

that φ_1 is the positive eigenfunction of the minus p -Laplacian associated to first eigenvalue λ_1 normalized in $W_0^{1,p}(\Omega)$ -norm), observing that

$$\frac{I_{k,\alpha}(t\varphi_1)}{t^p} = \frac{1}{p}a(\alpha) - \int_{\Omega} \frac{F_*(t\varphi_1)}{(t\varphi_1)^p} \varphi_1^p dx, \tag{2.4}$$

by assumption (H_3) , we can pass to the limit in the previous and obtain

$$\lim_{t \rightarrow 0^+} I_{k,\alpha}(t\varphi_1) = \frac{1}{p} \left(a(\alpha) - \frac{\gamma}{\lambda_1} \right) < 0,$$

namely, for $t > 0$ sufficiently small we obtain

$$I_{k,\alpha}(u_\alpha) \leq I_{k,\alpha}(t\varphi_1) < 0, \tag{2.5}$$

and so u_α is nontrivial.

First, let us verify that $0 \leq u_\alpha \leq t_*$. Indeed, if in (2.2) we take $v = -(u_\alpha)^-$, we get

$$a(\alpha)\|u_\alpha^-\|^p = - \int_{\Omega} f_*(x, u_\alpha)u_\alpha^- \leq 0,$$

and, since $a(\alpha) > 0$, $u_\alpha^- = 0$, that is $0 \leq u_\alpha$. Analogously, if in (2.2) we take $v = (u_\alpha - t_*)^+$, we get

$$a(\alpha)\|(u_\alpha - t_*)^+\|^p = \int_{\Omega} f_*(x, u_\alpha)(u_\alpha - t_*)^+ dx = 0,$$

so, again recalling that $a(\alpha) > 0$, $(u_\alpha - t_*)^+ = 0$ and $u_\alpha \leq t_*$. Hence, taking in mind the definition of f_* , u_α solves $(P_{k,\alpha,f})$.

Since u_α is bounded, by standard regularity arguments due to Lieberman [11, Theorem 1], we conclude that $u_\alpha \in C_0^{1,\beta}(\bar{\Omega})$ (for some $\beta \in (0, 1)$). Recalling that $f \in C([0, t_*])$ and observing that $f(u_\alpha) \geq 0$, one has $\Delta_p u_\alpha \in L^\infty$ and $\Delta_p u_\alpha \leq 0$. Hence, by the Maximum Principle (see Gasiński-Papageorgiou [8, Theorem 6.2.8], Vázquez [16] or Pucci-Serrin [14]) we get that $u_\alpha \in \text{int}(C_+)$.

Finally, suppose that v_α is a second solution to $(P_{k,\alpha})$, with $u_\alpha \neq v_\alpha$. Arguing as for u_α , one can point out that $0 < v_\alpha \leq t_*$ in Ω , as well as that v_α solves $(P_{k,\alpha,f})$. Hence, in view of assumption (H_1) , from [7] one can derive

$$0 \leq a(\alpha) \int_{\Omega} \left(\frac{-\Delta_p u_\alpha}{u_\alpha^{p-1}} + \frac{\Delta_p v_\alpha}{v_\alpha^{p-1}} \right) (u_\alpha^p - v_\alpha^p) dx = \int_{\Omega} \left(\frac{f(u_\alpha)}{u_\alpha^{p-1}} - \frac{f(v_\alpha)}{v_\alpha^{p-1}} \right) (u_\alpha^p - v_\alpha^p) dx < 0$$

and we get a contradiction. □

In order to better describe the strategy that we will follow, observe that when (H_0) , (H_1) and (H_3) hold, the previous proposition allow us to define suitable maps $\mathcal{S}_k : (t_{2k-1}, t_{2k}) \rightarrow C_0^1(\bar{\Omega})$ for every $k = 1, \dots, K$ by putting

$$\mathcal{S}_k(\alpha) = u_\alpha$$

for every $\alpha \in (t_{2k-1}, t_{2k})$, where u_α , that is a minimizer of $I_{k,\alpha}$, is the unique solution of $(P_{k,\alpha})$ such that $0 < u_\alpha \leq t_*$. At this point, for every $k = 1, \dots, K$, we introduce a real function $\mathcal{P}_k : (t_{2k-1}, t_{2k}) \rightarrow \mathbb{R}$ defined by

$$\mathcal{P}_k(\alpha) = |\mathcal{S}_k(\alpha)|_q^q = \int_{\Omega} u_\alpha^q dx \tag{2.6}$$

for all $\alpha \in (t_{2k-1}, t_{2k})$, where $q \geq 1$. The role of the map \mathcal{P}_k is revealed by the following claim

$$\text{if } \alpha \in \text{Fix}(\mathcal{P}_k), \text{ then } \mathcal{S}_k(\alpha) \text{ is a solution of problem } (P), \tag{C}$$

where $\text{Fix}(\mathcal{P}_k) = \{\alpha \in (t_{2k-1}, t_{2k}) : \mathcal{P}_k(\alpha) = \alpha\}$. Indeed, let $\alpha \in (t_{2k-1}, t_{2k})$ be such that $\mathcal{P}_k(\alpha) = \alpha$, then, recalling that $\mathcal{S}_k(\alpha) = u_\alpha$ is a solution of $(P_{k,\alpha,f})$ one can conclude that

$$-a \left(\int_{\Omega} u_\alpha^q dx \right) \Delta_p u_\alpha = -a(\alpha) \Delta_p u_\alpha = f(u_\alpha) \quad \text{in } \Omega$$

and the claim (C) holds.

Next lemma will be useful in the proof of Proposition 2.4 which provides the continuity of the map \mathcal{P}_k . First observe that, since the map $(0, t^*) \ni t \mapsto \psi(t) = f(t)/t^{p-1}$ is strictly decreasing (see hypothesis (H_1)), there exists the inverse $\psi^{-1} : (0, \gamma) \rightarrow (0, t^*)$, where γ is as in (H_3) .

Lemma 2.3. *Assume that hypotheses (H_0) , (H_1) and (H_3) hold. Put*

$$\bar{a} := \max_{1 \leq k \leq K} \max_{t \in [t_{2k-1}, t_{2k}]} a(t)$$

and let $\varepsilon \in (0, \gamma - \lambda_1 \bar{a})$. Then, for every $k = 1, \dots, K$ and $\alpha \in (t_{2k-1}, t_{2k})$ one has

$$c_\alpha \leq -\frac{1}{p} \varepsilon (\psi^{-1}(\lambda_1 a(\alpha) + \varepsilon))^p \int_{\Omega} e_1^p dx \tag{2.7}$$

where $c_\alpha = I_{k,\alpha}(u_\alpha) = \min_{u \in W_0^{1,p}(\Omega)} I_{k,\alpha}(u)$.

Proof. We only treat the case $\gamma < +\infty$. The case $\gamma = +\infty$ is analogous. First, by hypothesis (H_3) , since $0 < \bar{a} < \frac{\gamma}{\lambda_1}$, it makes sense the choice of $\varepsilon \in (0, \gamma - \lambda_1 \bar{a})$. Fix $k \in \{1, \dots, K\}$ and $\alpha \in (t_{2k-1}, t_{2k})$ and, observing that $\varepsilon < \gamma - \lambda_1 \bar{a} \leq \gamma - \lambda_1 a(\alpha)$, we can consider the function

$$y_\alpha := \psi^{-1}(\lambda_1 a(\alpha) + \varepsilon) e_1.$$

From assumption (H_1) , we have

$$F_*(t) \geq \frac{1}{p} f_*(t) t \quad \forall t \geq 0. \tag{2.8}$$

Hence,

$$\begin{aligned} \frac{I_{k,\alpha}(y_\alpha)}{(\psi^{-1}(\lambda_1 a(\alpha) + \varepsilon))^p} &= \frac{1}{p} a(\alpha) \|e_1\|^p - \int_{\Omega} \frac{F_*(y_\alpha)}{(\psi^{-1}(\lambda_1 a(\alpha) + \varepsilon))^p} dx \\ &\stackrel{(2.8)}{\leq} \frac{1}{p} \left[a(\alpha) \|e_1\|^p - \int_{\Omega} \frac{f_*(y_\alpha)}{(\psi^{-1}(\lambda_1 a(\alpha) + \varepsilon))^p} y_\alpha dx \right] \\ &= \frac{1}{p} \left[a(\alpha) \|e_1\|^p - \int_{\Omega} \frac{f_*(y_\alpha)}{y_\alpha^{p-1}} e_1^p dx \right] \\ &\stackrel{(H_1)}{\leq} \frac{1}{p} \left[a(\alpha) \|e_1\|^p - \int_{\Omega} \frac{f_*(\psi^{-1}(\lambda_1 a(\alpha) + \varepsilon))}{(\psi^{-1}(\lambda_1 a(\alpha) + \varepsilon))^{p-1}} e_1^p dx \right] \\ &= \frac{1}{p} \left[a(\alpha) \|e_1\|^p - (\lambda_1 a(\alpha) + \varepsilon) \int_{\Omega} e_1^p dx \right] = -\frac{1}{p} \varepsilon \int_{\Omega} e_1^p dx. \end{aligned}$$

Therefore, (2.7) holds. □

Proposition 2.4. *If hypotheses (H_0) , (H_1) , and (H_3) hold, then for each $k \in \{1, \dots, K\}$ the map $\mathcal{P}_k : (t_{2k-1}, t_{2k}) \rightarrow \mathbb{R}$ defined in (2.6) is continuous.*

Proof. Let $\{\alpha_n\} \subset (t_{2k-1}, t_{2k})$ be a sequence such $\alpha_n \rightarrow \alpha_*$, for some $\alpha_* \in (t_{2k-1}, t_{2k})$. For simplicity, put $u_n = S_k(\alpha_n)$. Taking in mind (2.5),

$$\frac{1}{p} a(\alpha_n) \|u_n\|^p - \int_{\Omega} F_*(u_n) dx = I_{k,\alpha_n}(u_n) < 0, \tag{2.9}$$

that implies

$$\|u_n\|^p \leq p \frac{F_*(t_*)|\Omega|}{a(\alpha_n)} \quad \forall n \in \mathbb{N}.$$

Therefore, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ and, passing to a subsequence if necessary, we may assume that

$$u_n \rightharpoonup u_* \text{ in } W_0^{1,p}(\Omega), \quad u_n \rightarrow u_* \text{ in } L^1(\Omega) \quad \text{and} \quad u_n(x) \rightarrow u_*(x) \text{ a.e. in } \Omega \tag{2.10}$$

for some $u_* \in W_0^{1,p}(\Omega)$. Moreover, for every $n \in \mathbb{N}$ one has

$$a(\alpha_n) \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla v dx = \int_{\Omega} f_*(u_n) v dx \quad \forall v \in W_0^{1,p}(\Omega). \tag{2.11}$$

Testing the previous with $v = u_n - u_*$ and passing to the limsup, in view of the continuity of α and f_* as well as (2.10) and the Lebesgue’s dominated convergence theorem, we get

$$a(\alpha_*) \limsup_{n \rightarrow \infty} \langle -\Delta_p u_n, u_n - u_* \rangle \leq 0.$$

The (S_+) property of $-\Delta_p u$ assures that $u_n \rightarrow u_*$ in $W_0^{1,p}(\Omega)$, so that, passing to the limit in (2.11), we can conclude that u_* is a nonnegative weak solution of $(P_{k,\alpha})$ with $\alpha = \alpha_*$. We need to show that $u_* \neq 0$. By Lemma 2.3, there exists $\varepsilon > 0$, small enough, such that

$$I_{k,\alpha_n}(u_n) \leq -\frac{1}{p} \varepsilon \psi^{-1}(\lambda_1 a(\alpha_n) + \varepsilon)^p \int_{\Omega} e_1^p dx \quad \forall n \in \mathbb{N}.$$

So, passing to the limit and using (2.10), we obtain

$$I_{k,\alpha}(u_*) \leq -\frac{1}{p} \varepsilon \psi^{-1}(\lambda_1 a(\alpha_*) + \varepsilon)^p \int_{\Omega} e_1^p dx < 0.$$

Therefore $u_* \neq 0$. Arguing as in the proof of Proposition 2.2 we can show that $u_* \in \text{int}(C_+)$. Since such a solution is unique, we conclude that $u_* = u_{\alpha_*} = S_k(\alpha_*)$. Finally, again from Proposition (2.2) one has that $0 < u_n \leq t_*$ for every $n \in \mathbb{N}$ and we can use (2.10) and the Lebesgue’s dominated convergence theorem in order to conclude that $\mathcal{P}_k(\alpha_n) \rightarrow \mathcal{P}_k(\alpha_*)$. This proves the continuity of \mathcal{P}_k . \square

Remark 2.5. A deeper analysis allow us to assure a strongest convergence of the sequence $\{u_n\}$ considered in the previous proof. Indeed, since every u_n solves $(P_{k,\alpha})$, with $\alpha = \alpha_n$, one has

$$-\Delta_p u_n = \frac{f_*(u_n)}{a(\alpha_n)} =: g_n.$$

Recalling that f_* is bounded and $a(\alpha_n)$ is away from zero, we have

$$|g_n|_{\infty} \leq C \quad \forall n \in \mathbb{N},$$

for some $C > 0$. By the regularity theory (see Lieberman [11]), we have that

$$\|u_n\|_{C_0^{1,\beta}(\overline{\Omega})} \leq C,$$

for some $C > 0$ and $\beta \in (0, 1)$. The compactness of the embedding $C^{1,\beta}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$, passing to a subsequence if necessary, permits to obtain that

$$u_n \rightarrow u_* \text{ in } C_0^1(\overline{\Omega}),$$

(the limit function u_* is the same as in (2.10), by the uniqueness of the limit function).

Next lemma will be needed in Proposition 2.7, where the fixed points of \mathcal{P}_k are provided.

Lemma 2.6. *If hypotheses (H_0) , (H_1) and (H_3) hold, then, for every $k \in \{1, \dots, K\}$ and $\alpha \in (t_{2k-1}, t_{2k})$ one has*

$$u_\alpha \geq z_\alpha := \psi^{-1}(\lambda_1 a(\alpha))e_1 \tag{2.12}$$

Proof. Fix $k \in \{1, \dots, K\}$ and $\alpha \in (t_{2k-1}, t_{2k})$. From hypothesis (H_1) , the definition of ψ^{-1} and observing that, in view of (H_3) , $0 < z_\alpha \leq \psi^{-1}(\lambda_1 a(\alpha)) < t_*$, we have

$$\lambda_1 a(\alpha) = \frac{f(\psi^{-1}(\lambda_1 a(\alpha)))}{\psi^{-1}(\lambda_1 a(\alpha))^{p-1}} \leq \frac{f(z_\alpha)}{z_\alpha^{p-1}},$$

so

$$-a(\alpha)\Delta_p z_\alpha = \lambda_1 a(\alpha)z_\alpha^{p-1} \leq f(z_\alpha) \text{ in } \Omega. \tag{2.13}$$

Therefore z_α is a subsolution of $(P_{k,\alpha})$. Now, put

$$\tilde{f}(x, t) = \begin{cases} f_*(z_\alpha(x)) & \text{if } t \leq z_\alpha(x), \\ f_*(t) & \text{if } z_\alpha(x) < t \end{cases} \tag{2.14}$$

for every $(x, t) \in \Omega \times \mathbb{R}^N$ and

$$\tilde{I}_{k,\alpha}(u) = \frac{a(\alpha)}{p} \|u\|^p - \int_\Omega \tilde{F}(x, u) \, dx$$

for every $u \in W_0^{1,p}(\Omega)$, where $\tilde{F}(x, t) = \int_0^t \tilde{f}(x, s) \, ds$. The direct methods assure the existence of a minimizer v_α for the functional $\tilde{I}_{k,\alpha}$. Moreover, since $\varphi_1, z_\alpha \in \text{int}(C_+)$, for $t > 0$ small enough one has $0 < t\varphi_1 \leq z_\alpha$ in Ω and

$$\tilde{I}_{k,\alpha}(v_\alpha) \leq \tilde{I}_{k,\alpha}(t\varphi_1) = t \left(\frac{t^{p-1}}{p} a(\alpha) - \int_\Omega f_*(z_\alpha)\varphi_1 \, dx \right) < 0.$$

Thus v_α is a nontrivial function such that

$$-a(\alpha) \int_\Omega |\nabla v_\alpha|^{p-2} |\nabla v_\alpha| \nabla w \, dx - \int_\Omega \tilde{f}(x, v_\alpha) w \, dx = 0 \quad \forall w \in W_0^{1,p}(\Omega). \tag{2.15}$$

Testing (2.15) with $-(v_\alpha)^-$ and $(v_\alpha - t_*)^+$ we obtain that

$$0 \leq v_\alpha \leq t_*.$$

In particular, we wish to show that

$$v_\alpha \geq z_\alpha. \tag{2.16}$$

Indeed, acting in (2.15) with $w = (z_\alpha - v_\alpha)^+$ and taking in mind (2.13), we obtain

$$\begin{aligned} a(\alpha) \langle -\Delta_p v_\alpha, (z_\alpha - v_\alpha)^+ \rangle &= \int_\Omega \tilde{f}(x, v_\alpha)(z_\alpha - v_\alpha)^+ \, dx \\ &= \int_\Omega f(z_\alpha)(z_\alpha - v_\alpha) \, dx \\ &\geq a(\alpha) \langle -\Delta_p z_\alpha, (z_\alpha - v_\alpha)^+ \rangle. \end{aligned}$$

Hence,

$$a(\alpha) \int_{\{z_\alpha > v_\alpha\}} \langle |\nabla z_\alpha|^{p-2} \nabla z_\alpha - |\nabla v_\alpha|^{p-2} \nabla v_\alpha, \nabla z_\alpha - \nabla v_\alpha \rangle_{\mathbb{R}^N} \, dx \leq 0.$$

and, taking into account [13, Lemma A.0.5], (2.16) holds. At this point, it is easy to observe that, because of the definition of \tilde{f} , one has

$$-a(\alpha)\Delta_p v_\alpha = \tilde{f}(x, v_\alpha) = f_*(v_\alpha) = f(v_\alpha),$$

namely v_α is a solution of $(P_{k,\alpha})$ such that $0 < v_\alpha \leq t_*$. Finally, Proposition 2.2 assures that $v_\alpha = u_\alpha$ and the proof is complete in view of (2.16). \square

Proposition 2.7. *If hypotheses (H_0) , (H_1) , (H_2) , (H_3) , and (H_4) hold, then, for every $k \in \{1, \dots, K\}$ the map \mathcal{P}_k has at least two fixed points $t_{2k-1} < \alpha_{1,k} < \alpha_{2,k} < t_{2k}$.*

Proof. Fix $k \in \{1, \dots, K\}$. First we show that

$$\lim_{\alpha \rightarrow t_{2k-1}^+} \mathcal{P}_k(\alpha) > t_{2k-1} \quad \text{and} \quad \lim_{\alpha \rightarrow t_{2k}^-} \mathcal{P}_k(\alpha) > t_{2k}. \tag{2.17}$$

From Lemma 2.6, we have

$$\mathcal{P}_k(\alpha) = \int_{\Omega} u_\alpha^q dx \geq \int_{\Omega} z_\alpha^q dx = (\psi^{-1}(\lambda_1 a(\alpha)))^q \int_{\Omega} e_1^q dx \quad \forall \alpha \in (t_{2k-1}, t_{2k}).$$

Hence, by hypotheses (H_1) and (H_2) , we have

$$\begin{aligned} \lim_{\alpha \rightarrow t_{2k-1}^+} \mathcal{P}_k(\alpha) &\geq t_*^q \int_{\Omega} e_1^q dx > t_{2K} > t_{2k-1}, \\ \lim_{\alpha \rightarrow t_{2k}^-} \mathcal{P}_k(\alpha) &\geq t_*^q \int_{\Omega} e_1^q dx > t_{2K} \geq t_{2k}, \end{aligned}$$

which proves (2.17).

Next, we show that for some $\alpha_0 \in (t_{2k-1}, t_{2k})$ we have

$$\mathcal{P}_k(\alpha_0) < \alpha_0. \tag{2.18}$$

For each $\alpha \in (t_{2k-1}, t_{2k})$, let w_α be the unique (positive) solution of the problem

$$\begin{cases} -\Delta_p u = u_\alpha^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.19}$$

where $u_\alpha = S_k(\alpha)$ is the (unique) positive solution of $(P_{k,\alpha})$. Multiplying by u_α , integrating by parts, using Cauchy-Schwarz inequality and Hölder inequality, we have

$$\begin{aligned} \mathcal{P}_k(\alpha) &= \int_{\Omega} u_\alpha^q dx = \int_{\Omega} |\nabla w_\alpha|^{p-2} \nabla w_\alpha \nabla u_\alpha dx \\ &\leq \int_{\Omega} |\nabla w_\alpha|^{p-1} |\nabla u_\alpha| dx \leq \|w_\alpha\|^{p-1} \|u_\alpha\|. \end{aligned} \tag{2.20}$$

By the variational property of u_α , we get

$$a(\alpha) \int_{\Omega} |\nabla u_\alpha|^p dx = \int_{\Omega} f_*(u_\alpha) u_\alpha dx,$$

thus

$$\|u_\alpha\| \leq \frac{|\Omega|^{1/p}}{a(\alpha)^{1/p}} \max_{[0,t_*]} [tf(t)]^{1/p}, \tag{2.21}$$

while, taking in mind that w_α solves (2.19), by the Hölder inequality and the variational characterization of λ_1 , we have

$$\begin{aligned} \|w_\alpha\|^p &= \int_\Omega |\nabla w_\alpha|^p dx = \int_\Omega u_\alpha^{q-1} w_\alpha \\ &\leq \left(\int_\Omega u_\alpha^{(q-1)p'} dx \right)^{1/p'} \left(\int_\Omega w_\alpha^p dx \right)^{1/p} \\ &\leq |\Omega|^{1/p'} t_*^{q-1} \lambda_1^{-1/p} \|w_\alpha\|, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, so

$$\|w_\alpha\|^{p-1} \leq |\Omega|^{1/p'} t_*^{q-1} \lambda_1^{-1/p}. \tag{2.22}$$

Applying (2.21) and (2.22) in (2.20), we obtain

$$\begin{aligned} \mathcal{P}_k(\alpha) &\leq \frac{1}{a(\alpha)^{1/p}} \left(\max_{[0,t_*]} [tf(t)]^{1/p} \right) |\Omega| t_*^{q-1} \lambda_1^{-1/p} \\ &= \frac{\theta^{1/p}}{a(\alpha)^{1/p}} \quad \forall \alpha \in (t_{2k-1}, t_{2k}). \end{aligned}$$

Using hypothesis (H_4) , for some $\alpha_0 \in (t_{2k-1}, t_{2k})$ one has

$$\alpha_0^p a(\alpha_0) = \max_{t \in [t_{2k-1}, t_{2k}]} [t^p a(t)] > \theta$$

and we get

$$\mathcal{P}_k(\alpha_0) \leq \frac{\theta^{1/p}}{a(\alpha_0)^{1/p}} < \alpha_0,$$

namely (2.18) holds.

From the continuity of \mathcal{P}_k (see Proposition 2.4), since (2.17) implies the existence of α_1 and α_2 in (t_{2k-1}, t_{2k}) with $P(\alpha_1) > t_{2k-1}$ and $P(\alpha_2) > t_{2k}$, (2.18) and the intermediate value theorem for continuous real functions, we conclude that \mathcal{P}_k has at least two fixed points in the interval (t_{2k-1}, t_{2k}) . \square

Proof of Theorem 2.1 Fix $k \in \{1, \dots, K\}$, recall claim (C) and observe that the two fixed points of \mathcal{P}_k obtained in Proposition 2.7 produce two positive solutions $u_{k,1}$ and $u_{k,2}$ of problem (P), with $u_{k,1}, u_{k,2} \in \text{int}(C_+)$ and satisfying

$$t_{2k-1} < \int_\Omega u_{k,1}^q dx < \int_\Omega u_{k,2}^q dx < t_{2k},$$

for any $k \in \{1, \dots, K\}$. This finishes the proof. \square

We conclude presenting a concrete example of possible nonlocal problem that can be considered

Example 2.8. Let K be a positive integer, $1 < p < +\infty$, $q \geq 1$ and fix t_* , $M > 0$ such that

$$t_* > \frac{((2K-1)\pi)^{1/q}}{|e_1|_q} \quad \text{and} \quad M \geq |\Omega|^p \frac{t_*^{pq}}{ep\lambda_1},$$

where e_1 and λ_1 are as defined in Section 1, so that (H_2) holds. Let $a : [0, +\infty) \rightarrow \mathbb{R}$ be a continuous function such that

$$a(t) = M \sin t$$

for $t \in [0, (2K-1)\pi]$. Obviously a satisfies hypothesis (H_0) for the partition

$$0 = t_0 = t_1 < t_2 = \pi < t_3 = 2\pi < \dots < t_{2K-1} = 2(K-1)\pi < t_{2K} = (2K-1)\pi.$$

Consider a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(t) = \begin{cases} -t^{p-1} \ln\left(\frac{t}{t_*}\right) & \text{if } t \in (0, t_*] \\ 0 & \text{if } t = 0. \end{cases}$$

It is simple to verify that

- $t \mapsto f(t)/t^{p-1}$ is decreasing on $(0, t_*)$,
- $\lim_{t \rightarrow 0^+} f(t)/t^{p-1} = +\infty$,
- $f(0) = f(t_*) = 0, f(t) > 0$ for $t \in (0, t_*)$.

Namely, hypotheses (H_1) and (H_3) hold. Finally, a direct computation shows that

$$\max_{t \in [0, t_*]} [tf(t)] = \frac{t_*^p}{ep}.$$

Hence, for every $k \in \{1, \dots, K\}$ one has

$$\max_{t \in [t_{2k-1}, t_{2k}]} [t^p a(t)] \geq M(\pi/2)^p > M \geq |\Omega|^p \frac{t_*^{pq}}{ep\lambda_1} = |\Omega|^p \frac{t_*^{p(q-1)}}{\lambda_1} \max_{t \in [0, t_*]} [tf(t)] = \theta.$$

That is (H_4) holds too.

Acknowledgements. The authors would like to thank the anonymous Referees for their valuable comments which helped to improve the manuscript.

The paper is partially supported by PRIN 2017 – Progetti di Ricerca di rilevante Interesse Nazionale, "Non-linear Differential Problems via Variational, Topological and Set-valued Methods" (2017AYM8XW).

Conflict of interest statement: Authors state no conflict of interest.

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