TWO NON-ZERO SOLUTIONS FOR STURM-LIOUVILLE EQUATIONS WITH MIXED BOUNDARY CONDITIONS

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Abstract. In this paper, we establish the existence of two non-zero solutions for a mixed boundary value problem with the Sturm-Liouville equation. The approach is based on a recent two critical point theorem.

1. Introduction

The aim of this paper is to establish existence results of two non-trivial solutions for Sturm-Liouville problems with mixed conditions and involving the ordinary $p-$Laplacian. We consider the following problem

$$\begin{cases} -(q(x)|u'(x)|^{p-2}u'(x))' + s(x)|u(x)|^{p-2}u(x) = \lambda f(x,u(x)) \quad \text{on} \quad]a,b[\,, \\ \\ u(a) = u'(b) = 0, \end{cases} \tag{P}$$

with $p > 1$, $q, s \in L^\infty([a,b])$, $s \not\equiv 0$, with $q_0 = \operatorname*{ess\,inf}_{[a,b]} q > 0$ and $s_0 = \operatorname*{ess\,inf}_{[a,b]} s \geq 0$. Here the nonlinearity $f : [a,b] \times \mathrm{I\!R} \to \mathrm{I\!R}$ is an L^1-Carathéodory function and λ is a real positive parameter. In this paper, under suitable assumptions on the nonlinearity f, we obtain the existence of two non-zero solutions to problem (P). Our main tool is a two critical point theorem (Theorem 2.1) established in [10]. Such critical point result is an appropriate combination of the local minimum theorem obtained in [7], with the classical and seminal Ambrosetti-Rabinowitz theorem (see [1]). A crucial assumption of the mountain pass theorem is the Palais-Smale condition. It is worth noticing that, here, to obtain the existence of two non-zero solutions, it is enough to assume together with the classical Ambrosetti-Rabinowitz condition only an algebraic condition on the nonlinearity (see condition (ii) in Theorem 3.1), which is more general than the $p-$sublinearity at zero. For more details on these subjects see [18], [19] and [20].

For other examples of results dealing with Sturm-Liouville equations see [3], [12], [15], for problems with mixed boundary conditions see [2], [4], [5], [6], [14], [16], [21], including elliptic case and systems. For other critical point results and applications, we refer for instance to [8], [9], [11], [13], [17].

As a special case of our main theorem (Theorem 3.1), here we point out the following result on the existence of two nonnegative solutions in the autonomous case.
Consider the following problem

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' + |u(x)|^{p-2}u(x) = \lambda f(u(x)) \quad \text{on} \quad]0,1[\,, \\ \\ u(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

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and put $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = +\infty,$$

and assume that there are two positive constants $\mu > p$ and $R > 0$ with $|t| \geq R$ such that

$$0 < \mu F(t) < tf(t).$$

Then, for each $\lambda \in]0, \lambda^*[$ where $\lambda^* = \dfrac{1}{p} \sup_{c>0} \dfrac{c^p}{F(c)}$, the problem (1.1) *has two nonnegative and non zero classical solutions.*

The paper is arranged as follows. In Section 2, we recall some basic definitions and our main tool, which is a two critical points theorem obtained in [10]. Finally, Section 3 is devoted to the main result (Theorem 3.1) and some of its consequences (Corollary 3.1 and Thoerem 3.2). Moreover some concrete examples of application are given (see Examples 3.1 and 3.2).

2. Preliminaries and basic notations

In this section, we recall definitions and theorems used in the paper.
Let $(X, ||\cdot||)$ be a real Banach space and I be a Gâteaux differentiable functional. We say that the functional I satisfies the *Palais-Smale condition* (in short (PS)-condition) if any sequence $\{u_n\}$ in X such that

(α_1) $\{I(u_n)\}$ is bounded,
(α_2) $\lim_{n\to+\infty} ||I'(u_n)||_{X^*} = 0$,

has a convergent subsequence.

Consider problem (P) and assume that $q, s \in L^\infty([a, b])$, with

$$q_0 = \operatorname*{ess\,inf}_{[a,b]} q > 0 \text{ and } s_0 = \operatorname*{ess\,inf}_{[a,b]} s \geq 0.$$

We use the following notations

$$\|q\|_\infty := \operatorname*{ess\,sup}_{x\in[a,b]} q(x), \qquad \|s\|_\infty := \operatorname*{ess\,sup}_{x\in[a,b]} s(x),$$

and put

$$k = \frac{1}{||q||_\infty + \frac{p+2}{p+1}\left(\frac{b-a}{2}\right)^p ||s||_\infty}. \tag{2.2}$$

Denote by $X = \{u \in W^{1,p}([a, b]) : u(a) = 0\}$ the Sobolev space endowed with the norm

$$\|u\| = \Big(\int_a^b q(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx\Big)^{\frac{1}{p}}.$$

It is well known that $(X, \|\cdot\|)$ is compactly embedded in $(C^0([a, b]), \|\cdot\|_\infty)$ and one has

$$\|u\|_\infty < \frac{(b-a)^{\frac{p-1}{p}}}{(q_0)^{\frac{1}{p}}}\|u\| \quad \text{for all} \quad u \in X. \tag{2.3}$$

Formula (2.3), can be obtained observing that, for all $x \in [a, b]$

$$|u(x)| \leq \int_a^b |u'(x)|dx$$

by Hölder inequality one has

$$\int_a^b |u'(x)|dx \le (b-a)^{(\frac{p-1}{p})}\|u'\|_{L^p} \le \frac{(b-a)^{\frac{p-1}{p}}}{(q_0)^{\frac{1}{p}}}\|u\|.$$

Throughout the sequel $f : [a,b]\times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function i.e. $f(\cdot,t)$ is measurable for all $t \in \mathbb{R}$, $f(x,\cdot)$ is continuous for almost every $x \in [a,b]$ and for all $\rho > 0$ one has $\sup_{|t|\le\rho} |f(x,t)| \in L^1([a,b])$. Clearly, if f is continuous in $[a,b]\times\mathbb{R}$, then it is L^1-Carathéodory.
Put

$$F(x,t) := \int_0^t f(x,\xi)d\xi \quad \text{for all} \quad (x,t) \in [a,b]\times\mathbb{R}.$$

We recall that $u : [a,b] \to \mathbb{R}$ is a weak solution of problem (P) if $u \in X$ satisfies the following condition

$$\int_a^b q(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dx = \lambda\int_a^b f(x,u(x))v(x)dx,$$

for all $v \in X$.

In order to study problem (P), we introduce the functionals Φ, $\Psi : X \to \mathbb{R}$ defined as follows

$$\Phi(u) = \frac{1}{p}\|u\|^p \quad \text{and} \quad \Psi(u) = \int_a^b F(x,u(x))dx, \quad \forall u \in X. \tag{2.4}$$

Clearly, Φ is coercive and continuously Gâteaux differentiable and its Gâteaux derivative at point $u \in X$ is defined by

$$\Phi'(u)(v) = \int_a^b q(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dt,$$

for every $v \in X$.
On the other hand Ψ is continuously Gâteaux differentiable and its Gâteaux derivative at point $u \in X$ is defined by

$$\Psi'(u)(v) = \int_a^b f(x,u(x))v(x)dx \quad \forall v \in X,$$

and

$$\Phi(0) = \Psi(0) = 0.$$

Clearly, the critical points of the functional $\Phi - \lambda\Psi$ on X are weak solutions of problem (P).
Moreover, if $f \in C^0([a,b])$, $q \in C^1([a,b])$ and $s \in C^0([a,b])$ the weak solutions for (P) are classical solutions.
Our main tool is the following theorem of existence of two non-zero critical points, Theorem 2.1, obtained in [10].

Theorem 2.1. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{2.5}$$

and, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$, *the functional* I_λ *admits at least two non-zero critical points* $u_{\lambda,1}$, $u_{\lambda,2}$ *such that* $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

3. Main Result

In this section, we present our main result. To be precise, we establish the existence result of at least two non zero weak solutions of problem (P).

Theorem 3.1. *Assume that there exist four positive constants* c, d, μ *and* R *with* $\mu > p$ *and* $d < c$, *such that*

(i) $\displaystyle\int_a^{\frac{a+b}{2}} F(x,t)dx \geq 0 \quad \forall t \in [0,d]$,

(ii) $\dfrac{\displaystyle\int_a^b \max_{|\xi|\leq c} F(x,\xi)dx}{c^p} < \dfrac{q_0}{2^{p-1}}k\dfrac{\displaystyle\int_{\frac{a+b}{2}}^b F(x,d)dx}{d^p}$,
where the constat k *is given by* (2.2),

(iii) $0 < \mu F(x,t) \leq tf(x,t)$ *for all* $x \in [a,b]$ *and for all* $|t| \geq R$.

Then, for each $\lambda \in \left]\dfrac{2^{p-1}d^p}{p(b-a)^{p-1}k\displaystyle\int_{\frac{a+b}{2}}^b F(x,d)dx}, \dfrac{q_0c^p}{p(b-a)^{p-1}\displaystyle\int_a^b \max_{|\xi|\leq c} F(x,\xi)dx}\right[$ *problem* (P) *has at least two non-zero weak solutions.*

Proof. Our goal is to apply Theorem 2.1. Consider the Sobolev space X and the operators Φ and Ψ defined in (2.4). We observe that the regularity assumptions on Φ and Ψ are satisfied, then taking into account (iii) by standard computations, for each $\lambda > 0$, $\Phi - \lambda\Psi$ is unbounded from below and satisfies the $(PS)-$condition, (see also the proof of Theorem 3.1 of reference [16]).

Consider the function $\bar{u} \in X$ defined by putting

$$\bar{u}(x) := \begin{cases} \dfrac{2d}{b-a}(x-a) & \text{if } x \in \left[a, \dfrac{a+b}{2}\right[, \\ \\ d & \text{if } x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases} \tag{3.6}$$

We observe that

$$\begin{aligned} \Phi(\bar{u}) \quad &:= \quad \frac{1}{p}||\bar{u}||^p = \frac{1}{p}\Big(\frac{2^pd^p}{(b-a)^p}\int_a^{\frac{a+b}{2}} q(x)dx \\ &+ \quad \frac{2^pd^p}{(b-a)^p}\int_a^{\frac{a+b}{2}}(x-a)^p s(x)dx + d^p\int_{\frac{a+b}{2}}^b s(x)dx\Big). \end{aligned} \tag{3.7}$$

Moreover, since $0 < d < c$ and by virtue of (ii), we obtain that

$$d^p < \frac{q_0}{2^{p-1}}kc^p. \tag{3.8}$$

Indeed, arguing by contradiction, if we assume that $d^p \geq \dfrac{q_0}{2^{p-1}}kc^p$, we have

$$\frac{\int_a^b \max_{|\xi|\le c} F(x,\xi)dx}{c^p} \ge \frac{\int_{\frac{a+b}{2}}^b F(x,d)dx}{c^p} \ge \frac{q_0}{2^{p-1}} k \frac{\int_{\frac{a+b}{2}}^b F(x,d)dx}{d^p},$$

which contradicts (ii). Put $r = \dfrac{q_0 c^p}{p(b-a)^{p-1}}$, by (3.7) and (3.8) we obtain

$$0 < \Phi(\bar{u}) \le \frac{1}{p}\left(\frac{2}{b-a}\right)^{p-1}\left[||q||_\infty + \frac{p+2}{p+1}\left(\frac{b-a}{2}\right)^p ||s||_\infty\right] d^p = \frac{1}{pk}\left(\frac{2}{b-a}\right)^{p-1} d^p < r.$$

By virtue of (i) we have

$$\Psi(\bar{u}) \ge \int_{\frac{a+b}{2}}^b F(x,d)dx.$$

Therefore, one has

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \ge \frac{p(b-a)^{p-1}k}{2^{p-1}d^p}\int_{\frac{a+b}{2}}^b F(x,d)dx. \tag{3.9}$$

From (2.3) if $\Phi(u) \le r$, we have $||u||_\infty \le c$ therefore

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) \le \int_a^b \max_{|\xi|\le c} F(x,\xi)dx. \tag{3.10}$$

Hence, owing to (3.9), (3.10) and (ii) condition (2.1) of Theorem 2.1 is verified.
All assumptions of Theorem 2.1 are satisfied and the proof is complete.

Then, for each $\lambda \in \left]\dfrac{2^{p-1}d^p}{p(b-a)^{p-1}k\int_{\frac{a+b}{2}}^b F(x,d)dx}, \dfrac{q_0 c^p}{p(b-a)^{p-1}\int_a^b \max_{|\xi|\le c} F(x,\xi)dx}\right[$, problem (P) has at least two non-zero weak solutions. □

Remark 3.1. We observe that in the problem (P) we can consider also the case $s \not\equiv 0$ and $\operatorname{ess\,inf}_\Omega s = 0$. Moreover, even when $s \equiv 0$, it is a simple computation to verify that the previous result can be proved in the same way by endowing the space X with the equivalent norm $\|u'\|_{L^p([a,b])}$.

Now, we suppose that $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative function. We note the following Lemma, that is useful to obtain results of existence of nonnegative solutions.

Lemma 3.1. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a nonnegative function. Suppose that $u \in X$ is a weak solution of problem (P). Then u is nonnegative.*

Proof. Put $u^- = -\min\{u, 0\}$, and one has $u^- \in X$. Taking into account that u is a weak solution and choosing $v = u^-$, one has

$$0 \le \lambda \int_a^b f(x,u(x))u^-(x)dx = \int_a^b q(x)|(u^-)'(x)|^p dx + \int_a^b s(x)|u^-(x)|^p dx =$$

$$= -\|u^-\|^p.$$

That is $u^- = 0$ a.e. in $[a,b]$. Hence, our claim is proved. □

Now, we point out a result when the nonlinear term has separable variables. To be precise, let $h : [a, b] \to \mathbb{R}$ be a function such that $h \in L^1([a, b])$, $h(x) \geq 0$ a.e. $x \in [a, b]$, $h \neq 0$ and let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative and continuous function. Consider the following problem

$$\begin{cases} -(q|u'|^{p-2}u')' + r|u|^{p-2}u = \lambda h(x)g(u) \quad \text{on} \quad]a, b[\,, \\ \\ u(a) = u'(b) = 0, \end{cases} \tag{P_1}$$

and put $G(t) = \displaystyle\int_0^t g(\xi)d\xi$ for all $t \in \mathbb{R}$,

$$k' = \frac{\frac{q_0}{2^{p-1}} k \|h\|_{L^1([\frac{a+b}{2}, b])}}{\|h\|_{L^1([a,b])}}, \tag{3.11}$$

where k is given by (2.2).

Taking into account Theorem 3.1 and Lemma 3.1 we have the following result.

Corollary 3.1. *Assume that there exist two positive constants c, d with $d < c$, such that*

$$\frac{G(c)}{c^p} < k' \frac{G(d)}{d^p}$$

where the constants k' is given by (3.11).

Further, suppose that there exist $\mu > p$, $R > 0$ such that

$$0 < \mu G(t) \leq t g(t),$$

for all $|t| \geq R$.

Then, for each $\lambda \in \left] \frac{2^{p-1} d^p}{p(b-a)^{p-1} k \|h\|_{L^1([\frac{a+b}{2}, b])} G(d)}, \frac{q_0 c^p}{p(b-a)^{p-1} \|h\|_{L^1([a,b])} G(c)} \right[$, the problem (P_1) admits two nonnegative and non-zero weak solutions.

A consequence of Corollary 3.1 is the following result.

Theorem 3.2. *Assume that*

$$\lim_{t \to 0^+} \frac{g(t)}{t^{p-1}} = +\infty, \tag{3.12}$$

and suppose that there exist $\mu > p$, $R > 0$ such that

$$0 < \mu G(t) \leq t g(t), \tag{3.13}$$

for all $t \geq R$. Then, for each $\lambda \in]0, \lambda^*[$ where $\lambda^* = \frac{q_0}{p(b-a)^{p-1} \|h\|_{L^1([a,b])}} \sup_{c>0} \frac{c^p}{G(c)}$, the problem (P_1) has at least two nonnegative and non zero weak solutions.

Proof. Fix $\lambda \in]0, \lambda^*[$. Then there is $c > 0$ such that $\lambda < \frac{q_0}{p(b-a)^{p-1} \|h\|_{L^1([a,b])}} \frac{c^p}{G(c)}$. Since g is non negative one has that $\max_{|\xi| \leq c} G(\xi) = G(c)$ for every $c \in \mathbb{R}_+$. From (3.12) there is $d < c$ such that

$$\frac{p(b-a)^{p-1} k \|h\|_{L^1([\frac{a+b}{2}, b])}}{2^{p-1}} \frac{G(d)}{d^p} > \frac{1}{\lambda},$$

where k is given by (2.2). Hence, Corollary 3.1 ensures the conclusion. $\square$

Remark 3.2. We observe that, in literature the existence of at least two nontrivial solutions for differential problems is obtained associating to the classical Ambrosetti-Rabinowitz condition a hypothesis on the nonlinear term of the type $f(x,0) > 0$, see for instance [16]. Instead in our results we can also assume $f(x,0) = 0$ (see Theorem 3.1 and Theorem 3.2).

Remark 3.3. Theorem 1.1 in the Introduction is a consequence of Theorem 3.2 with $q(x) = s(x) = h(x) = 1$ for all $x \in [0,1]$.

Now, we present examples that illustrate our results.

Example 3.1. *Consider $p = 4$ and the function $g(t) = 5t^4 + 1$. We have*

$$\lim_{t\to 0^+} \frac{g(t)}{t^3} = +\infty,$$

and (3.13) *is satisfied. Moreover, one has* $\lambda^* = \frac{1}{4}\sup_{c>0} \frac{c^4}{G(c)} = \frac{\sqrt[4]{27}}{4}$. *Due to Theorem 3.2, for each* $\lambda \in \left]0, \frac{\sqrt[4]{27}}{4}\right[$ *the problem*

$$\begin{cases} -((x+1)|u'|^2u')' + x|u|^2u = \lambda g(u) \quad \text{on} \quad]0,1[\,, \\ \\ u(0) = u'(1) = 0, \end{cases}$$

admits at least two non-zero and nonnegative solutions.

Example 3.2. *Consider $p = 3$ and the function*

$$g(t) = \begin{cases} \frac{3}{2}\sqrt{t} + 5t^4 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

We observe that it is enought to pick for instance $\mu = 4$ and (3.13) *is verified. Due to Theorem 3.2, for each* $\lambda \in \left]0, \frac{2}{21}\sqrt[7]{54}\right[$ *the problem*

$$\begin{cases} -(|u'|u')' + |u|u = \lambda g(u) \quad \text{on} \quad]0,1[\,, \\ \\ u(0) = u'(1) = 0, \end{cases}$$

admits at least two non-zero and nonnegative solutions.

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