.

Minimal star-varieties of polynomial growth and bounded colength

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Abstract. Let $\mathcal{V}$ be a variety of associative algebras with involution $*$ over a field F of characteristic zero. Giambruno and Mishchenko proved in [**6**] that the $*$-codimension sequence of $\mathcal{V}$ is polynomially bounded if and only if $\mathcal{V}$ does not contain the commutative algebra $D = F \oplus F$, endowed with the exchange involution, and M, a suitable 4-dimensional subalgebra of the algebra of 4×4 upper triangular matrices, endowed with the reflection involution. As a consequence the algebras D and M generate the only varieties of almost polynomial growth. In [**20**] the authors completely classify all subvarieties and all minimal subvarieties of the varieties $\mathrm{var}^*(D)$ and $\mathrm{var}^*(M)$. In this paper we exhibit the decompositions of the $*$-cocharacters of all minimal subvarieties of $\mathrm{var}^*(D)$ and $\mathrm{var}^*(M)$ and compute their $*$-colengths. Finally we relate the polynomial growth of a variety to the $*$-colengths and classify the varieties such that their sequence of $*$-colengths is bounded by three.

1. Introduction

Let A be an associative algebra with involution ($*$-algebra) over a field F of characteristic zero and let $c_n^*(A), n = 1, 2, \ldots,$ be its sequence of $*$-codimensions. In case A satisfies a nontrivial identity, it was proved in [**8**] that $c_n^*(A)$ is exponentially bounded. In order to capture the exponential rate of growth of the sequence of $*$-codimensions, recently, in [**7**] the authors proved that for any associative $*$-algebra A, satisfying an ordinary identity,

$$\exp^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^*(A)}$$

exists and is an integer called the $*$-exponent of A.

Given a variety of $*$-algebras $\mathcal{V}$, the growth of $\mathcal{V}$ is the growth of the sequence of $*$-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^*(A)$. In this paper we are interested in varieties of polynomial growth, i.e., varieties of $*$-algebras such that $c_n^*(\mathcal{V}) = c_n^*(A)$ is polynomially bounded.

In such a case, if A is an algebra with 1, in [**19**] it was proved that $c_n^*(A) = qn^k + O(n^{k-1})$ is a polynomial with rational coefficients whose leading term satisfies the inequalities $\frac{1}{k!} \leq q \leq \sum_{i=0}^{k} 2^{k-i} \frac{(-1)^i}{i!}$.

In case of polynomial growth Giambruno and Mishchenko proved in [**6**] that a variety $\mathcal{V}$ has polynomial growth if and only if $\mathcal{V}$ does not contain the commutative algebra $D = F \oplus F$, endowed with the exchange involution, and M, a suitable 4-dimensional subalgebra of the algebra of 4×4 upper triangular matrices, endowed with the reflection involution. As a consequence the $*$-algebras D and M generate the only varieties of almost polynomial growth, i.e, they grow exponentially but any proper subvariety is polynomially bounded.

In [**20**] the authors completely classify all subvarieties of the varieties $\mathrm{var}^*(D)$ and $\mathrm{var}^*(M)$. They also classify all their minimal subvarieties of polynomial growth. We recall that $\mathcal{V}$ is a minimal variety of polynomial growth n^k if asymptotically $c_n^*(\mathcal{V}) \approx an^k$, for some $a \neq 0$, and $c_n^*(\mathcal{U}) \approx bn^t$, with $t < k$, for any proper subvariety $\mathcal{U}$ of $\mathcal{V}$.

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The relevance of the minimal varieties of polynomial growth relies in the fact that these were the building blocks that allowed the authors to give a complete classification of the subvarieties of the varieties of almost polynomial growth (see also [**5, 11, 13, 14, 16, 17**]).

An equivalent formulation of Giambruno-Mishchenko's result can be given as follows. Let P_n^* be the vector space of multilinear polynomials of degree n and $\mathrm{Id}^*(A)$ the ideal of identities satisfied by a $*$-algebra A. The space $\dfrac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)}$ has a structure of $\mathbb{Z}_2 \wr S_n$-module and its character $\chi_n^*(A)$, by complete reducibility, decomposes as

$$\chi_n^*(A) = \sum_{|\lambda| + |\mu| = n} m_{\lambda,\mu}\chi_{\lambda,\mu},$$

where $\chi_{\lambda,\mu}$ is the irreducible $\mathbb{Z}_2 \wr S_n$-character associated to the pair of partitions (λ, μ) and $m_{\lambda,\mu} \geq 0$ is the corresponding multiplicity. Then

$$l_n^*(A) = \sum_{|\lambda| + |\mu| = n} m_{\lambda,\mu},$$

is called the nth $*$-colength of A. If A satisfies a non-trivial identity then $l_n^*(A)$, $n = 1, 2, \ldots$, is polynomially bounded [**1**].

In this paper we state the Giambruno-Mishchenko's result as follows: if A is any $*$-algebra, $c_n^*(A)$ is polynomially bounded if and only if the sequence of $*$-colengths is bounded by a constant, i.e., $l_n^*(A) \leq k$, for some $k \geq 0$ and for all $n \geq 1$. Such result was proved for finite dimensional $*$-algebras in [**24**].

Moreover we exhibit the decompositions of the $*$-cocharacters of all minimal subvarieties of $\mathrm{var}^*(D)$ and $\mathrm{var}^*(M)$, compute their $*$-colengths and complete their $*$-codimensions. Finally we classify the varieties such that their sequence of $*$-colengths is bounded by three, for n large enough. Furthermore we show that if $l_n^*(A) \leq 3$, then for n large enough, $l_n^*(A)$ is always constant.

2. Generalities and basic tools

Throughout this paper we shall denote by F a field of characteristic zero and by A an associative algebra, not necessarily with 1, endowed with an involution $*$ over F. Let us write $A = A^+ \oplus A^-$, where $A^+ = \{a \in A \mid a^* = a\}$ and $A^- = \{a \in A \mid a^* = -a\}$ denote the sets of symmetric and skew elements of A, respectively.

Let $F\langle X, *\rangle$ be the free associative algebra with involution on a countable set $X = \{x_1, x_1^*, x_2, x_2^*, \ldots\}$ of noncommutative variables over F (see [**10**]). It is useful to consider $F\langle X, *\rangle$ as generated by symmetric and skew variables: if we let $y_i = x_i + x_i^*$ and $z_i = x_i - x_i^*$ for $i = 1, 2, \ldots$, then $F\langle X, *\rangle = F\langle y_1, z_1, y_2, z_2, \ldots\rangle$. We say that a polynomial $f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F\langle X, *\rangle$ is a $*$-identity of A, and we write $f \equiv 0$, if $f(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$ for all $a_1, \ldots, a_n \in A^+$ and $b_1, \ldots, b_m \in A^-$.

The set $\mathrm{Id}^*(A)$ of all $*$-identities of A is a T^*-ideal of $F\langle X, *\rangle$, i.e., an ideal invariant under all endomorphisms of the free algebra commuting with the involution and is completely determined by its multilinear polynomials. We denote by P_n^* the space of all multilinear polynomials of degree n in the variables $y_1, z_1, \ldots, y_n, z_n$, i.e,

$$P_n^* = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i = y_i \text{ or } w_i = z_i, i = 1, \ldots, n\}.$$

The dimension of the space $P_n^*(A) = \dfrac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)}$ is called the n-th $*$-codimension of A and is denoted by $c_n^*(A)$.

For $0 \leq r \leq n$, let $P_{r,n-r}^*$ denote the space of multilinear polynomials in the variables $y_1, \ldots, y_r, z_{r+1}, \ldots, z_n$. In order to study the space $P_n^* \cap \mathrm{Id}^*(A)$ it is enough to study $P_{r,n-r}^* \cap \mathrm{Id}^*(A)$, for all $r \geq 0$.

Setting $P_{r,n-r}^*(A) = \dfrac{P_{r,n-r}^*}{P_{r,n-r}^* \cap \mathrm{Id}^*(A)}$ and $c_{r,n-r}^*(A) = \dim P_{r,n-r}^*(A)$ we have that

$$c_n^*(A) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}^*(A). \tag{2.1}$$

Remark 2.1. If A and B are $*$-algebras, it is well known that $A \oplus B$ is a $*$-algebra and $\mathrm{Id}^*(A \oplus B) = \mathrm{Id}^*(A) \cap \mathrm{Id}^*(B)$. Furthermore, $c_n^*(A \oplus B) \le c_n^*(A) + c_n^*(B)$ and the equality holds if and only if

$$\dim \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A) \cap \mathrm{Id}^*(B)} = \dim \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)} + \dim \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(B)}.$$

This is equivalent to saying that $\dim P_n^* = \dim(P_n^* \cap \mathrm{Id}^*(A) + P_n^* \cap \mathrm{Id}^*(B))$, and, so, any polynomial in P_n^* can be written as a sum of multilinear polynomials in $\mathrm{Id}^*(A)$ and in $\mathrm{Id}^*(B)$.

Similarly $c_{r,n-r}^*(A \oplus B) = c_{r,n-r}^*(A) + c_{r,n-r}^*(B)$ if and only if any polynomial in $P_{r,n-r}^*$ can be written as a sum of multilinear polynomials in $\mathrm{Id}^*(A)$ and in $\mathrm{Id}^*(B)$ with r symmetric and $n-r$ skew variables.

Let H_n be the hyperoctahedral group of degree n, i.e., $H_n = \mathbb{Z}_2 \wr S_n$, the wreath product of the multiplicative group of order two with S_n. The space P_n^* has a natural left H_n-module structure induced by defining for $h = (a_1, \dots, a_n; \sigma) \in H_n$, $hy_i = y_{\sigma(i)}$, $hz_i = z_{\sigma(i)}^{a_{\sigma(i)}} = \pm z_{\sigma(i)}$.

Since $P_n^* \cap \mathrm{Id}^*(A)$ is invariant under this H_n-action, the space $P_n^*/(P_n^* \cap \mathrm{Id}^*(A))$ has the structure of a left H_n-module and its character $\chi_n^*(A)$, called the nth $*$-cocharacter of A, decomposes as

$$\chi_n^*(A) = \sum_{|\lambda| + |\mu| = n} m_{\lambda,\mu} \chi_{\lambda,\mu}, \tag{2.2}$$

where $\lambda \vdash r$, $\mu \vdash n-r$, $r = 0, 1, \dots, n$ and $m_{\lambda,\mu} \ge 0$ is the multiplicity of the irreducible H_n-character $\chi_{\lambda,\mu}$ associated to the pair (λ, μ).

Also

$$l_n^*(A) = \sum_{|\lambda| + |\mu| = n} m_{\lambda,\mu}$$

is called the nth $*$-colength of A.

Let $F_m \langle X, * \rangle = \langle y_1, \dots, y_m, z_1, \dots, z_m \rangle$ denote the free associative algebra with involution in m symmetric and skew variables and let $U = \mathrm{span}_F\{y_1, \dots, y_m\}, V = \mathrm{span}_F\{z_1, \dots, z_m\}$. There is a natural left action of the group $GL(U) \times GL(V) \cong GL_m \times GL_m$ on the space $U \oplus V$ and we can extend this action diagonally to get an action on $F_m \langle X, * \rangle$. Note that for any algebra A with involution, the space $F_m \langle X, * \rangle \cap \mathrm{Id}^*(A)$ is invariant under this action.

So by considering $F_m^n \langle X, * \rangle$, the space of all homogeneous polynomials of degree n in the variables $y_1, \dots, y_m, z_1, \dots, z_m$, we have that

$$F_m^n(A) := F_m^n \langle X, * \rangle / (F_m^n \langle X, * \rangle \cap \mathrm{Id}^*(A))$$

is a $GL_m \times GL_m$-module and we denote its character by $\psi_n^*(A)$. It is well known (see [**2**, Theorem 12.4.4]) that there is a one-to-one correspondence between irreducible $GL_m \times GL_m$-characters and pairs of partitions (λ, μ), with $\lambda \vdash n-r$ and $\mu \vdash r$, $r = 0, \dots, n$ where λ and μ are partitions with at most m parts.

If $\psi_{\lambda,\mu}$ denotes the irreducible $GL_m \times GL_m$-character corresponding to (λ, μ) then we can write

$$\psi_n^*(A) = \sum_{\substack{|\lambda| + |\mu| = n \\ h(\lambda), h(\mu) \le m}} \tilde{m}_{\lambda,\mu} \psi_{\lambda,\mu} \tag{2.3}$$

where $\tilde{m}_{\lambda,\mu}$ are the corresponding multiplicities and $h(\lambda)$ (respectively $h(\mu)$) denotes the height of the Young diagram corresponding to λ (respectively μ).

In order to calculate the multiplicity $m_{\lambda,\mu}$ of an irreducible character $\chi_{\lambda,\mu}$ in the decomposition (2.2), we use the following relationship proved by Giambruno in [**3**, Theorem 3]

$$m_{\lambda,\mu} = \tilde{m}_{\lambda,\mu}, \quad \text{for all } \lambda \vdash n-r \text{ and } \mu \vdash r \text{ with } h(\lambda),\ h(\mu) \le m. \tag{2.4}$$

It is well known that an irreducible submodule of $F_m^{n\,*}(A)$ corresponding to the pair (λ, μ) is generated by a non-zero polynomial $f_{\lambda,\mu}$, called *highest weight vector*, of the form (see for instance [**2**, Theorem 12.4.12])

$$\begin{aligned} &f_{\lambda,\mu}(y_1, \dots, y_p, z_1, \dots, z_q) \\ &\quad = \prod_{i=1}^{\lambda_1} St_{h_i(\lambda)}(y_1, \dots, y_{h_i(\lambda)}) \prod_{i=1}^{\mu_1} St_{h_i(\mu)}(z_1, \dots, z_{h_i(\mu)}) \sum_{\sigma \in S_n} \alpha_\sigma \sigma, \end{aligned} \tag{2.5}$$

where $\alpha_\sigma \in F$, $St_k(x_1, \dots, x_k) = \sum_{\sigma \in S_k} (\text{sign } \sigma) x_{\sigma(1)} \cdots x_{\sigma(k)}$ is the standard polynomial of degree k and S_n acts from right by permuting places in which the variables occur.

Let T_λ and T_μ be two Young tableaux. We denote by f_{T_λ, T_μ} the highest weight vector obtained from (2.5) by considering the only permutation $\sigma \in S_n$ such that the integers $\sigma(1), \dots, \sigma(h_1(\lambda))$, in this order, fill in from top to bottom the first column of T_λ, $\sigma(h_1(\lambda)+1), \dots, \sigma(h_1(\lambda)+h_2(\lambda))$ the second column of T_λ, $\dots$, $\sigma(h_1(\lambda)+\cdots+h_{\lambda_1-1}(\lambda)+1), \dots, \sigma(r)$ the last column of T_λ; also $\sigma(r+1), \dots, \sigma(r+h_1(\mu))$ fill in the first column of T_μ, $\dots$, $\sigma(r+h_1(\mu)+\cdots+h_{\mu_1-1}(\mu)+1), \dots, \sigma(n)$ the last column of T_μ.

Remark 2.2. (see [**2**]) In the decomposition (2.3) we have $\tilde{m}_{\lambda,\mu} \neq 0$ if and only if there exists a pair of tableaux (T_λ, T_μ) such that the corresponding highest weight vector f_{T_λ, T_μ} is not a $*$-identity of A. Moreover $\tilde{m}_{\lambda,\mu}$ is the maximal number of linearly independent highest weight vectors f_{T_λ, T_μ} in $F_m^n(A)$.

3. Varieties of almost polynomial growth and their subvarieties

The purpose of this section is to study the sequences of $*$-cocharacters, $*$-codimensions and $*$-colengths of the minimal subvarieties of polynomial growth of the varieties of almost polynomial growth, which are classified in [**20**].

We denote by $UT_s = UT_s(F)$ the algebra of the $s \times s$ upper triangular matrices over F and by I_s the $s \times s$ identity matrix. Recall that the varieties of almost polynomial growth are generated by the following two algebras (see [**6**])

1) $F \oplus F$, the two-dimensional commutative algebra with the exchange involution $(a,b)^* = (b,a)$;
2) $M = \left\{ \begin{pmatrix} u & r & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & v \\ 0 & 0 & 0 & u \end{pmatrix} \mid u, r, s, v \in F \right\}$, the subalgebra of UT_4 with the reflection involution, i.e., the involution obtained by reflecting a matrix along its secondary diagonal: if $a = \alpha(e_{11}+e_{44}) + \beta(e_{22}+e_{33}) + \gamma e_{12} + \delta e_{34}$ then $a^* = \alpha(e_{11}+e_{44}) + \beta(e_{22}+e_{33}) + \delta e_{12} + \gamma e_{34}$, where the e_{ij}s denote the usual matrix units.

The above algebras characterize the varieties of $*$-algebras of polynomial growth.

Theorem 3.1. [**6**, Theorem 4.7] *Let A be a $*$-algebra. Then the sequence $c_n^*(A)$, $n = 1, 2, \dots$, is polynomially bounded if and only if $M, D \notin \operatorname{var}^*(A)$.*

We start by presenting $*$-algebras belonging to the variety generated by D and generating minimal varieties of polynomial growth (see [**20**]).

For $k \geq 2$, let
$$C_k = \{\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F\}$$
be the commutative subalgebra of UT_k with involution given by
$$(\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i)^* = \alpha I_k + \sum_{1 \leq i < k} (-1)^i \alpha_i E_1^i.$$
Here $E_1 = \sum_{i=1}^{k-1} e_{i,i+1}$.

Since D is commutative, any antiautomorphism of D is an automorphism and, so, D can be viewed as a superalgebra with grading $(D^{(0)}, D^{(1)})$, where $D^{(0)} = D^+$ and $D^{(1)} = D^-$. Hence, the classification of the $*$-algebras, up to T^*-equivalence, inside $\operatorname{var}^*(D)$ and the classification of the superalgebras inside $\operatorname{var}^{gr}(D)$ are equivalent. In the light of these considerations we have the following.

Theorem 3.2. [**20**, Lemma 9],[**23**, Theorem 8.3] *Let $k \geq 2$. Then*

(1) $\operatorname{Id}^*(C_k) = \langle [y_1, y_2], [y, z], [z_1, z_2], z_1 \cdots z_k \rangle_{T^*}$.

(2) $c_n^*(C_k) = \sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!} n^{k-1}, n \to \infty$.

(3) $\chi_n^*(C_k) = \sum_{j=0}^{k-1} \chi_{(n-j),(j)}$ *and* $l_n^*(C_k) = k$.

Given two $*$-algebras A and B, we say that A is T^*-equivalent to B, and we write $A \sim_{T^*} B$, in case $\mathrm{Id}^*(A) = \mathrm{Id}^*(B)$.

The following theorem classifies the subvarieties and the minimal varieties of $\mathrm{var}^*(D)$.

THEOREM 3.3. [**20**, Theorem 7 and Corollary 3] *Let A be a $*$-algebra such that $\mathrm{var}^*(A) \subsetneq \mathrm{var}^*(D)$. Then*

(1) *either $A \sim_{T^*} N$ or $A \sim_{T^*} C \oplus N$ or $A \sim_{T^*} C_k \oplus N$, for some $k \geq 2$, where N is a nilpotent $*$-algebra and C is a non-nilpotent commutative $*$-algebra with trivial involution.*

(2) *The algebra A generates a minimal variety of polynomial growth if and only if $A \sim_{T^*} C_k$, for some $k \geq 2$.*

Next we exhibit the decomposition of the $*$-cocharacter of all minimal subvarieties of $\mathrm{var}^*(M)$.

We start by recalling $*$-algebras inside $\mathrm{var}^*(M)$ generating minimal varieties of polynomial growth.

For any $k \geq 2$, consider the following subalgebras of UT_{2k} endowed with the reflection involution:

$$N_k = \mathrm{span}_F\{I_{2k}, E, \dots, E^{k-2}; e_{12} - e_{2k-1,2k}, e_{13}, \dots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \dots, e_{2k-2,2k}\}$$

$$U_k = \mathrm{span}_F\{I_{2k}, E, \dots, E^{k-2}; e_{12} + e_{2k-1,2k}, e_{13}, \dots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \dots, e_{2k-2,2k}\},$$

$$A_k = \mathrm{span}_F\{e_{11} + e_{2k,2k}, E, \dots, E^{k-2}; e_{12}, e_{13}, \dots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \dots, e_{2k-1,2k}\},$$

where $E = \sum_{i=2}^{k-1} e_{i,i+1} + e_{2k-i,2k-i+1}$.

Notice that in case $k = 2$, we have that U_2 is T^*-equivalent to the commutative algebra with trivial involution, so $\mathrm{Id}^*(U_2) = \langle [y_1, y_2], z_1 \rangle_{T^*}$ and $c_n^*(U_2) = 1$.

The following results describe the T^*-ideals of the above algebras and explicit the $*$-codimensions of N_k and U_k.

LEMMA 3.4. [**20**, Lemma 2] *Let $k \geq 2$. Then*

(1) $\mathrm{Id}^*(N_k) = \langle [y_1, \dots, y_{k-1}], z_1 z_2 \rangle_{T^*}$, *in case $k \geq 3$ and* $\mathrm{Id}^*(N_k) = \langle [y_1, y_2], [y, z], z_1 z_2 \rangle_{T^*}$, *in case $k = 2$.*

(2) $c_n^*(N_k) = 1 + \sum_{j=1}^{k-2} \binom{n}{j}(2j-1) + \binom{n}{k-1}(k-1) \approx qn^{k-1}$, *for some $q > 0$.*

LEMMA 3.5. [**20**, Lemma 3] *Let $k \geq 3$. Then*

(1) $\mathrm{Id}^*(U_k) = \langle [z, y_1, \dots, y_{k-2}], z_1 z_2 \rangle_{T^*}$.

(2) $c_n^*(U_k) = 1 + \sum_{j=1}^{k-2} \binom{n}{j}(2j-1) + \binom{n}{k-1}(k-2) \approx qn^{k-1}$, *for some $q > 0$.*

LEMMA 3.6. [**20**, Lemma 3] *Let $k \geq 2$. Then*

$$\mathrm{Id}^*(A_k) = \langle y_1 \cdots y_{k-2} St_3(y_{k-1}, y_k, y_{k+1}) y_{k+2} \cdots y_{2k-1}, y_1 \cdots y_{k-1} z y_k \cdots y_{2k-2}, z_1 z_2 \rangle_{T^*}.$$

The relevance of the above $*$-algebras is shown in the following.

THEOREM 3.7. [**20**, Theorem 6 and Corollary 1] *Let A be a $*$-algebra such that $\mathrm{var}^*(A) \subsetneq \mathrm{var}^*(M)$. Then*

(1) *A is T^*-equivalent to one of the following $*$-algebras: N, $N_k \oplus N$, $U_k \oplus N$, $N_k \oplus U_k \oplus N$, $A_t \oplus N$, $N_k \oplus A_t \oplus N$, $U_k \oplus A_t \oplus N$, $N_k \oplus U_k \oplus A_t \oplus N$, for some $k, t \geq 2$, where N is a nilpotent $*$-algebra.*

(2) *A generates a minimal variety if and only if either $A \sim_{T^*} U_r$ or $A \sim_{T^*} N_k$ or $A \sim_{T^*} A_k$, for some $k \geq 2, r > 2$.*

Next we determine the $*$-codimensions of the algebra A_k, for any $k \geq 2$. We start by considering the case $k = 2$.

LEMMA 3.8. *$c_n^*(A_2) = 4n - 1$, for $n \geq 3$.*

PROOF. We have $\mathrm{Id}^*(A_2) = \langle St_3(y_1, y_2, y_3), y_1 z y_2, z_1 z_2\rangle_{T^*}$. Since $z_1 z_2 \in \mathrm{Id}^*(A_2)$, by [**21**, Remark 8], we have $z_1 w z_2 \in \mathrm{Id}^*(A_2)$ for any monomial w of $F\langle X, *\rangle$, and, so $c^*_{n-r,r}(A_k) = 0$ for all $r \geq 2$. Thus by (2.1)

$$c^*_n(A_2) = c^*_{n,0}(A_2) + nc^*_{n-1,1}(A_2). \tag{3.1}$$

We start by considering $P^*_{n,0}(A_2)$. By the Poincaré-Birkhoff-Witt theorem (see [**2**]), every monomial in $y_1, \dots, y_n$ can be written as a linear combination of products of the type

$$y_{i_1} \cdots y_{i_s} w_1 \cdots w_m \tag{3.2}$$

where $w_1, \dots, w_m$ are left normed Lie commutators in y_i's and $i_1 < \cdots < i_s$. Since $[y_1, y_2][y_3, y_4] \in \mathrm{Id}^*(A_2)$, we get that, modulo $\langle [y_1, y_2][y_3, y_4]\rangle_{T^*}$, at most one commutator can appear in (3.2) and the elements in (3.2) are polynomials of type

$$y_1 \cdots y_n \text{ or } y_{i_1} \cdots y_{i_s}[y_r, y_{j_1}, \dots, y_{j_t}] \text{ with } r > j_i < \cdots < j_t.$$

Moreover, modulo $\langle y_1[y_2, y_3]y_4\rangle_{T^*}$, we have that

$$[y_r, y_{j_1} \dots, y_{j_t}] = [y_r, y_{j_1}]y_{j_2} \cdots y_{j_t} \pm y_{j_t} \cdots y_{j_2}[y_r, y_{j_1}].$$

Then modulo $\mathrm{Id}^*(A_2)$, every polynomial in $P^*_{n,0}$ can be written as a linear combination of elements of the type

$$[y_r, y_1]y_2 \cdots \widehat{y}_r \cdots y_n, \quad y_{i_1} \cdots y_{i_{n-2}}[y_i, y_j] \text{ and } y_1 \cdots y_n, \tag{3.3}$$

$2 \leq r \leq n, 1 \leq i \leq j \leq n$, where the symbol $\widehat{y}_r$ means that the variable y_r is omitted. Notice that elements of the first type only appear in case $s = 0$ in (3.2). Because of $[y_1, y_2][y_3, y_4] \in \mathrm{Id}^*(A_2)$ the variables out of the commutator in the polynomials of the second type in (3.3) can be ordered. Moreover, since $St_3(y_1, y_2, y_3) \in \mathrm{Id}^*(A_2)$, $y_1[y_2, y_3] \equiv y_2[y_1, y_3] + y_3[y_2, y_1]$ can be applied and we obtain that the polynomials

$$[y_r, y_1]y_2 \cdots \widehat{y}_r \cdots y_n, \quad y_2 \cdots \widehat{y}_r \cdots y_n[y_r, y_1] \text{ and } y_1 \cdots y_n, \quad 2 \leq r \leq n \tag{3.4}$$

generate $P^*_{n,0}$ modulo $P^*_{n,0} \cap \mathrm{Id}^*(A_2)$.

We claim that these polynomials form a basis of $P^*_{n,0}(A_2)$. Suppose that $f \in P^*_{n,0} \cap \mathrm{Id}^*(A_2)$ is a linear combination of the polynomials in (3.4) and write

$$f = \alpha y_1 \cdots y_n + \sum_{j=2}^{n} \alpha_j [y_j, y_1]y_2 \cdots \widehat{y}_j \cdots y_n + \sum_{j=2}^{n} \beta_j y_2 \cdots \widehat{y}_j \cdots y_n[y_j, y_1].$$

By making the evaluation $y_i = e_{11} + e_{44}$, for all $i = 1, \dots, n$, we get $\alpha(e_{11} + e_{44}) = 0$, and, so, $\alpha = 0$. Now for a fixed j, the evaluation $y_j = e_{12} + e_{34}$ and $y_i = e_{11} + e_{44}$, for all $i \neq j$ gives $\alpha_j e_{34} - \beta_j e_{12} = 0$, and so, $\alpha_j = \beta_j = 0$ and the claim is proved. Thus $c^*_{n,0}(A_2) = 1 + 2(n-1) = 2n-1$.

We now consider $P^*_{n-1,1}(A_2)$. Since $y_1 z y_2 \in \mathrm{Id}^*(A_2)$, then, modulo $P^*_{n-1,1} \cap \mathrm{Id}^*(A_2)$, $P^*_{n-1,1}$ can be generated by the monomials

$$z_n y_1 \cdots y_{n-1} \text{ and } y_1 \cdots y_{n-1} z_n. \tag{3.5}$$

We claim that these polynomials form a basis of $P^*_{n-1,1}$ modulo $P^*_{n-1,1} \cap \mathrm{Id}^*(A_2)$. Let $f = \alpha z_n y_1 \cdots y_{n-1} + \beta y_1 \cdots y_{n-1} z_n \in P^*_{n-1,1} \cap \mathrm{Id}^*(A_2)$. By making the evaluation $z_n = e_{12} - e_{34}$ and $y_i = e_{11} + e_{44}$, for all $i \neq n$, we get $-\alpha e_{34} + \beta e_{12} = 0$ and so $\alpha = \beta = 0$. Thus $c^*_{n-1,1}(A_2) = 2$.

Hence, from (3.1) it follows that $c^*_n(A_2) = 2n - 1 + 2n = 4n - 1$. □

REMARK 3.9. For $k \geq 3$, let

$$I_1 = \langle [y_1, y_2]\,[y_3, y_4]\,, [y_1, y_2]\, y_3 \cdots y_{k+1}\rangle_{T^*} \text{ and } I_2 = \langle [y_1, y_2]\,[y_3, y_4]\,, y_3 \cdots y_{k+1}\,[y_1, y_2]\rangle_{T^*}\,.$$

By [**13**, Lemma 3.1],

$$c^*_{n,0}(I_1) = c^*_{n,0}(I_2) = 1 + \sum_{j=0}^{k-2} \binom{n}{j}(n-j-1).$$

Moreover, if I is the T^*-ideal $I_1 \cap I_2$ then, by [**13**, Lemma 3.4],

$$I = \langle [y_1, y_2]\,[y_3, y_4]\,, y_1 \cdots y_{k-1}\,[y_k, y_{k+1}]\, y_{k+2} \cdots y_{2k}\rangle_{T^*}\,.$$

From Remark 2.1, we have the strict inequality $c^*_{n,0}(I) < c^*_{n,0}(I_1)+c^*_{n,0}(I_2)$ since $y_1 \cdots y_n$ is a polynomial in $P^*_{n,0}$ which is not in $(P^*_{n,0} \cap I_1) + (P^*_{n,0} \cap I_2)$. Moreover, since $I \cap P^*_{n,0} \subset \mathrm{Id}^*(A_k) \cap P^*_{n,0}$, we have

$$c^*_{n,0}(A_k) \leq c^*_{n,0}(I) < c^*_{n,0}(I_1) + c^*_{n,0}(I_2) = 2 + 2\sum_{j=0}^{k-2}\binom{n}{j}(n-j-1). \tag{3.6}$$

LEMMA 3.10. *Let $k \geq 2$. Then*

$$c^*_n(A_k) = 1 + 2\sum_{j=0}^{k-2}\binom{n}{j}(n-j) + 2\sum_{j=0}^{k-2}\binom{n}{j}(n-j-1) \approx qn^{k-1}, \text{ for some } \ q > 0.$$

PROOF. The result has already been proved for $k = 2$ in Lemma 3.8 so we consider $k \geq 3$. Since $z_1z_2 \in \mathrm{Id}^*(A_k)$, by [**21**, Remark 8] we have that $z_1wz_2 \in \mathrm{Id}^*(A_k)$, for any monomial w of $F\langle X, *\rangle$, and, so $P^*_{n-r,r}(A_k) = \{0\}$ for all $r \geq 2$ and

$$c^*_n(A_k) = c^*_{n,0}(A_k) + nc^*_{n-1,1}(A_k). \tag{3.7}$$

Let us study the dimensions of $P^*_{n,0}(A_k)$ and $P^*_{n-1,1}(A_k)$. We start by considering $P^*_{n,0}(A_k)$. We claim that the following polynomials in $P^*_{n,0}$

$$y_1 \cdots y_n, \ y_{i_1} \cdots y_{i_t}[y_r, y_m]y_{j_1} \cdots y_{j_s}, \ y_{p_1} \cdots y_{p_u}[y_a, y_b]y_{q_1} \cdots y_{q_v} \tag{3.8}$$

where $t < k-1$, $i_1 < \cdots < i_t$, $r > m < j_1 < \cdots < j_s$ and $v < k-1$, $a > b < p_1 < \cdots < p_u$, $q_1 < \cdots < q_v$ are linearly independent modulo $\mathrm{Id}^*(A_k)$. Suppose that $f \in P^*_{n,0} \cap \mathrm{Id}^*(A_k)$ is a linear combination of the above polynomials and write

$$f = \alpha y_1 \cdots y_n + \sum_{\substack{t<k-1 \\ \text{or} \\ s<k-1}} \sum_{r,I,J} \alpha_{r,I,J} y_{i_1} \cdots y_{i_t}[y_r, y_m]y_{j_1} \cdots y_{j_s},$$

where $t + s = n-2$ and for any fixed t and s, $I = \{i_1, \ldots, i_t\}$ and $J = \{j_1, \ldots, j_s\}$. If $t < k-1$ then $i_1 < \cdots < i_t$ and $r > m < j_1 < \cdots < j_s$ and if $s < k-1$ then $r > m < i_1 < \cdots < i_t$ and $j_1 < \cdots < j_s$.

First suppose that $\alpha \neq 0$. Then by making the evaluation $y_1 = \cdots = y_n = e_{11} + e_{2k,2k}$ we get $\alpha(e_{11} + e_{2k,2k}) = 0$ and so $\alpha = 0$, a contradiction.

Now suppose that $\alpha_{r,I,J} \neq 0$, for some $t < k-1, r, I$ and J. Then by making the evaluation $y_{i_1} = \cdots = y_{i_t} = E$, $y_r = e_{12}+e_{2k-1,2k}$ and $y_m = y_{j_1} = \cdots = y_{j_s} = e_{11}+e_{2k,2k}$ we get $\alpha_{r,I,J}e_{2k-t-1,2k} - \alpha_{r,J,I}e_{1,2+t} = 0$, and, so, $\alpha_{r,I,J} = \alpha_{r,J,I} = 0$, a contradiction. Similarly, if $\alpha_{r,J,I} \neq 0$, for some $s < k-1, r, I$ and J, by making the evaluation $y_m = y_{i_1} = \cdots = y_{i_t} = e_{11} + e_{2k,2k}$, $y_r = e_{12} + e_{2k-1,2k}$ and $y_{j_1} = \cdots = y_{j_s} = E$ we get $\alpha_{r,I,J} = \alpha_{r,J,I} = 0$, a contradiction as before.

In (3.8) we have $1+2\sum\limits_{j=0}^{k-2}\binom{n}{j}(n-j-1)$ polynomials which are linearly independent modulo $P^*_{n,0}\cap\mathrm{Id}^*(A_k)$ so we have

$$1 + 2\sum_{j=0}^{k-2}\binom{n}{j}(n-j-1) \leq c^*_{n,0}(A_k).$$

On the other hand, by (3.6) we get

$$c^*_{n,0}(A_k) < 2 + 2\sum_{j=0}^{k-2}\binom{n}{j}(n-j-1).$$

Thus we conclude that $c^*_{n,0}(A_k) = 1 + 2\sum\limits_{j=0}^{k-2}\binom{n}{j}(n-j-1)$.

Now we consider $P^*_{n-1,1}(A_k)$. Since $y_1 \cdots y_{k-1}zy_k \cdots y_{2k-2} \in \mathrm{Id}^*(A_k)$, then $P^*_{n-1,1}$ can be generated modulo $\mathrm{Id}^*(A_k)$ by the monomials

$$y_{i_1} \cdots y_{i_t} z_n y_{j_1} \cdots y_{j_s} \tag{3.9}$$

where $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$ and we have $t < k-1$ or $s < k-1$.

We next show that these polynomials are linearly independent modulo $\mathrm{Id}^*(A_k)$. Suppose that $f \in P^*_{n-1,1} \cap \mathrm{Id}^*(A)$ is a linear combination of the polynomials above and write

$$f = \sum_{\substack{t<k-1 \\ \text{or} \\ s<k-1}} \sum_{I,J} \alpha_{I,J} y_{i_1} \cdots y_{i_t} z_n y_{j_1} \cdots y_{j_s}$$

where $t+s = n-1$ and for any fixed t and s, $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$, $I = \{i_1, \ldots, i_t\}$ and $J = \{j_1, \ldots, j_s\}$.

Suppose $\alpha_{I,J} \neq 0$, for some $t < k-1, I$ and J. By making the evaluation $z_n = e_{12} - e_{2k-1,2k}$, $y_{i_1} = \cdots = y_{i_t} = E$ and $y_{j_1} = \cdots = y_{j_s} = e_{11} + e_{2k,2k}$ we get $-\alpha_{I,J} e_{2k-t-1,2k} + \alpha_{J,I} e_{1,2+t} = 0$, thus $\alpha_{I,J} = \alpha_{J,I} = 0$, a contradiction.

Suppose now $\alpha_{J,I} \neq 0$, for some $s < k-1, I$ and J. Then the evaluation $z_n = e_{12} - e_{2k-1,2k}$, $y_{i_1} = \cdots = y_{i_t} = e_{11} + e_{2k,2k}$ and $y_{j_1} = \cdots = y_{j_s} = E$ gives $\alpha_{J,I} = 0$, a contradiction. Thus the polynomials in (3.9) form a basis of $P^*_{n-1,1}(A_k)$ and by counting we get $c^*_{n-1,1}(A_k) = 2\sum_{j=0}^{k-2} \binom{n-1}{j}$. So $nc^*_{n-1,1}(A_k) = 2\sum_{j=0}^{k-2} \binom{n}{j}(n-j)$.

Finally, by (3.7), we have

$$c^*_n(A_k) = 1 + 2\sum_{j=0}^{k-2} \binom{n}{j}(n-j-1) + 2\sum_{j=0}^{k-2} \binom{n}{j}(n-j).$$

□

Next we explicitly determine the sequences of $*$-cocharacters and $*$-colengths of the minimal varieties $\mathrm{var}^*(A) \subseteq \mathrm{var}^*(M)$. If $\chi^*_n(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}$ is the decomposition of the nth $*$-cocharacter of A, we denote by $d_{\lambda,\mu}$ the degree of the H_n-character $\chi_{\lambda,\mu}$.

We shall prove all theorems by using induction on k, so for each class of algebras N_k, U_k and A_k we start with a lemma about the sequence of $*$-cocharacters in a particular case. We start with the study of $*$-cocharacters and $*$-colengths of the minimal varieties $\mathrm{var}^*(A_k)$.

Lemma 3.11. *$\chi^*_n(A_2) = \chi_{(n),\emptyset} + 2\chi_{(n-1,1),\emptyset} + 2\chi_{(n-1),(1)}$ and $l^*_n(A_2) = 5$*

Proof. Let $\chi^*_n(A_2) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}$ be the decomposition of the nth $*$-cocharacter of A_2. Notice that

$$d_{(n),\emptyset} + 2d_{(n-1),(1)} + 2d_{(n-1,1),\emptyset} = 1 + 2n + 2(n-1) = c^*_n(A_2).$$

Then, since that $m_{(n),\emptyset} = 1$, if we find two linearly independent highest weight vectors for each pair of partitions $((n-1),(1))$ and $((n-1,1),\emptyset)$ which are not $*$-identities of A_2 we may conclude that $\chi^*_n(A_2)$ has the desired decomposition.

In fact, let

$$f_1 = y^{n-1}z \quad \text{and} \quad f_2 = zy^{n-1}$$

be highest weight vectors associated to the pair of partitions $((n-1),(1))$ and corresponding to the pairs of tableaux:

(3.10) $\left(\boxed{1 \mid 2 \mid \cdots \mid n-1}\ ,\ \boxed{n}\right)$ and $\left(\boxed{2 \mid 3 \mid \cdots \mid n}\ ,\ \boxed{1}\right)$,

respectively. It is clear that by making the evaluation $y = e_{11}+e_{44}$ and $z = e_{12}-e_{34}$, we get that $f_1 = e_{12} \neq 0$ and $f_2 = -e_{34} \neq 0$. This says that f_1 and f_2 are not $*$-identities of A_2. Moreover by making the same evaluation we have that $\alpha f_1 + \beta f_2 = 0$ implies $\alpha = \beta = 0$, so these polynomials are linearly independent modulo $\mathrm{Id}^*(A_2)$.

On the other hand,

$$g_1 = [y_1, y_2]y_1^{n-2} \quad \text{and} \quad g_2 = y_1^{n-2}[y_1, y_2]$$

are the highest weight vector associated to the pair of partitions $((n-1,1),\emptyset)$ and corresponding to the pairs of tableaux:

(3.11) $\left(\begin{array}{|c|c|c|c|}\hline 1 & 3 & \cdots & n \\ \hline 2 \\ \cline{1-1}\end{array}\ ,\ \emptyset\right)$ and $\left(\begin{array}{|c|c|c|c|}\hline n-1 & 1 & \cdots & n-2 \\ \hline n \\ \cline{1-1}\end{array}\ ,\ \emptyset\right)$,

respectively.

By making the evaluation $y_1 = e_{11}+e_{44}$ and $y_2 = e_{12}+e_{34}$, we get that $g_1 = -e_{34} \neq 0$ and $g_2 = e_{12} \neq 0$. It shows that g_1 and g_2 are not $*$-identities of A_2 and by making the same evaluation we have that $\alpha g_1+\beta g_2 = 0$ implies $\alpha = \beta = 0$, so these polynomials are linearly independent modulo $\mathrm{Id}^*(A_2)$.

Thus $\chi_n^*(A_2) = \chi_{(n),\emptyset} + 2\chi_{(n-1),(1)} + 2\chi_{(n-1,1),\emptyset}$ and $l_n^*(A_2) = 5$. □

Before giving the decomposition of the $\chi_n^*(A_k)$, for any $k \geq 2$, we prove the following.

REMARK 3.12. Let $k \geq 2$. Then

$$\begin{aligned} c_n^*(A_k) &= d_{(n),\emptyset} + \sum_{j=1}^{k-1} 2(k-j)d_{(n-j,j),\emptyset} + \sum_{j=1}^{k-2} 2(k-j-1)d_{(n-j-1,j,1),\emptyset} \\ &\quad + \sum_{j=0}^{k-2} 2(k-j-1)d_{(n-j-1,j),(1)}. \end{aligned}$$

PROOF. We use induction on k. By Lemma 3.11, we have that $\chi_n^*(A_2) = \chi_{(n),\emptyset}+2\chi_{(n-1,1),\emptyset}+2\chi_{(n-1),(1)}$. This says that $c_n^*(A_2) = d_{(n),\emptyset} + 2d_{(n-1,1),\emptyset} + 2d_{(n-1),(1)}$ and, so the result is true for $k = 2$.

Now we suppose the result is true for some $k \geq 2$. By Lemma 3.10, we have that

$$c_n^*(A_{k+1}) = c_n^*(A_k) + 2\binom{n}{k-1}(n-k) + 2\binom{n}{k-1}(n-k+1).$$

Hence, by using that

$$\sum_{j=1}^{k} d_{(n-j,j),\emptyset} + \sum_{j=1}^{k-1} d_{(n-j,j-1,1),\emptyset} = \binom{n}{k-1}(n-k) \text{ and } \sum_{j=0}^{k-1} d_{(n-j,j-1),(1)} = \binom{n}{k-1}(n-k+1),$$

we have

$$\begin{aligned} c_n^*(A_{k+1}) &= c_n^*(A_k) + 2\tbinom{n}{k-1}(n-k) + 2\tbinom{n}{k-1}(n-k+1) \\ &= c_n^*(A_k) + 2\sum_{j=1}^{k} d_{(n-j,j),\emptyset} + 2\sum_{j=1}^{k-1} d_{(n-j-1,j,1),\emptyset} + 2\sum_{j=0}^{k-1} d_{(n-j-1,j),(1)} \\ &= d_{(n),\emptyset} + \sum_{j=1}^{k} 2(k+1-j)d_{(n-j,j),\emptyset} + \sum_{j=1}^{k-1} 2(k-j)d_{(n-j-1,j,1),\emptyset} \\ &\quad + \sum_{j=0}^{k-1} 2(k-j)d_{(n-j-1,j),(1)}. \end{aligned}$$

Thus the result is true for any $k \geq 2$. □

In the next lemmas, we shall adopt the convention that the symbols $\bar{\ }$, $\bar{\bar{\ }}$ and $\tilde{\ }$ indicate alternation on a given set of variables. Thus, for instance, the notation $\bar{\bar{y}}_1\bar{y}_1\tilde{y}_1y_4\bar{y}_2\bar{\bar{y}}_2\tilde{y}_2\bar{y}_3$ indicates the polynomial

$$\sum_{\substack{\sigma\in S_3 \\ \rho,\tau\in S_2}} (\mathrm{sign}\rho)(\mathrm{sign}\sigma)(\mathrm{sign}\tau)y_{\rho(1)}y_{\sigma(1)}y_{\tau(1)}y_4y_{\sigma(2)}y_{\rho(2)}y_{\tau(2)}y_{\sigma(3)}.$$

Now we are in position to compute the $*$-cocharacter and the $*$-colength of A_k, for any $k \geq 2$.

THEOREM 3.13. *For $k \geq 2$, we have*

(1) $\chi_n^*(A_k) = \chi_{(n),\emptyset} + \sum_{j=1}^{k-1} 2(k-j)\chi_{(n-j,j),\emptyset} + \sum_{j=1}^{k-2} 2(k-j-1)\chi_{(n-j-1,j,1),\emptyset}$
$+ \sum_{j=0}^{k-2} 2(k-j-1)\chi_{(n-j-1,j),(1)}$.

(2) $l_n^*(A_k) = 3k^2 - 5k + 3$.

PROOF. By the previous remark, we have that, for any $k \geq 2$,

$$\begin{aligned} c_n^*(A_k) &= d_{(n),\emptyset} + \sum_{j=1}^{k-1} 2(k-j)d_{(n-j,j),\emptyset} + \sum_{j=1}^{k-2} 2(k-j-1)d_{(n-j-1,j,1),\emptyset} \\ &\quad + \sum_{j=0}^{k-2} 2(k-j-1)d_{(n-j-1,j),(1)}. \end{aligned}$$

It is clear that $m_{(n),\emptyset} = 1$. In order to prove the desired decomposition of $\chi_n^*(A_k)$, we shall prove that the irreducible characters $\chi_{(n-j,j),\emptyset}$, $\chi_{(n-l-1,l,1),\emptyset}$ and $\chi_{(n-t-1,1)(1)}$, for $1 \le j \le k-1$, $1 \le l \le k-2$ and $0 \le t \le k-2$, appear in the decomposition of the $*$-cocharacter $\chi_n^*(A_k)$ with multiplicity $m_{(n-j,j),\emptyset} = 2(k-j)$, $m_{(n-l-1,l,1),\emptyset} = 2(k-l-1)$ and $m_{(n-t-1,1)(1)} = 2(k-t-1)$, respectively.

(i) For the pair of partitions $((n-1,1),\emptyset)$, for any $0 \le p \le k-2$ we consider the following pairs of tableaux:

$$\left(\begin{array}{|c|c|c|c|c|c|c|}\hline p+1 & 1 & \cdots & p & p+3 & \cdots & n \\ \hline p+2 \\ \cline{1-1}\end{array}, \emptyset\right)$$

$$\left(\begin{array}{|c|c|c|c|c|c|c|}\hline n-p-1 & 1 & \cdots & n-p-2 & n-p+1 & \cdots & n \\ \hline n-p \\ \cline{1-1}\end{array}, \emptyset\right)$$

and their corresponding highest weight vectors, respectively,

$$f_p = y_1^p [y_1, y_2] y_1^{n-p-2} \text{ and } g_p = y_1^{n-p-2}[y_1, y_2] y_1^p.$$

By making the evaluation $y_1 = e_{11} + e_{2k,2k} + E$ and $y_2 = e_{12} + e_{2k-1,2k}$, we get that

$$f_p(y_1, y_2) = e_{2k-p-2,2k} - e_{2k-p-1,2k} \text{ and } g_p(y_1, y_2) = e_{1,p+2} - e_{1,p+3}.$$

Then f_p and g_p are not $*$-identities of A_k, for any $0 \le p \le k-2$, and these $2(k-1)$ polynomials are linearly independent modulo $\mathrm{Id}^*(A_k)$. Hence $m_{(n-1,1),\emptyset} \ge 2(k-1)$.

(ii) For fixed $2 \le j \le k-1$, for the pair of partitions $((n-j,j),\emptyset)$ and for $0 \le p \le k-j-1$ and $w = n-p$, we consider the following pairs of tableaux:

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline p+1 & p+2 & \cdots & p+j & 1 & \cdots & p & p+2j+1 & \cdots & n \\ \hline p+j+1 & p+j+2 & \cdots & p+2j \\ \cline{1-4}\end{array}, \emptyset\right)$$

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline w-2j+1 & w-2j+2 & \cdots & w-j & 1 & \cdots & w-2j & w+1 & \cdots & n \\ \hline w-j+1 & w-j+2 & \cdots & w \\ \cline{1-4}\end{array}, \emptyset\right) \tag{3.12}$$

and their corresponding highest weight vectors, respectively,

$$f_p = y_1^p \underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{j} \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{j} y_1^{n-2j-p} \qquad \text{and} \qquad g_p = y_1^{n-2j-p} \underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{j} \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{j} y_1^p.$$

We have, by making the evaluation $y_1 = e_{11} + e_{2k,2k} + E$ and $y_2 = e_{11} + e_{2k,2k} + e_{12} + e_{2k-1,2k}$, that $f_p(y_1, y_2) = \alpha e_{2k-p-j,2k}$ and $g_p(y_1, y_2) = \beta e_{1,j+p+1}$, with $\alpha \ne 0$ and $\beta \ne 0$. Then , for any $0 \le p \le k-j-1$, f_p and g_p are not $*$-identities of A_k. Moreover, the same evaluation shows that these $2(k-j)$ polynomials are linearly independent modulo $\mathrm{Id}^*(A_k)$. Thus $m_{(n-j,j),\emptyset} \ge 2(k-j)$, for any $2 \le j \le k-1$.

(iii) Now, for fixed $1 \le l \le k-2$, for the pair of partitions $((n-l-1,l,1),\emptyset)$ and for $0 \le p \le k-j-2$ and $w = n-p$, we consider the following pairs of tableaux:

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline p+l & p+1 & \cdots & p+l-1 & 1 & \cdots & p & p+2l+2 & \cdots & n \\ \hline p+l+1 & p+l+3 & \cdots & p+2l+1 \\ \cline{1-4} p+l+2 \\ \cline{1-1}\end{array}, \emptyset\right)$$

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline w-l-1 & w-2l & \cdots & w-l-2 & 1 & \cdots & w-2l-1 & w+1 & \cdots & n \\ \hline w-l & w-l+2 & \cdots & w \\ \cline{1-4} w-l+1 \\ \cline{1-1}\end{array}, \emptyset\right) \tag{3.13}$$

and their corresponding highest weight vectors, respectively,

$$f_p = y_1^p \underbrace{\bar{y}_1 \cdots \bar{\bar{y}}_1}_{l-1} \tilde{y}_1 \tilde{y}_2 \tilde{y}_3 \underbrace{\bar{y}_2 \cdots \bar{\bar{y}}_2}_{l-1} y_1^{n-p-2l-1} \text{ and } g_p = y_1^{n-p-2l-1} \underbrace{\bar{y}_1 \cdots \bar{\bar{y}}_1}_{l-1} \tilde{y}_1 \tilde{y}_2 \tilde{y}_3 \underbrace{\bar{y}_2 \cdots \bar{\bar{y}}_2}_{l-1} y_1^p.$$

Evaluating $y_1 = e_{11} + e_{2k,2k} + E$, $y_2 = E$ and $y_3 = e_{12} + e_{2k-1,2k}$, we get that $f_p(y_1, y_2, y_3) = \alpha e_{2k-l-p-1,2k}$ and $g_p(y_1, y_2, y_3) = \beta e_{1,l+p+2}$, with $\alpha \ne 0$ and $\beta \ne 0$. Thus f_p and g_p, for any $0 \le p \le k-j-2$, are not $*$-identities of A_k and these $2(k-l-1)$ polynomials are linearly independent modulo $\mathrm{Id}^*(A_k)$. Hence we have that $m_{(n-l-1,l,1)} \ge 2(k-l-1)$, for any $1 \le l \le k-2$.

(iv) Finally, for fixed $0 \leq t \leq k-2$, for the pair of partitions $((n-t-1,t),(1))$ and for $0 \leq p \leq k-j-2$ and $w = n-p$, we consider the following pairs of tableaux:

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline p+1 & \cdots & p+t & 1 & \cdots & p & p+2t+2 & \cdots & n \\ \hline p+t+2 & \cdots & p+2t+1 \\ \cline{1-3} \end{array} \ , \ \begin{array}{|c|} \hline p+t+1 \\ \hline \end{array} \right)$$

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline w-2t & \cdots & w-t-1 & 1 & \cdots & w-2t-1 & w+1 & \cdots & n \\ \hline w-t+1 & \cdots & w \\ \cline{1-3} \end{array} \ , \ \begin{array}{|c|} \hline w-t \\ \hline \end{array} \right) \tag{3.14}$$

and their corresponding highest weight vectors, respectively,

$$f_p = y_1^p \underbrace{\bar{y}_1 \cdots \bar{\bar{y}}_1}_{t} z \underbrace{\bar{y}_2 \cdots \bar{\bar{y}}_2}_{t} y_1^{n-p-2t-1} \quad \text{and} \quad g_p = y_1^{n-p-2t-1} \underbrace{\bar{y}_1 \cdots \bar{\bar{y}}_1}_{t} z \underbrace{\bar{y}_2 \cdots \bar{\bar{y}}_2}_{t} y_1^p.$$

By making the evaluation $y = e_{11} + e_{2k,2k} + E$ and $z = e_{12} - e_{2k-1,2k}$, in case $t = 0$, and $y_1 = e_{11} + e_{2k,2k} + E$, $y_2 = E$ and $z = e_{12} - e_{2k-1,2k}$ otherwise, we get that $f_p(y_1, y_2, z) = \alpha e_{2k-t-p-1,2k}$ and $g_p(y_1, y_2, z) = \beta e_{1,t+p+1}$, with $\alpha \neq 0$ and $\beta \neq 0$. Thus $m_{(n-t-1,t),(1)} \geq 2(k-t-1)$, for any $0 \leq t \leq k-2$, since f_p and g_p are not $*$-identities of A_k, for all $0 \leq p \leq k-t-2$, and these $2(k-t-1)$ polynomials are linearly independent modulo $\mathrm{Id}^*(A_k)$.

Thus we have that

$$\begin{aligned} c_n^*(A_k) \quad \geq \quad & d_{(n),\emptyset} + \sum_{j=1}^{k-1} 2(k-j) d_{(n-j,j),\emptyset} + \sum_{j=1}^{k-2} 2(k-j-1) d_{(n-j-1,j,1),\emptyset} \\ & + \sum_{j=0}^{k-2} 2(k-j-1) d_{(n-j-1,j),(1)} = c_n^*(A_k). \end{aligned}$$

Hence $\chi_n^*(A_k)$ has the desired decomposition. It is easy to show that $l_n^*(A_k) = 3k^2 - 5k + 3, \forall k \geq 2$, and the result is proved. $\square$

Now we study the $*$-cocharacters and the $*$-colengths of the minimal variety $\mathrm{var}^*(N_k)$, for all $k \geq 2$.

Lemma 3.14. *$\chi_n^*(N_2) = \chi_{(n),\emptyset} + \chi_{(n-1),(1)}$ and $l_n^*(N_2) = 2$.*

Proof. Notice that we have

$$d_{(n),\emptyset} + d_{(n-1),(1)} = 1 + n = c_n^*(N_2).$$

Then, since $m_{(n),\emptyset} = 1$, if we find a highest weight vector for the pair of partitions $((n-1),(1))$ which is not a $*$-identity of N_2 we may conclude that $\chi_n^*(N_2)$ has the desired decomposition.

In fact, let $f_1 = y^{n-1}z$ be the highest weight vector associated to the pair of partitions $((n-1),(1))$ and corresponding to the pair of tableaux:

$$\left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & n-1 \\ \hline \end{array} \ , \ \begin{array}{|c|} \hline n \\ \hline \end{array} \right). \tag{3.15}$$

By making the evaluation $y = I$ and $z = e_{12} - e_{34}$, we get that $f = e_{12} - e_{34} \neq 0$. This says that f is not a $*$-identity of N_2. Hence we have $\chi_n^*(N_2) = \chi_{(n),\emptyset} + \chi_{(n-1),(1)}$ and $l_n^*(N_2) = 2$. $\square$

Remark 3.15. Let $k \geq 2$. Then

$$c_n^*(N_k) = d_{(n),\emptyset} + \sum_{j=1}^{k-3} (k-j-2)[d_{(n-j,j),\emptyset} + d_{(n-j-1,j,1),\emptyset}] + \sum_{j=0}^{k-2} (k-j-1) d_{(n-j-1,j),(1)}.$$

Proof. We shall use induction on k. From Lemma 3.14 it follows that the result is true for $k = 2$.

Now we suppose the result is true for some $k \geq 2$. By Lemma 3.4, we have that

$$c_n^*(N_{k+1}) = c_n^*(N_k) + \binom{n}{k-1}(k-2) + \binom{n}{k} k.$$

Hence, by using that, for all $r \geq 1$,

$$\sum_{j=0}^{r} d_{(n-j,j-1),(1)} = \binom{n}{r}(n-r) = \binom{n}{r+1}(r+1) \quad \text{and} \quad \sum_{j=1}^{r} \left[d_{(n-j,j),\emptyset} + d_{(n-j-1,j,1),\emptyset} \right] = \binom{n}{r+1} r,$$

we get the following:

$$\begin{aligned}c_n^*(N_{k+1}) &= c_n^*(N_k)+\tbinom{n}{k-1}(k-2)+\tbinom{n}{k}k\\ &= c_n^*(A_k)+\sum_{j=1}^{k-2}[d_{(n-j,j),\emptyset}+d_{(n-j-1,j,1),\emptyset}]+\sum_{j=0}^{k-1}d_{(n-j-1,j),(1)}\\ &= d_{(n),\emptyset}+\sum_{j=1}^{k-2}(k-j-1)[d_{(n-j,j),\emptyset}+d_{(n-j-1,j,1),\emptyset}]+\sum_{j=0}^{k-1}(k-j)d_{(n-j-1,j),(1)}\end{aligned}$$

Thus the result is true for any $k\geq 2$. □

THEOREM 3.16. *For $k\geq 3$, we have*

(1) $\chi_n^*(N_k)=\chi_{(n),\emptyset}+\sum_{j=1}^{k-3}(k-j-2)\left[\chi_{(n-j,j),\emptyset}+\chi_{(n-j-1,j,1),\emptyset}\right]+\sum_{j=0}^{k-2}(k-j-1)\chi_{(n-j-1,j),(1)}$.

(2) $l_n^*(N_k)=\dfrac{3k^2-11k+14}{2}$.

PROOF. The proof is similar to the proof of Lemma 3.13. By the previous remark, we have that, for any $k\geq 3$,

$$c_n^*(N_k)=d_{(n),\emptyset}+\sum_{j=1}^{k-3}(k-j-2)[d_{(n-j,j),\emptyset}+d_{(n-j-1,j,1),\emptyset}]+\sum_{j=0}^{k-2}(k-j-1)d_{(n-j-1,j),(1)}.$$

It is clear that $m_{(n),\emptyset}=1$. In order to prove the desired decomposition of $\chi_n^*(N_k)$, we shall prove that the characters $\chi_{(n-j,j),\emptyset}$, $\chi_{(n-l-1,l,1),\emptyset}$ and $\chi_{(n-t-1,1),(1)}$, for $1\leq j,l\leq k-3$ and $0\leq t\leq k-2$, appear in the decomposition of the $*$-cocharacter $\chi_n^*(N_k)$ with multiplicity $m_{(n-j,j),\emptyset}=k-j-2$, $m_{(n-l-1,l,1),\emptyset}=k-l-2$ and $m_{(n-t-1,1)(1)}=k-t-1$, respectively.

(i) For fixed $1\leq j\leq k-3$, for the pair of partitions $((n-j,j),\emptyset)$ and for $0\leq p\leq k-j-3$, we consider the pair of tableaux (3.12) given in Lemma 3.13 whose corresponding highest weight vector is

$$f_p=y_1^{n-2j-p}\underbrace{\bar{y_1}\cdots\tilde{y_1}}_{j}\underbrace{\bar{y_2}\cdots\tilde{y_2}}_{j}y_1^p.$$

By making the evaluation $y_1=I+E$ and $y_2=I+e_{13}+e_{2k-2,2k}$ we get

$$f_p(y_1,y_2)=\alpha\sum_{i=0}^{k-2}\binom{n-2j-p}{i}e_{2k-j-i-2,2k}+\beta\sum_{i=0}^{p}\binom{p}{i}e_{1,3+j+i},$$

with α and β non-zero values. Then , for any $0\leq p\leq k-j-3$, f_p is not a $*$-identity of N_k. Moreover, the same evaluation shows that these $(k-j-2)$ polynomials are linearly independent modulo $\mathrm{Id}^*(N_k)$. Thus $m_{(n-j,j),\emptyset}\geq k-j-2$, for any $1\leq j\leq k-3$.

(ii) Now, for fixed $1\leq l\leq k-3$, for the pair of partitions $((n-l-1,l,1),\emptyset)$ and $0\leq p\leq k-j-3$, we consider the pair of tableaux (3.13) with the following corresponding highest weight vector:

$$g_p=y_1^{n-p-2l-1}\underbrace{\bar{y_1}\cdots\bar{\bar{y_1}}}_{l-1}\tilde{y_1}\tilde{y_2}\tilde{y_3}\underbrace{\bar{y_2}\cdots\bar{\bar{y_2}}}_{l-1}y_1^p.$$

Evaluating $y_1=I+E$, $y_2=E$ and $y_3=e_{13}+e_{2k-2,2k}$, we also get that

$$g_p(y_1,y_2,y_3)=\alpha\sum_{i=0}^{k-2}\binom{n-2j-p}{i}e_{2k-j-i-2,2k}+\beta\sum_{i=0}^{p}\binom{p}{i}e_{1,3+j+i},$$

with α and β non-zero values. Thus g_p, for any $0\leq p\leq k-j-3$, is not a $*$-identity of N_k and these $(k-l-2)$ polynomials are linearly independent modulo $\mathrm{Id}^*(N_k)$. Hence we have that $m_{(n-l-1,l,1)}\geq(k-l-2)$, for any $1\leq l\leq k-3$.

(iii) Finally, for fixed $0\leq t\leq k-2$, for the pair of partitions $((n-t-1,t),(1))$ and for $0\leq p\leq k-j-2$, we consider the pair of tableaux (3.14) and its corresponding highest weight vector

$$h_p = y_1^{n-p-2t-1}\,\underbrace{\bar{y}_1 \cdots \bar{\bar{y}}_1}_{t}\, z\, \underbrace{\bar{y}_2 \cdots \bar{\bar{y}}_2}_{t}\, y_1^p.$$

By making the evaluation $y_1 = I + E$ and $z = e_{12} - e_{2k-1,2k}$, in case $t = 0$, and $y_1 = I + E$, $y_2 = E$ and $z = e_{12} - e_{2k-1,2k}$ otherwise, we get that

$$h_p(y_1, y_2, z) = \alpha \sum_{i=0}^{k-2} \binom{n-2j-p}{i} e_{2k-j-i-1,2k} + \beta \sum_{i=0}^{p} \binom{p}{i} e_{1,2+j+i},$$

with α and β non-zero values. Thus $m_{(n-t-1,t),(1)} \geq (k-t-1)$, for any $0 \leq t \leq k-2$, since that h_p is not a $*$-identity of N_k, for all $0 \leq p \leq k-t-2$, and these $(k-t-1)$ polynomials are linearly independent modulo $\mathrm{Id}^*(N_k)$.

Thus we have that

$$c_n^*(N_k) \geq d_{(n),\emptyset} + \sum_{j=1}^{k-3} (k-j-2)[d_{(n-j,j),\emptyset} + d_{(n-j-1,j,1),\emptyset}] + \sum_{j=0}^{k-2} (k-j-1) d_{(n-j-1,j),(1)}.$$

Hence, by the previous remark, $\chi_n^*(N_k)$ has the desired decomposition and $l_n^*(N_k) = \dfrac{3k^2 - 11k + 14}{2}$. $\square$

We finish this section by calculating the $*$-cocharacters and $*$-colengths of $\mathrm{var}^*(U_k)$, for all $k \geq 3$.

Lemma 3.17. *$\chi_n^*(U_3) = \chi_{(n),\emptyset} + \chi_{(n-1,1),\emptyset} + \chi_{(n-2,1,1),\emptyset} + \chi_{(n-1),(1)}$ and $l_n^*(U_3) = 4$.*

Proof. Notice that

$$d_{(n),\emptyset} + d_{(n-1),(1)} + d_{(n-1,1),\emptyset} + d_{(n-1,1^2),\emptyset} = 1 + n + (n-1) + \frac{(n-1)(n-2)}{2} = c_n^*(U_3).$$

Then, since $m_{(n),\emptyset} = 1$, if we find a highest weight vector for each pair of partitions $((n-1),(1))$, $((n-1,1),\emptyset)$ and $((n-1,1^2),\emptyset)$ which is not a $*$-identity of U_3 we may conclude that $\chi_n^*(U_3)$ has the desired decomposition.

In fact, let $f = y^{n-1}z$ be the highest weight vector associated to the pair of partitions $((n-1),(1))$ and corresponding to the pair of tableaux:

$$\left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n-1 \\ \hline \end{array}\ ,\ \begin{array}{|c|}\hline n \\ \hline \end{array} \right). \tag{3.16}$$

By making the evaluation $y = I$ and $z = e_{13} - e_{46}$, we get that $f = e_{13} - e_{46} \neq 0$ and, so, f is not a $*$-identity of U_3.

Now we consider $g = [y_1, y_2]y_1^{n-2}$ the highest weight vector associated to the pair of partitions $((n-1,1),\emptyset)$ and corresponding to the pair of tableaux:

$$\left(\begin{array}{|c|c|c|c|}\hline 1 & 3 & \cdots & n \\ \hline 2 \\ \cline{1-1} \end{array}\ ,\ \emptyset \right). \tag{3.17}$$

By making the evaluation $y_1 = I + e_{12} + e_{56}$ and $y_2 = e_{23} + e_{45}$, we get that $g = e_{13} - e_{46} \neq 0$. It shows that g is not a $*$-identity of U_3.

Finally we consider $h = St_3(y_1, y_2, y_3)y_1^{n-3}$ the highest weight vector associated to the pair of partitions $((n-1,1^2),\emptyset)$ and corresponding to the pair of tableaux:

$$\left(\begin{array}{|c|c|c|c|}\hline 1 & 4 & \cdots & n \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}\ ,\ \emptyset \right). \tag{3.18}$$

By making the evaluation $y_1 = I$, $y_2 = e_{23} + e_{45}$ and $y_3 = e_{12} + e_{56}$, we get that $h = -e_{13} + e_{46} \neq 0$ and this says that h is not a $*$-identity of U_3. Hence we have that $\chi_n^*(U_3) = \chi_{(n),\emptyset} + \chi_{(n-1,1),\emptyset} + \chi_{(n-2,1,1),\emptyset} + \chi_{(n-1),(1)}$ and $l_n^*(U_3) = 4$. $\square$

The proof of the next result is similar to the proof of Lemma 3.16.

Theorem 3.18. *For $k \geq 3$, we have*

(1) $\chi_n^*(U_k) = \chi_{(n),\emptyset} + \sum_{j=1}^{k-2} (k-j-1) \left[\chi_{(n-j,j),\emptyset} + \chi_{(n-j-1,j,1),\emptyset}\right] + \sum_{j=0}^{k-3} (k-j-2) \chi_{(n-j-1,j),(1)}$.

(2) $l_n^*(U_k) = \dfrac{3k^2 - 9k + 8}{2}$.

4. Characterizing varieties of small $*$-colength

In this section we shall classify the varieties such that their sequence of $*$-colengths is bounded by three, for n large enough. We start by considering the algebra G_2^*, the Grassmann algebra with 1 generated by the elements e_1, e_2 over F subject to the condition $e_1e_2 + e_2e_1 = e_1^2 = e_2^2 = 0$, and endowed with the involution $*$ such that $e_i^* = -e_i$, for $i = 1, 2$. We have the following.

Lemma 4.1. *For the algebra G_2^* we have*

1. $\mathrm{Id}^*(G_2^*) = \langle [y_1, y_2], [y, z], z_1z_2 + z_2z_1, z_1z_2z_3 \rangle_{T^*}$.
2. $c_n^*(G_2^*) = 1 + n + \dfrac{n(n-1)}{2}$.
3. $\chi_n^*(G_2^*) = \chi_{(n),\emptyset} + \chi_{(n-1),(1)} + \chi_{(n-2),(1^2)}$ *and* $l_n^*(G_2^*) = 3$.

Proof. In [**21**, Lemma 16] the authors determined the T^*-ideal and computed the nth $*$-codimension of the algebra G_2^*. Here we shall prove that $\chi_n^*(G_2^*) = \chi_{(n),\emptyset} + \chi_{(n-1),(1)} + \chi_{(n-2),(1^2)}$. We start by noticing that $(G_2^*)^+ = \mathrm{span}_F\{1\}$ and $(G_2^*)^- = \mathrm{span}_F\{e_1, e_2, e_1e_2\}$. Moreover, we have

$$d_{(n),\emptyset} + d_{(n-1),(1)} + d_{(n-2),(1^2)} = 1 + n + \frac{n(n-1)}{2} = c_n^*(G_2^*).$$

Then, since $m_{(n),\emptyset} = 1$, we just need to find a highest weight vector for each pair of partitions $((n-1),(1))$ and $((n-2),(1^2))$ which is not a $*$-identity of G_2^* to conclude that $\chi_n^*(G_2^*)$ has the desired decomposition.

In fact, let $f = y^{n-1}z_1$ and $g = y^{n-2}[z_1, z_2]$ be the highest weight vectors associated to the pairs of partitions $((n-1),(1))$ and $((n-2),(1^2))$ and corresponding to the pairs of tableaux, respectively:

$$\left(\boxed{1}\boxed{2}\boxed{\cdots}\boxed{n-1}\ ,\ \boxed{n}\right) \text{ and } \left(\boxed{1}\boxed{2}\boxed{\cdots}\boxed{n-2}\ ,\ \begin{array}{|c|}\hline n-1 \\ \hline n \\ \hline\end{array}\right). \tag{4.1}$$

By making the evaluation $y = 1, z_1 = e_1$ and $z_2 = e_2$, we get that $f = e_1 \neq 0$ and $g = 2e_1e_2 \neq 0$; then f and g are not $*$-identities of G_2^* and the proof is complete. □

Next we consider the algebra $G_2^* \oplus C_3$ and the algebra G_3^*, the Grassmann algebra with 1 generated by the elements e_1, e_2, e_3 over F subject to the condition $e_ie_j + e_je_i = e_i^2 = 0$, for all $i, j = 1, 2, 3$, and endowed with the involution $*$ such that $e_i^* = -e_i$, for $i = 1, 2, 3$. The next lemma can be proved as the previous one.

Lemma 4.2. *For the algebras G_3^* and $G_2^* \oplus C_3$ we have*

1. $\mathrm{Id}^*(G_3^*) = \langle [y_1, y_2], [y, z], z_1z_2 + z_2z_1, z_1z_2z_3z_4 \rangle_{T^*}$.
2. $c_n^*(G_3^*) = 1 + n + \dfrac{n(n-1)}{2} + \dfrac{n(n-1)(n-2)}{6}$.
3. $\chi_n^*(G_3^*) = \chi_{(n),\emptyset} + \chi_{(n-1),(1)} + \chi_{(n-2),(1^2)} + \chi_{(n-3),(1^3)}$.
4. $\mathrm{Id}^*(G_2^* \oplus C_3) = \langle [y_1, y_2], [y, z], z_1z_2z_3 \rangle_{T^*}$.
5. $c_n^*(G_2^* \oplus C_3) = n^2 + 1$.
6. $\chi_n^*(G_2^* \oplus C_3) = \chi_{(n),\emptyset} + \chi_{(n-1),(1)} + \chi_{(n-2),(1^2)} + \chi_{(n-2),(2)}$.
7. $l_n^*(G_3^*) = l_n^*(G_2^* \oplus C_3) = 4$.

Recall that if $A = F + J$ is a finite dimensional algebra over F where $J = J(A)$ is its Jacobson radical, then J can be decomposed into the direct sum of B-bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11} \tag{4.2}$$

where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. In a similar way, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, r, s \in \{0, 1\}$, $J_{ir}J_{rs} \subseteq J_{is}$, $J_{ik}J_{rs} = 0$ for $k \neq r$ and $J_{11} = BN$ for some nilpotent subalgebra N of A commuting with B [**9**].

Notice that if the algebra A has an involution $*$, then J_{00} and J_{11} are stable under the involution whereas $J_{01}^* = J_{10}$.

In what follows we use the following result.

Proposition 4.3. [**20**, Theorem 2] *Let A be an algebra with involution over a field F of characteristic zero and suppose that $c_n^*(A)$, $n=1,2,\ldots,$ is polynomially bounded. Then $A\sim_{T^*} B_1\oplus\cdots\oplus B_m$ where, for each $i\in\{1,\cdots,m\}$, B_i is a finite-dimensional algebra with involution over F and $\dim B_i/J(B_i)\le 1$, for all $i=1,\ldots,m$.*

Now, by applying [**24**, Corollary 5.5] we get the following.

Theorem 4.4. *Let A be an algebra with involution over a field F of characteristic zero. Then $c_n^*(A)$, $n=1,2,\ldots,$ is polynomially bounded if and only if $l_n^*(A)\le k$, for some constant k and for all $n\ge 1$.*

Proof. If $c_n^*(A)$, $n=1,2,\ldots,$ is polynomially bounded then by Proposition 4.3, A satisfies the same $*$-identities as a finite dimensional algebra and the result follows by applying Corollary 5.5 in [**24**]. Conversely, suppose that $l_n^*(A)\le k$, for some constant k and for all $n\ge 1$. Then by [**22**] and [**6**], M and D do not belong to the variety generated by A since their $*$-colengths are not bounded by any constants. Then, by Theorem 3.1, $c_n^*(A)$, $n=1,2,\ldots,$ is polynomially bounded. □

Much effort has been put into the study of algebras with colengths bounded by a constant (see [**4, 15, 12, 18**] for the ordinary and graded cases). Here we deal with the case of algebras with involution.

Lemma 4.5. [**21**, Lemma 14] *If $A=F+J$ is a finite-dimensional algebra with involution where $J=J_{00}\oplus J_{01}\oplus J_{10}\oplus J_{11}$ and $A_2\notin \operatorname{var}^*(B)$ then $J_{10}=J_{01}=0$.*

Next we study $*$-algebras of the type $F+J_{11}$.

Lemma 4.6. *Let $B=F+J_{11}$. If $C_i\notin\operatorname{var}^*(B)$, for $i\ge 2$, then $z^{i-1}\equiv 0$ on B.*

Proof. We give a proof of the result by following closely the proof of [**21**, Lemma 27]. Suppose that there exists $a\in J_{11}^-$ such that $a^{i-1}\neq 0$ and consider the $*$-subalgebra R of B generated by 1 and a over F. Then if I is the $*$-ideal generated by a^i, we have that the algebra $\overline{R}=R/I$ has induced involution and $\overline{R}=\operatorname{span}\{\overline{1},\overline{a},\overline{a}^2,\ldots,\overline{a}^{i-1}\}$. It is easily seen that $\overline{R}\cong C_i$ through the isomorphism φ such that $\varphi(\overline{1})=e_{11}+\cdots+e_{ii}$, $\varphi(\overline{a})=e_{12}+\cdots+e_{i-1\ i}$. Hence $C_i\in var^*(B)$ and we have reached a contradiction. □

Lemma 4.7. *Let $B=F+J_{11}$.*

1. *If $U_3\notin\operatorname{var}^*(B)$ then $[y_1,y_2]\equiv 0$ on B.*
2. *If $N_3\notin\operatorname{var}^*(B)$ then $[y,z]\equiv 0$ on B.*

Proof. Suppose, for a contradiction, that $[y_1,y_2]\not\equiv 0$. Let $a,b\in J_{11}^+$ be such that $[a,b]\neq 0$ and consider the $*$-subalgebra R generated by $1,a,b$ over F and let I be the $*$-ideal generated by $a^2,b^2,ab+ba$. So the $*$-algebra $\overline{R}=R/I$ is linearly generated by $\{\overline{1},\overline{a},\overline{b},\overline{a}\overline{b}\}$ and we claim that $\operatorname{Id}^*(\overline{R})=\operatorname{Id}^*(U_3)$. Clearly $z_1z_2\equiv 0$ and $[z,y]\equiv 0$ are $*$-identities of $\overline{R}$, and so, $\operatorname{Id}^*(U_3)\subseteq\operatorname{Id}^*(\overline{R})$.

Let $f\in P_n^*\cap\operatorname{Id}^*(\overline{R})$ a multilinear polynomial of degree n. By [**21**, Lemma 19] we can write f (mod $\operatorname{Id}^*(U_3)$) as:

$$f=\alpha y_1\cdots y_n+\sum_{1\le i<j\le n}\alpha_{ij}y_{i_1}\cdots y_{i_{n-2}}[y_i,y_j]+\sum_{i=1}^{n}\alpha_i y_{j_1}\cdots y_{j_{n-1}}z_i,$$

where $i_1<i_2<\cdots<i_{n-2}$ and $j_1<j<_2\cdots<j_{n-1}$. By making the evaluations $y_1=\cdots=y_n=\overline{1}$ and $z_i=0$ for $i=1,\ldots,n$, we get $\alpha=0$. Also, for a fixed $i<j$ the evaluation $y_i=\overline{a}$, $y_j=\overline{b}$, $y_k=\overline{1}$ for $k\notin\{i,j\}$ and $z_l=0$ for $l=1,\ldots,n$, gives $\alpha_{ij}=0$. Finally the evaluation $z_i=[\overline{a},\overline{b}]$, $y_j=\overline{1}$ for $j\neq i$ gives $\alpha_i=0$. Hence $f\in\operatorname{Id}^*(U_3)$ and, so, $\operatorname{Id}^*(\overline{R})\subseteq\operatorname{Id}^*(U_3.)$ Thus $U_3\in\operatorname{var}^*(B)$ and the proof of the first part is complete.

The second part of the lemma is proved similarly. □

Lemma 4.8. *Suppose that $B=F+J_{11}$ satisfies $z_1z_2+z_2z_1\equiv 0$. If $z_1z_2z_3\not\equiv 0$ then $G_3^*\in\operatorname{var}^*(B)$.*

Proof. Consider $a,b,c\in J_{11}^-$ such that $abc\neq 0$. Let R be the subalgebra of B generated by $1,a,b,c$. Since $z_1z_2+z_2z_1\equiv 0$ in R we have $a^2=b^2=c^2=0$ and so $R=\operatorname{span}\{1,a,b,c,ab,ac,bc,abc\}$. As a consequence, the correspondence

$$1\mapsto 1,\ a\mapsto e_1,\ b\mapsto e_2,\ c\mapsto e_3$$

defines an isomorphism between R and G_3^*. □

Lemma 4.9. *If $B = F + J_{11}$ is such that $[z_1, z_2] \not\equiv 0$ then $G_2^* \in \mathrm{var}^*(B)$.*

Proof. Consider $a, b \in J_{11}^-$ such that $[a, b] \neq 0$. Let R be the subalgebra of B generated by $1, a, b$ and let I be the $*$-ideal generated by $a^2, b^2, ab + ba$. So the $*$-algebra $\overline{R} = R/I$ is linearly generated by $\{\overline{1}, \overline{a}, \overline{b}, \overline{a}\overline{b}\}$. We have $\overline{R}$ is isomorphic to G_2^* and so $G_2^* \in \mathrm{var}^*(B)$. □

Now we are in position to prove the main result of this section which allows us to classify the varieties with $*$-colengths bounded by 3, for n large enough.

Theorem 4.10. *Let A be an algebra with involution over a field F of characteristic zero. The following conditions are equivalent.*

(1) $l_n^*(A) \leq 3$, *for n large enough.*
(2) $A_2, N_3, U_3, C_4, G_3^*, G_2^* \oplus C_3 \notin \mathrm{var}^*(A)$.
(3) *A is T^*-equivalent to N or $C \oplus N$ or $C_2 \oplus N$ or $C_3 \oplus N$, $G_2^* \oplus N$, where N is a nilpotent $*$-algebra and C is a commutative non-nilpotent algebra with trivial involution.*

Proof. First, notice that the condition (1) implies the condition (2) since by Lemmas 3.11, 3.17, 3.16, 4.2 and Theorem 3.2 we have that $l_n^*(A_2) = 5$, $l_n^*(N_3) = l_n^*(U_3) = l_n^*(G_2^* \oplus C_3) = l_n^*(G_3^*) = l_n^*(C_4) = 4$. Also, the condition (3) implies the condition (1), by Lemmas 3.14, 3.2 and 4.1.

Suppose now that $A_2, N_3, U_3, C_4, G_3^*, G_2^* \oplus C_3 \notin \mathrm{var}^*(A)$. Since $C_4 \in \mathrm{var}^*(D)$ and $A_2 \in \mathrm{var}^*(M)$, it follows that $D, M \notin \mathrm{var}^*(A)$. Hence, by Theorem 3.1, the $*$-codimensions of A are polynomially bounded and by Proposition 4.3, we may assume that

$$A = B_1 \oplus \cdots \oplus B_m$$

is a direct sum of finite-dimensional $*$-algebras where either B_i is nilpotent or $B_i = F + J(B_i)$.

If B_i is nilpotent for all i, then A is a nilpotent $*$-algebra and we are done in this case.

Therefore we may assume that there exists $i = 1, \ldots, m$ such that $B_i = F + J(B_i)$ and $J(B_i) = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$.

Since $A_2 \notin \mathrm{var}^*(B_i)$, by Lemma 4.5, we have that $J_{01} = J_{10} = 0$ and, so, $B_i = (F + J_{11}) \oplus J_{00}$ is a direct sum of $*$-algebras and we study $B = F + J_{11}$.

Since $N_3, U_3 \notin \mathrm{var}^*(B)$, by Lemma 4.7, it follows that $[y_1, y_2] \equiv 0$ and $[y, z] \equiv 0$ are $*$-identities of B.

Now we have to consider two different cases:

(1) $[z_1, z_2] \equiv 0$ on B
(2) $[z_1, z_2] \not\equiv 0$ on B.

In case (1), we have that $B \in \mathrm{var}^*(D)$. Since $C_4 \notin \mathrm{var}^*(B)$, by Theorem 3.3 we must have that B is T^*-equivalent to either C or C_2 or C_3.

Now assume that $[z_1, z_2] \not\equiv 0$ on B. So, by Lemma 4.9, $G_2^* \in \mathrm{var}^*(B)$. On the other hand, since $G_2^* \oplus C_3 \notin \mathrm{var}^*(A)$ we must have that $C_3 \notin \mathrm{var}^*(A)$. Hence, by Lemma 4.6, $z^2 \equiv 0$ on B and after linearizing we get that $z_1 z_2 + z_2 z_1 \equiv 0$ on B. Finally, since $G_3^* \notin \mathrm{var}^*(B)$, by Lemma 4.8, we have that $z_1 z_2 z_3 \equiv 0$. Hence $\mathrm{Id}^*(G_2^*) \subseteq \mathrm{Id}^*(B)$ and it follows that B is T^*-equivalent to G_2^*.

Recalling that $A = B_1 \oplus \cdots \oplus B_m$ and putting together all pieces, we get the desired conclusion. □

Actually, notice that if $l_n^*(A) \leq 3$, then for n large enough, $l_n^*(A)$ is always constant.

In conclusion we have the following classification: for any $*$-algebra A and n large enough,

1. $l_n(A) = 0$ if and only if $A \sim_{T^*} N$.
2. $l_n(A) = 1$ if and only if $A \sim_{T^*} C \oplus N$.
3. $l_n(A) = 2$ if and only if $A \sim_{T^*} C_2 \oplus N$.
4. $l_n(A) = 3$ if and only if either $A \sim_{T^*} C_3 \oplus N$ or $A \sim_{T^*} G_2^* \oplus N$,

where N is a nilpotent $*$-algebra and C is a commutative non-nilpotent algebra with trivial involution.

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