Probabilistic squares and hexagons of opposition under coherence[☆]

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Abstract

Various semantics for studying the square of opposition and the hexagon of opposition have been proposed recently. We interpret *sentences* by imprecise (set-valued) probability assessments on a finite sequence of conditional events. We introduce the *acceptability* of a sentence within coherence-based probability theory. We analyze the relations of the square and of the hexagon in terms of acceptability. Then, we show how to construct probabilistic versions of the square and of the hexagon of opposition by forming suitable tripartitions of the set of all coherent assessments on a finite sequence of conditional events. Finally, as an application, we present new versions of the square and of the hexagon involving generalized quantifiers.

Keywords: coherence, conditional events, hexagon of opposition, imprecise probability, square of opposition, quantified sentences, tripartition

1. Introduction

There is a long history of investigations on the square of opposition spanning over two millennia ([5, 49]). A *square of opposition* represents logical key relations among basic (syllogistic) sentence types in a diagrammatic way. The basic sentence types, traditionally denoted by *A* (universal affirmative: "Every S is *P*"), *E* (universal negative: "No S is *P*"), *I* (particular affirmative: "Some S is *P*"), and *O* (particular negative: "Some S is not *P*"), constitute the corners of the square. The diagonals and the sides of the square of opposition are formed by the following logical relations among the basic sentence types: *A* and *E* are *contraries* (i.e., they cannot both be true), *I* and *O* are *subcontraries* (i.e., they cannot both be false), *A* and *O* as well as *E* and *I* are *contradictories* (i.e., they cannot both be true and they cannot both be false), *I* is a *subaltern* of *A* and *O* is a *subaltern* of *E* (i.e., *A* entails *I* and *E* entails *O*); for a visual representation see Figure 1 below, and cover the probabilities for seeing the traditional square of opposition). In the early 1950ies, the square of opposition was expanded to the *hexagon of opposition*, by adding the sentence $U : A \vee E$ at the top and the sentence $Y : I \wedge O$ at the bottom of the square (see Figure 2). Recently, the square of opposition as well as the hexagon of opposition and its extensions have been investigated from various semantic points of view (see, e.g., [4, 5, 14, 24, 25, 26, 34, 45, 46, 47]). In this paper we present a probabilistic analysis of the square of opposition under coherence, introduce the hexagon of opposition under coherence, and study the semantics of basic key relations among quantified statements.

After preliminary notions (Section 2), we introduce, based on g-coherence, a (probabilistic) notion of sentences and their acceptability and show how to construct squares of opposition under coherence from suitable tripartitions (Section 3). Then, we present an application of our square to the study of generalized quantifiers (Section 4). In Section 5 we introduce the *hexagon of opposition* under coherence. Section 6 concludes the paper by some remarks on future work.

[☆] This is a substantially extended version of a paper ([58]) presented at the 8th International Conference Soft Methods in Probability and Statistics 2016 (SMPS 2016) held in Rome in September 12–14, 2016.

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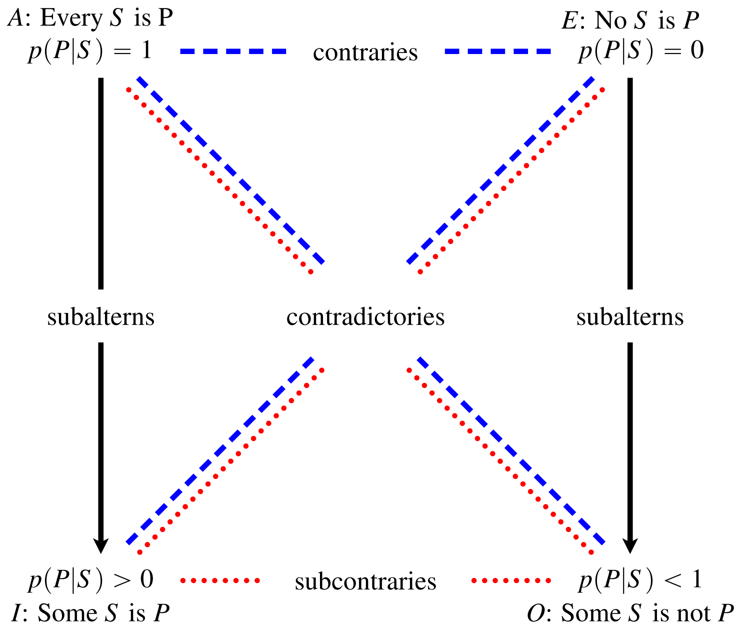

Figure 1: Traditional and probabilistic square of opposition defined on the four classical sentence types A, E, I, O and their relations in between. The probabilistic semantics of the basic sentence types involving the predicate term P and the subject term S is interpreted by a suitable probability assessment on the conditional event $P|S$ (see Table 1). For the relations see Definition 9.

2. Preliminary Notions

The coherence-based approach to probability and to other uncertain measures has been adopted by many authors (see, e.g., [6, 8, 12, 13, 17, 18, 20, 19, 21, 22, 30, 32, 37, 38, 39, 50, 54, 55, 57]); we therefore recall only selected key features of coherence and its generalizations in this section.

An event E is a two-valued logical entity which can be either true or false. The indicator of E is a two-valued numerical quantity which is 1, or 0, according to whether the event E is true, or false, respectively. We use the same symbols for events and their indicators. We denote by $\top$ the sure event (i.e., tautology or logical truth) and by $\bot$ the impossible event (i.e., contradiction or logical falsehood). Moreover, given two events E and H, we denote by $E \wedge H$ (resp., $E \vee H$) conjunction (resp., disjunction). To simplify notation, we will use the product EH to denote the conjunction $E \wedge H$, which also denotes the indicator of $E \wedge H$. We denote by $\bar{E}$ the negation of E.

Given two events E and H, with $H \neq \bot$, the *conditional event* $E|H$ is defined as a three-valued logical entity which is *true* if EH (i.e., $E \wedge H$) is true, *false* if $\bar{E}H$ is true, and *indetermined* (void) if H is false ([23, p. 307]). In terms of the betting metaphor, if you assess $p(E|H) = p$, then you are willing to pay (resp., to receive) an amount p and to receive (resp., to pay) 1, or 0, or p, according to whether EH is true, or $\bar{E}H$ is true, or $\bar{H}$ is true (the bet is called off), respectively. For defining coherence, consider a real-valued function $p : \mathcal{K} \rightarrow \mathbb{R}$, where $\mathcal{K}$ is an arbitrary (possibly not finite) family of conditional events. Consider a finite sequence $\mathcal{F} = (E_1|H_1, \dots, E_n|H_n)$, with $E_i|H_i \in \mathcal{K}$, $i = 1, \dots, n$, and the vector $\mathcal{P} = (p_1, \dots, p_n)$, where $p_i = p(E_i|H_i)$, $i = 1, \dots, n$. We denote by $\mathcal{H}_n$ the disjunction $H_1 \vee \cdots \vee H_n$. With the pair $(\mathcal{F}, \mathcal{P})$ we associate the random gain $\mathcal{G} = \sum_{i=1}^{n} s_i H_i (E_i - p_i)$, where $s_1, \dots, s_n$ are n arbitrary real numbers. $\mathcal{G}$ represents the net gain of n transactions, where for each transaction its meaning is specified by the sign of s_i (*plus* for buying or *minus* for selling) and its scaling is specified by the magnitude of s_i. Denoting by $G_{\mathcal{H}_n}$ the set of values of $\mathcal{G}$ restricted to $\mathcal{H}_n$, we recall

Definition 1. The function p defined on $\mathcal{K}$ is called *coherent* if and only if, for every integer n, for every sequence $\mathcal{F}$ of n conditional events in $\mathcal{K}$ and for every $s_1, \dots, s_n$, it holds that: $\min G_{\mathcal{H}_n} \leqslant 0 \leqslant \max G_{\mathcal{H}_n}$.

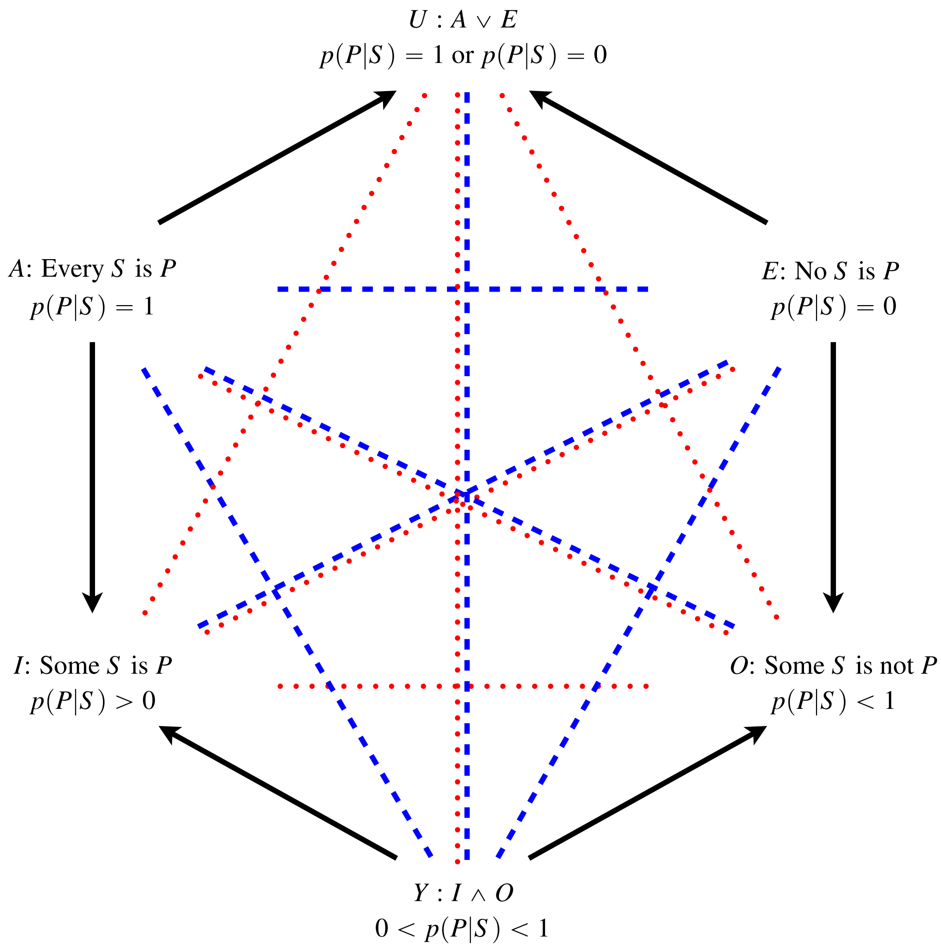

Figure 2: Probabilistic hexagon of opposition on the six sentence types A, E, I, O, U, Y, where A, E, I, O is a square of opposition, $U = A \vee E$ and $Y = I \wedge O$. The arrows indicate subalternation, dashed lines indicate contraries, and dotted lines indicate sub-contraries. Contradictories are indicated by combined dotted and dashed lines.

We say that p is *incoherent* if and only if p is not coherent.
As shown by Definition 1, a probability assessment is coherent if and only if, in any finite combination of n bets, it does not happen that the values in the set $G_{\mathcal{H}_n}$ are all positive, or all negative (*no Dutch Book*). In particular, coherence of $p(E|H)$ requires that $p(E|H) \in [0,1]$ for every $E|H \in \mathcal{K}$. If p on $\mathcal{K}$ is coherent, we call it a *conditional probability on* $\mathcal{K}$ (see, e.g., [2, 20, 21, 61]). Notice that, if p is coherent, then p also satisfies all the well known properties of finitely additive conditional probability (while the converse does not hold; see, e.g., [21, Example 13] or [27, Example 8]). Moreover, coherence can be characterized in terms of proper scoring rules ([9, 35]), which can be related also to the notion of entropy in information theory ([42, 43]).
In what follows $\mathcal{F}$ refers to a finite sequence of conditional events, $(E_1|H_1, \ldots, E_n|H_n)$ and $\mathbf{F} = \{E_j|H_j, j = 1, \ldots, n\}$ denotes the family of events in $\mathcal{F}$. Moreover, we denote by $\mathcal{P}$ a (precise) probability assessment $\mathcal{P} = (p_1, \ldots, p_n)$ on $\mathcal{F}$, where $p_j = p(E_j|H_j)$, $j = 1, \ldots, n$. Then, we say that a probability assessment $\mathcal{P}$ on $\mathcal{F}$ is "coherent" if and only if the corresponding function $p : \mathbf{F} \to \mathbb{R}$, defined as $p(E_j|H_j) = p_j$, $j = 1, \ldots, n$, is coherent (see Definition 1). As coherence requires that $p(E_j|H_j) \in [0,1]$, in what follows we will only consider probability assessment $\mathcal{P}$ on $\mathcal{F}$ such that: $\mathcal{P} \in [0,1]^n$. Let Π denote the set of *all coherent precise* assessments on $\mathcal{F}$. We recall that when there are no logical relations among the events $E_1, H_1, \ldots, E_n, H_n$ involved in $\mathcal{F}$, that is $E_1, H_1, \ldots, E_n, H_n$ are logically independent, then the set Π associated with $\mathcal{F}$ is the whole unit hypercube $[0,1]^n$. If there are logical relations, then the set Π *could be* a strict subset of $[0,1]^n$. As it is well known $\Pi \neq \emptyset$; therefore, $\emptyset \neq \Pi \subseteq [0,1]^n$.

Definition 2. *An* imprecise, or set-valued, probability assessment $\mathcal{I}$ *on a sequence of n conditional events* $\mathcal{F}$ *is a (possibly empty) set of precise probability assessments* $\mathcal{P}$ *on* $\mathcal{F}$.

Definition 2 states that an *imprecise (probability) assessment* $\mathcal{I}$ on a sequence of n conditional events $\mathcal{F}$ is just a (possibly empty) subset of $[0,1]^n$ ([31, 33, 34]). Of course, any n-dimensional rectangle $\mathcal{I} = [\alpha_1,\beta_1] \times \cdots \times [\alpha_n,\beta_n] \subseteq [0,1]^n$ is an imprecise assessment on a sequence of n conditional events $\mathcal{F} = (E_1|H_1, \ldots, E_n|H_n)$. Such an assessment is usually denoted by the interval-valued probability assessment $([\alpha_1,\beta_1], \ldots, [\alpha_n,\beta_n])$ on $\mathcal{F}$ or by the lower and upper probability constraints: $\alpha_j \leqslant p(E_j|H_j) \leqslant \beta_j$, $j = 1, \ldots, n$. Moreover, By Definition 2 an imprecise assessment could also be a subset of $[0,1]^n$ which is not an n-dimensional rectangle. Then, in our approach, imprecise probability assessments are not merely interval-valued probability assessments. For instance, think about an agent (like Pythagoras) who considers only rational numbers to evaluate the probability of an event $E|H$. Pythagoras' evaluation can be represented by the imprecise assessment $\mathcal{I} = [0,1] \cap \mathbb{Q}$ on $E|H$. Moreover, a constraint like $p(E|H) > 0$ (resp., $p(E|H) = \{0\} \cup \{1\}$) can be represented by the imprecise assessment $\mathcal{I} =]0,1]$ (resp., $\mathcal{I} = \{0,1\}$) on $E|H$. A generalized notion of coherence for interval-valued probability assessments was introduced in [28], which became known as "g-coherence" ([6]). We recall that, by replacing each upper probability bound $P(E|H) \leqslant \beta$ by the lower bound $P(\bar{E}|H) \geqslant 1-\beta$, a given interval-valued conditional probability assessments can be represented as a conditional lower probability assessment. Moreovoer, it can be shown that ([7, 29]) an interval-valued conditional probability assessments assessment is g-coherent if and only if the corresponding conditional lower probability "avoids uniform loss" ("avoids sure loss" in the unconditional case) ([64, 65, 66]). In this sense, the notion of g-coherence for interval-valued conditional probability assessments coincides with the property of "avoiding uniform loss" (AUL) for lower conditional probabilities. In this context upper and lower probabilities are defined as special cases of upper and lower previsions. For surveys on imprecise probability and lower previsions see [1, 44, 63]).
We now recall the notions of g-coherence and total coherence in the general case of imprecise (in the sense of set-valued) probability assessments ([34]).

Definition 3 (g-coherence)**.** *Given a sequence of n conditional events* $\mathcal{F}$. *An imprecise assessment* $\mathcal{I} \subseteq [0,1]^n$ *on* $\mathcal{F}$ *is* g-coherent *if and only if there exists a coherent precise assessment* $\mathcal{P}$ *on* $\mathcal{F}$ *such that* $\mathcal{P} \in \mathcal{I}$.

Definition 4 (t-coherence)**.** *An imprecise assessment* $\mathcal{I}$ *on* $\mathcal{F}$ *is* totally coherent *(t-coherent) if and only if the following two conditions are satisfied: (i)* $\mathcal{I}$ *is non-empty; (ii) if* $\mathcal{P} \in \mathcal{I}$, *then* $\mathcal{P}$ *is a coherent precise assessment on* $\mathcal{F}$.

Definition 5 (t-coherent part)**.** *Let* Π *be the set of all coherent assessments on a sequence of n conditional events* $\mathcal{F}$. *For each subset* $\mathcal{I} \subseteq [0,1]^n$, *the* t-coherent part *of* $\mathcal{I}$, *denoted by* $\pi(\mathcal{I})$, *is defined as* $\pi(\mathcal{I}) = \Pi \cap \mathcal{I}$.

Of course, $\pi(\mathcal{I}) \subseteq \mathcal{I}$. Moreover, if $\pi(\mathcal{I}) \neq \emptyset$, then $\mathcal{I}$ is g-coherent and $\pi(\mathcal{I})$ is t-coherent.

3. From Imprecise Assessments to the Square of Opposition

In this section we consider imprecise assessments on a given sequence $\mathcal{F}$ of n conditional events. In our approach, a sentence s is a pair $(\mathcal{F}, \mathcal{I})$, where $\mathcal{I} \subseteq [0,1]^n$ is an imprecise assessment on $\mathcal{F}$. Given an imprecise assessment $\mathcal{I}$ we denote by $\overline{\mathcal{I}}$ the *complementary imprecise assessment* of $\mathcal{I}$, i.e., $\overline{\mathcal{I}} = [0,1]^n \setminus \mathcal{I}$.

We introduce the following equivalence relation under t-coherence:

Definition 6. *Given two sentences $s_1 : (\mathcal{F}, \mathcal{I}_1)$ and $s_2 : (\mathcal{F}, \mathcal{I}_2)$, s_1 and s_2 are* equivalent*, denoted by $s_1 = s_2$, if and only if $\pi(\mathcal{I}_1) = \pi(\mathcal{I}_2)$.*

Definition 7. *Given three sentences $s : (\mathcal{F}, \mathcal{I})$, $s_1 : (\mathcal{F}, \mathcal{I}_1)$, and $s_2 : (\mathcal{F}, \mathcal{I}_2)$. We define $s_1 \wedge s_2 : (\mathcal{F}, \mathcal{I}_1 \cap \mathcal{I}_2)$ (conjunction); $s_1 \vee s_2 : (\mathcal{F}, \mathcal{I}_1 \cup \mathcal{I}_2)$ (disjunction); $\bar{s} : (\mathcal{F}, \overline{\mathcal{I}})$ (negation).*

Remark 1. *As the basic operations among sentences are defined by set-theoretical operations, they inherit the corresponding properties (including associativity, commutativity, De Morgan's law, etc.). Moreover, given two sentences $s_1 : (\mathcal{F}, \mathcal{I}_1)$ and $s_2 : (\mathcal{F}, \mathcal{I}_2)$, as $\pi(\mathcal{I}_1 \cap \mathcal{I}_2) = \pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_2)$, by setting $s_1^* : (\mathcal{F}, \pi(\mathcal{I}_1))$, $s_2^* : (\mathcal{F}, \pi(\mathcal{I}_2))$ and $(s_1 \wedge s_2)^* : (\mathcal{F}, \pi(\mathcal{I}_1 \cap \mathcal{I}_2))$, it follows that $(s_1 \wedge s_2) = (s_1 \wedge s_2)^* = s_1^* \wedge s_2^*$. Likewise, $s_1 \vee s_2 = (s_1 \vee s_2)^* = s_1^* \vee s_2^*$.*

As we interpret the basic sentence types involved in the square of opposition by imprecise probability assessments on sequences of conditional events, we will introduce the following notion of acceptability, which serves as a semantic bridge between basic sentence types and imprecise assessments:

Definition 8. *A sentence $s : (\mathcal{F}, \mathcal{I})$ is (resp., is not)* acceptable *if and only if the assessment $\mathcal{I}$ on $\mathcal{F}$ is (resp., is not) g-coherent, i.e., $\pi(\mathcal{I})$ is not (resp., is) empty.*

Remark 2. *If $s_1 \wedge s_2$ is acceptable, then s_1 is acceptable and s_2 is acceptable. However, the converse does not hold. Indeed, given a conditional event $E|H$, with $E \wedge H \neq \bot$ and $E \wedge H \neq H$, as the set of all coherent assessments on $E|H$ is $\Pi = [0,1]$, we notice that $s_1 : (E|H, \{1\})$ is acceptable and that $s_2 : (E|H), \{0\})$ is acceptable. However, $s_1 \wedge s_2 : (E|H, \emptyset)$ is not acceptable because $\pi(\emptyset) = \emptyset$.*

Definition 9. *Given two sentences $s_1 : (\mathcal{F}, \mathcal{I}_1)$ and $s_2 : (\mathcal{F}, \mathcal{I}_2)$, we say: s_1 and s_2 are* contraries *if and only if the sentence $s_1 \wedge s_2$ is not acceptable;*[2] *s_1 and s_2 are* subcontraries *if and only if $\bar{s}_1 \wedge \bar{s}_2$ is not acceptable; s_1 and s_2 are* contradictories *if and only if s_1 and s_2 are both contraries and subcontraries; s_2 is a* subaltern *of s_1 if and only if the sentence $s_1 \wedge \bar{s}_2$ is not acceptable.*

Remark 3. *By Remark 1, we observe that two sentences s_1 and s_2 are contraries if and only if $\pi(\mathcal{I}_1 \cap \mathcal{I}_2) = \pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_2) = \emptyset$. Moreover, two sentences s_1 and s_2 are subcontraries if and only if $\pi(\overline{\mathcal{I}}_1 \cap \overline{\mathcal{I}}_2) = \pi(\overline{\mathcal{I}}_1) \cap \pi(\overline{\mathcal{I}}_2) = \emptyset$, that is (by De Morgan's law) if and only if $\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_2) = \Pi$. Then, two sentences s_1 and s_2 are contradictories if and only if $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_2) = \emptyset$ and $\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_2) = \Pi$, that is if and only if $s_2 = \bar{s}_1$ (and, of course, $s_1 = \bar{s}_2$). Given two sentences s_1, s_2, we also observe that s_2 is a subaltern of s_1 if and only if $\Pi \cap (\mathcal{I}_1 \cap \overline{\mathcal{I}}_2) = \emptyset$, which also amounts to say that $\Pi \cap \mathcal{I}_1 \subseteq \Pi \cap \mathcal{I}_2$, that is if and only if $\pi(\mathcal{I}_1) \subseteq \pi(\mathcal{I}_2)$. For instance, $s_1 \vee s_2$ is a subaltern of s_1 and also of s_2; similarly, s_1 is a subaltern of $s_1 \wedge s_2$, and s_2 is a subaltern of $s_1 \wedge s_2$. Furthermore, if s_1 is not acceptable, that is $\pi(\mathcal{I}_1) = \emptyset$, then any sentence s_2 is a subaltern of s_1. For example, the sentence $s_1 : (E|\bar{E}, \{1\})$ is not acceptable because $\Pi = \{0\}$ and then any sentence $s_2 : (E|\bar{E}, \mathcal{I})$, where $\mathcal{I} \subseteq [0,1]$, is a subaltern of s_1.*

Based on the relations given in Definition 9 we define a square of opposition as follows.

Definition 10. *Let $s_k : (\mathcal{F}, \mathcal{I}_k)$, $k = 1,2,3,4$, be four sentences. We call the ordered quadruple (s_1, s_2, s_3, s_4) a* square of opposition *(under coherence) if and only if the following relations among the four sentences hold:*

(a) s_1 and s_2 are contraries, i.e., $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_2) = \emptyset$;

(b) s_3 and s_4 are subcontraries, i.e., $\pi(\mathcal{I}_3) \cup \pi(\mathcal{I}_4) = \Pi$;

[2] Some definitions of contrariety additionally require that "s_1 and s_2 can both be acceptable". For reasons stated in [34], we omit this additional requirement. Similarly, *mutatis mutandis*, in our definition of subcontrariety.

(c) s_1 *and* s_4 *are contradictories, i.e.,* $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_4) = \emptyset$ *and* $\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_4) = \Pi$*;*
s_2 *and* s_3 *are contradictories, i.e.,* $\pi(\mathcal{I}_2) \cap \pi(\mathcal{I}_3) = \emptyset$ *and* $\pi(\mathcal{I}_2) \cup \pi(\mathcal{I}_3) = \Pi$*;*

(d) s_3 *is a subaltern of* s_1*, i.e.,* $\pi(\mathcal{I}_1) \subseteq \pi(\mathcal{I}_3)$*;*
s_4 *is a subaltern of* s_2*, i.e.,* $\pi(\mathcal{I}_2) \subseteq \pi(\mathcal{I}_4)$*.*

Figure 3 shows the square of opposition based on Definition 10.

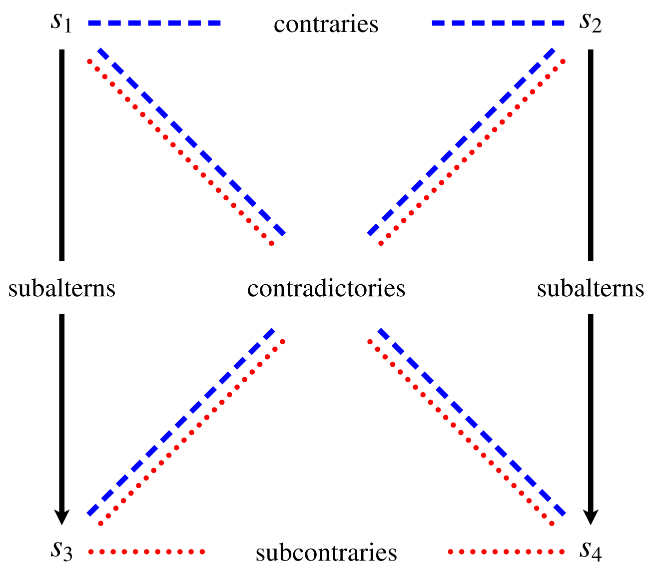

Figure 3: Probabilistic square of opposition defined by the quadruple (s_1, s_2, s_3, s_4).

Remark 4. *Based on Definition 10, we observe that in order to verify if a quadruple of sentences* (s_1, s_2, s_3, s_4)*, where* $s_k : (\mathcal{F}, \mathcal{I}_k)$*,* $k = 1, 2, 3, 4$*, is a square of opposition, it is necessary and sufficient to check that the quadruple* $(s_1^*, s_2^*, s_3^*, s_4^*)$*, where* $s_k^* : (\mathcal{F}, \mathcal{I}_k^*)$*,* $\mathcal{I}_k^* = \pi(\mathcal{I}_k)$*,* $k = 1, 2, 3, 4$*, is a square of opposition. Then, we observe that two squares* (s_1, s_2, s_3, s_4) *and* $(s_1^*, s_2^*, s_3^*, s_4^*)$ coincide *when* $\pi(\mathcal{I}_k) = \pi(\mathcal{I}_k^*)$ *for each k.*

Remark 5. *Based on Definition 10, we observe the following equivalence between two squares:* (s_1, s_2, s_3, s_4) *is a square of opposition if and only if* (s_2, s_1, s_4, s_3) *is a square of opposition. However, we cannot say in general that a generic permutation* $(s_{i_1}, s_{i_2}, s_{i_3}, s_{i_4})$ *of a square of opposition* (s_1, s_2, s_3, s_4) *is also a square of opposition. For instance, as subalternation is asymmetric, we cannot say in general that* (s_3, s_4, s_1, s_2) *is a square of opposition, if* (s_1, s_2, s_3, s_4) *is a square of opposition.*

Definition 11. *An (ordered)* tripartition *of a set* $\mathfrak{S}$ *is a triple* $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$*, where* $\mathcal{D}_1$*,* $\mathcal{D}_2$*, and* $\mathcal{D}_3$ *are subsets of* $\mathfrak{S}$*, such that the following conditions are satisfied: (i)* $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$*,* $i \neq j$ *for all* $i, j = 1, 2, 3$*; (ii);* $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 = \mathfrak{S}$*.*

Theorem 1. *Let any sequence of n conditional events* $\mathcal{F}$ *and a quadruple* (s_1, s_2, s_3, s_4) *of sentences, with* $s_k : (\mathcal{F}, \mathcal{I}_k)$*,* $k = 1, 2, 3, 4$*, be given. Define* $\mathcal{D}_1 = \pi(\mathcal{I}_1)$*,* $\mathcal{D}_2 = \pi(\mathcal{I}_2)$*, and* $\mathcal{D}_3 = \pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4)$*. Then, the quadruple* (s_1, s_2, s_3, s_4) *is a square of opposition if and only if* $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ *is a tripartition of (the non-empty set)* Π *such that:* $\pi(\mathcal{I}_3) = \mathcal{D}_1 \cup \mathcal{D}_3$*,* $\pi(\mathcal{I}_4) = \mathcal{D}_2 \cup \mathcal{D}_3$*.*

Proof. ($\Rightarrow$). We assume that $\mathcal{D}_1 = \pi(\mathcal{I}_1)$, $\mathcal{D}_2 = \pi(\mathcal{I}_2)$, and $\mathcal{D}_3 = \pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4)$. Of course, $\mathcal{D}_k \subseteq \Pi$, $k = 1, 2, 3$. We now prove that: *(i)* $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$; *(ii)* $\mathcal{D}_3 = \Pi \backslash (\mathcal{D}_1 \cup \mathcal{D}_2)$. *(i)* From condition (a) in Definition 10, as s_1 and s_2 are contraries, it follows that $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$. *(ii)* We first prove that $\mathcal{D}_3 \subseteq \Pi \backslash (\mathcal{D}_1 \cup \mathcal{D}_2)$. This trivially follows when $\mathcal{D}_3 = \emptyset$. If $\mathcal{D}_3 \neq \emptyset$, then let $x \in \mathcal{D}_3 = \pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4)$. As $x \in \pi(\mathcal{I}_3)$, from condition (c) in Definition 10, we obtain $x \notin \pi(\mathcal{I}_2)$. Likewise, as $x \in \pi(\mathcal{I}_4)$, from condition (c) in Definition 10, we obtain $x \notin \pi(\mathcal{I}_1)$. Then, $x \in \Pi$ and $x \notin (\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_2))$, that is $x \in \Pi \backslash (\mathcal{D}_1 \cup \mathcal{D}_2)$. We now prove that $\Pi \backslash (\mathcal{D}_1 \cup \mathcal{D}_2) \subseteq \mathcal{D}_3$. This trivially follows when $\Pi \backslash (\mathcal{D}_1 \cup \mathcal{D}_2) = \emptyset$. If $\Pi \backslash (\mathcal{D}_1 \cup \mathcal{D}_2) \neq \emptyset$, let $x \in \Pi \backslash (\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_2))$. As $x \in \Pi \backslash \pi(\mathcal{I}_1)$, from condition (c) in Definition 10, we obtain $x \in \pi(\mathcal{I}_4)$. Likewise, as $x \in \Pi \backslash \pi(\mathcal{I}_2)$ from condition (c) in Definition 10, we obtain $x \in \pi(\mathcal{I}_3)$. Then, $x \in (\pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4)) = \mathcal{D}_3$. Therefore $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ is a tripartition of Π. By our assumption, $\pi(\mathcal{I}_1) = \mathcal{D}_1$ and $\pi(\mathcal{I}_2) = \mathcal{D}_2$. We observe that $\pi(\mathcal{I}_3) \cap \mathcal{D}_3 = \mathcal{D}_3$; moreover, from conditions (c) and (d), we obtain $\pi(\mathcal{I}_3) \cap \mathcal{D}_2 = \pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_2) = \emptyset$ and $\pi(\mathcal{I}_3) \cap \mathcal{D}_1 = \pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_3) = \pi(\mathcal{I}_1) = \mathcal{D}_1$; then $\pi(\mathcal{I}_3) = \pi(\mathcal{I}_3) \cap (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3) = \mathcal{D}_1 \cup \mathcal{D}_3$. Likewise, we observe that $\pi(\mathcal{I}_4) \cap \mathcal{D}_3 = \mathcal{D}_3$; moreover, from conditions (c),(d) in Definition 10, we obtain $\mathcal{D}_1 \cap \pi(\mathcal{I}_4) = \pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_4) = \emptyset$ and $\mathcal{D}_2 \cap \pi(\mathcal{I}_4) = \pi(\mathcal{I}_2) \cap \pi(\mathcal{I}_4) = \pi(\mathcal{I}_2) = \mathcal{D}_2$; then $\pi(\mathcal{I}_4) = \pi(\mathcal{I}_4) \cap (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3) = \mathcal{D}_2 \cup \mathcal{D}_3$.
($\Leftarrow$) Assume that $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, where $\mathcal{D}_1 = \pi(\mathcal{I}_1)$, $\mathcal{D}_2 = \pi(\mathcal{I}_2)$, $\mathcal{D}_3 = \pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4)$, is a tripartition of Π such that $\mathcal{D}_1 \cup \mathcal{D}_3 = \pi(\mathcal{I}_3)$ and $\mathcal{D}_2 \cup \mathcal{D}_3 = \pi(\mathcal{I}_4)$, we prove that the quadruple (s_1, s_2, s_3, s_4) satisfies conditions (a), (b), (c), and (d) in Definition 10. We observe that $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_2) = \mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, which coincides with (a). Condition (b) is satisfied because $\pi(\mathcal{I}_3) \cup \pi(\mathcal{I}_4) = \mathcal{D}_1 \cup \mathcal{D}_3 \cup \mathcal{D}_2 \cup \mathcal{D}_3 = \Pi$. Moreover, $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_4) = \mathcal{D}_1 \cap (\mathcal{D}_2 \cup \mathcal{D}_3) = \emptyset$ and $\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_4) = \mathcal{D}_1 \cup (\mathcal{D}_2 \cup \mathcal{D}_3) = \Pi$; likewise, $\pi(\mathcal{I}_2) \cap \pi(\mathcal{I}_3) = \mathcal{D}_2 \cap (\mathcal{D}_1 \cup \mathcal{D}_3) = \emptyset$ and $\pi(\mathcal{I}_2) \cup \pi(\mathcal{I}_3) = \mathcal{D}_2 \cup (\mathcal{D}_1 \cup \mathcal{D}_3) = \Pi$. Thus, the conditions in (c) are satisfied. Finally, $\pi(\mathcal{I}_1) = \mathcal{D}_1 \subseteq \mathcal{D}_1 \cup \mathcal{D}_3 = \pi(\mathcal{I}_3)$ and $\pi(\mathcal{I}_2) = \mathcal{D}_2 \subseteq \mathcal{D}_2 \cup \mathcal{D}_3 = \pi(\mathcal{I}_4)$ which satisfy conditions in (d). □

A method to construct a square of opposition by starting from a tripartition of Π is given in the following result (see also [24]).

Corollary 1. *Given any sequence of n conditional events $\mathcal{F}$ and a tripartition $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ of Π, then the quadruple (s_1, s_2, s_3, s_4), with $s_k : (\mathcal{F}, \mathcal{I}_k)$, $k = 1, 2, 3, 4$, and $\pi(\mathcal{I}_1) = \mathcal{D}_1$, $\pi(\mathcal{I}_2) = \mathcal{D}_2$, $\pi(\mathcal{I}_3) = \mathcal{D}_1 \cup \mathcal{D}_3$, $\pi(\mathcal{I}_4) = \mathcal{D}_2 \cup \mathcal{D}_3$ is a square of opposition.*

Proof. The proof immediately follows by observing that $\pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4) = \mathcal{D}_3$ and by the ($\Leftarrow$) side proof of Theorem 1. □

The following result allows to construct a square of opposition by starting from a tripartition of the whole set $[0, 1]^n$:

Corollary 2. *Given a tripartition $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ of $[0, 1]^n$, let $\mathcal{I}_1 = \mathcal{B}_1$, $\mathcal{I}_2 = \mathcal{B}_2$, $\mathcal{I}_3 = \mathcal{B}_1 \cup \mathcal{B}_3$, and $\mathcal{I}_4 = \mathcal{B}_2 \cup \mathcal{B}_3$. For any sequence of n conditional events $\mathcal{F}$, the quadruple (s_1, s_2, s_3, s_4), where $s_k : (\mathcal{F}, \mathcal{I}_k)$, $k = 1, 2, 3, 4$, is a square of opposition.*

Proof. Let $\mathcal{F}$ be any sequence of n conditional events and Π be the associated set of all coherent precise assessments. We set $\mathcal{D}_i = \pi(\mathcal{B}_i)$, $i = 1, 2, 3$. Of course, $(\pi(\mathcal{B}_1), \pi(\mathcal{B}_2), \pi(\mathcal{B}_3))$ is a tripartition of Π. Moreover, $\pi(\mathcal{I}_1) = \mathcal{D}_1$, $\pi(\mathcal{I}_2) = \mathcal{D}_2$, $\pi(\mathcal{I}_3) = \mathcal{D}_1 \cup \mathcal{D}_3$, $\pi(\mathcal{I}_4) = \mathcal{D}_2 \cup \mathcal{D}_3$. Then, by Corollary 1 and Remark 4 we obtain that (s_1, s_2, s_3, s_4) is a square of opposition. □

Traditionally the square of opposition can be constructed based on the fragmented square of opposition which requires only the contrariety and contradiction relations (which goes back to Aristotle's *De Interpretatione* 6–7, 17b.17–26, see [49, Section 2]). This result also holds in our framework:

Theorem 2. *The quadruple (s_1, s_2, s_3, s_4) of sentences, with $s_k : (\mathcal{F}, \mathcal{I}_k)$, $k = 1, 2, 3, 4$, is a square of opposition if and only if relations (a) and (c) in Definition 10 are satisfied.*

Proof. ($\Rightarrow$) It follows directly from Definition 10. ($\Leftarrow$) We prove that (d) and (b) in Definition 10 follow from (a) and (c). If $\pi(\mathcal{I}_1) = \emptyset$, then of course $\pi(\mathcal{I}_1) \subseteq \pi(\mathcal{I}_3)$. If $\pi(\mathcal{I}_1) \neq \emptyset$, let $x \in \pi(\mathcal{I}_1) \subseteq \Pi$, from (a) it follows that $x \notin \pi(\mathcal{I}_2)$, and since (c) requires $\pi(\mathcal{I}_2) \cup \pi(\mathcal{I}_3) = \Pi$, we obtain $x \in \pi(\mathcal{I}_3)$. Thus, $\pi(\mathcal{I}_1) \subseteq \pi(\mathcal{I}_3)$; likewise, $\pi(\mathcal{I}_2) \subseteq \pi(\mathcal{I}_4)$. Therefore, (d) is satisfied. Now we prove that (b) is satisfied, i.e., $\pi(\mathcal{I}_3) \cup \pi(\mathcal{I}_4) = \Pi$. Of course,

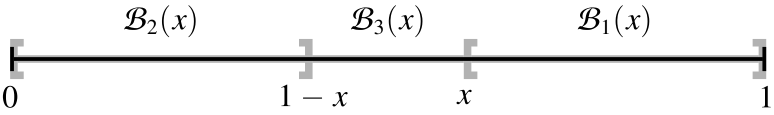

Figure 4: Example of a tripartition $(\mathcal{B}_1(x), \mathcal{B}_2(x), \mathcal{B}_3(x))$ of $[0,1]$, with $x \in]\frac{1}{2}, 1]$.

$\pi(\mathcal{I}_3) \cup \pi(\mathcal{I}_4) \subseteq \Pi$. Let $x \in \Pi$. If $x \notin \pi(\mathcal{I}_3)$, then, $x \in \pi(\mathcal{I}_2)$ from (c). Moreover, from (d), $x \in \pi(\mathcal{I}_4)$. Then, $\Pi \subseteq \pi(\mathcal{I}_3) \cup \pi(\mathcal{I}_4)$. Therefore, (b) is satisfied. □

Corollary 3. *The quadruple (s_1, s_2, s_3, s_4) of sentences, with $s_k : (\mathcal{F}, \mathcal{I}_k)$, $k = 1, 2, 3, 4$, is a square of opposition if and only if $(s_1, s_2, s_3, s_4) = (s_1, s_2, \bar{s}_2, \bar{s}_1)$ with s_1 and s_2 being contraries.*

Proof. Of course, if (s_1, s_2, s_3, s_4) is a square of opposition, then s_1 and s_2 are contraries. Moreover, s_1 and s_4 are contradictories, that is: $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_4) = \emptyset$ and $\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_4) = \Pi$. Therefore, $\Pi \backslash \pi(\mathcal{I}_4) = \pi(\mathcal{I}_1)$, which amounts to $s_4 = \bar{s}_1$. Similarly, as s_2 and s_3 are contradictories, it holds that $s_3 = \bar{s}_2$. Conversely, assume that s_1 and s_2 are contraries. By instantiating Theorem 2 with $s_3 = \bar{s}_2$ and with $s_4 = \bar{s}_1$, it follows that the quadruple $(s_1, s_2, \bar{s}_2, \bar{s}_1)$ is a square of opposition. □

In the next section we consider the case where $\mathcal{F}$ consists of one conditional event only.

4. Square of Opposition and Generalized Quantifiers

From a traditional logical point of view, the quantifier *some* is interpreted in the context of the square of opposition to mean *at least one*. From a natural language point of view, however, speakers usually use generalized quantifiers, like *(at least) most* and *(at least) almost all*. Even if speakers mention words like "every" or "all", they often mean quantifiers which allow for exceptions (like *(at least) almost all*; see, e.g., [15, 16, 48, 53, 56, 59]). Of course, generalized quantifiers allow for many more applications compared to the "extreme" quantifiers *at least one*, *no* and *every* ([51]).

In this section we extend our semantics to deal with generalized quantifiers by introducing a threshold which makes a criterion for asserting such quantifiers explicit. Of course, the selection of the threshold may depend on the speaker's degrees of belief and on the context. One could, for example, set the threshold x equal to .6; then, $p(P|S) \geqslant .6$ represents *most S are P* (in the sense of "*at least* most S are P"). In other contexts, the threshold could be lower or higher than .6 to represent *most S are P*. Likewise, $p(P|S) \geqslant .9$ could represent *almost all S are P* (in the sense of "*at least* almost all S are P").

Let a conditional event $P|S$ (where $S \neq \bot$) and a threshold $x \in]\frac{1}{2}, 1]$ be given. We denote by $(\mathcal{B}_1(x), \mathcal{B}_2(x), \mathcal{B}_3(x))$ a tripartition of $[0,1]$, where $\mathcal{B}_1(x) = [x, 1]$, $\mathcal{B}_2(x) = [0, 1-x]$, $\mathcal{B}_3(x) =]1-x, x[$ and $x \in]\frac{1}{2}, 1]$ (see Figure 4). Consider the quadruple of sentences $(A(x), E(x), I(x), O(x))$, with $A(x) : (P|S, \mathcal{I}_{A(x)})$, $E(x) : (P|S, \mathcal{I}_{E(x)})$, $I(x) : (P|S, \mathcal{I}_{I(x)})$, $O(x) : (P|S, \mathcal{I}_{O(x)})$, where $\mathcal{I}_{A(x)} = \mathcal{B}_1(x) = [x, 1]$, $\mathcal{I}_{E(x)} = \mathcal{B}_2(x) = [0, 1-x]$, $\mathcal{I}_{I(x)} = \mathcal{B}_1(x) \cup \mathcal{B}_3(x) =]1-x, 1]$, and $\mathcal{I}_{O(x)} = \mathcal{B}_2(x) \cup \mathcal{B}_3(x) = [0, x[$. By applying Corollary 2 with $(s_1, s_2, s_3, s_4) = (A(x), E(x), I(x), O(x))$, it follows that $(A(x), E(x), I(x), O(x))$ is a square of opposition for any $x \in]\frac{1}{2}, 1]$ (see Figure 5). We recall that in presence of some logical relations between P and S the set Π could be a strict subset of $[0,1]$. More precisely, we have the following three cases (see, [36, 37]): (i) if $P \wedge S \neq \bot$ and $P \wedge S \neq S$, then $\Pi = [0,1]$; (ii) if $P \wedge S = S$, then $\Pi = \{1\}$; (iii) if $P \wedge S = \bot$, then $\Pi = \{0\}$. The quadruple $(A(x), E(x), I(x), O(x))$, with the threshold $\frac{1}{2} < x \leqslant 1$, is a square of opposition in each of the three cases. In particular we obtain: case (i) $\pi(\mathcal{I}_{A(x)}) = \mathcal{I}_{A(x)}$, $\pi(\mathcal{I}_{E(x)}) = \mathcal{I}_{E(x)}$,$\pi(\mathcal{I}_{I(x)}) = \mathcal{I}_{I(x)}$, and $\pi(\mathcal{I}_{O(x)}) = \mathcal{I}_{O(x)}$; case (ii): $\pi(\mathcal{I}_{A(x)}) = \{1\}$, $\pi(\mathcal{I}_{E(x)}) = \emptyset$,$\pi(\mathcal{I}_{I(x)}) = \{1\}$, and $\pi(\mathcal{I}_{O(x)}) = \emptyset$; case (iii): $\pi(\mathcal{I}_{A(x)}) = \emptyset$, $\pi(\mathcal{I}_{E(x)}) = \{1\}$,$\pi(\mathcal{I}_{I(x)}) = \emptyset$, and $\pi(\mathcal{I}_{O(x)}) = \{1\}$. We note that in cases (ii) and (iii) we obtain degenerated squares each, where—apart from the contradictory relations—all relations are strengthened (see Figure 6). Specifically, both contrary and the subcontrary become contradictory relations. Moreover, both subalternation relations become symmetric. As by coherence $p(P|S) + p(\bar{P}|S) = 1$, the probability constraint $p(P|S) \leqslant 1 - x$ is equivalent to $p(\bar{P}|S) \geqslant x$, likewise, $p(P|S) < x$ is equivalent to $p(\bar{P}|S) > 1 - x$. Table 1

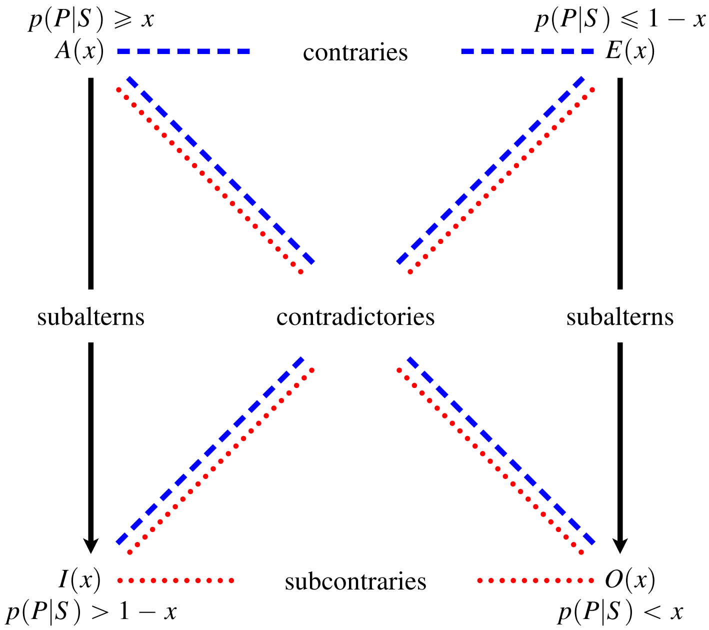

Figure 5: Probabilistic square of opposition $\mathbf{S}(x)$ involving generalized quantifiers defined on the four sentence types $(A(x), E(x), I(x), O(x))$ with the threshold $x \in]\frac{1}{2}, 1]$ (see also Table 1). In the extreme case where $x = 1$, we obtain a new interpretation of the traditional square of opposition (see also Figure 1), where the corners are labeled by "Every S is P" (A), "No S is P" (E), "Some S is P" (I), and "Some S is not P" (O).

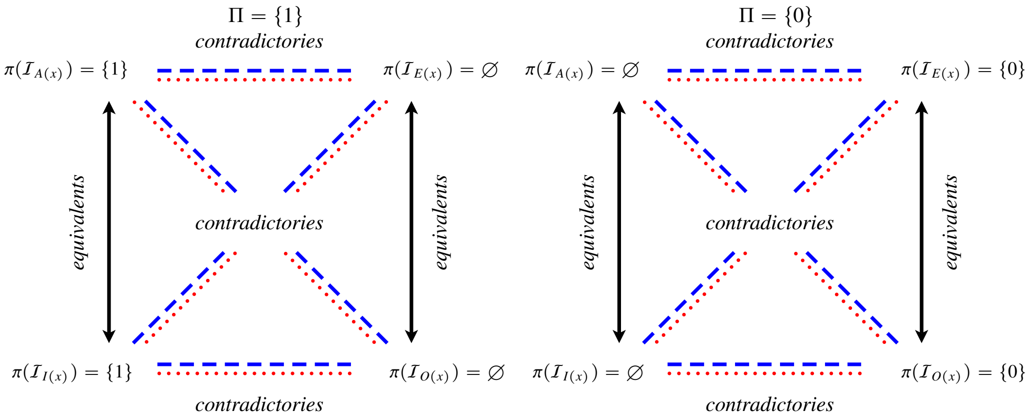

Figure 6: Degenerated squares of opposition $(\pi(\mathcal{I}_{A(x)}), \pi(\mathcal{I}_{E(x)}), \pi(\mathcal{I}_{I(x)}), \pi(\mathcal{I}_{O(x)}))$ when $\mathcal{F}$ consists of the conditional event $P|S$ and the set of all coherent assessments on $P|S$ is $\Pi = \{1\}$ (i.e., $P \wedge S = S$; left) or $\Pi = \{0\}$ (i.e., $P \wedge S = \bot$; right).

presents generalization of basic sentence types $A(x)$, $E(x)$, $I(x)$, and $O(x)$ involving generalized quantifiers Q. The generalized quantifiers are defined on a threshold $x > \frac{1}{2}$. The value of the threshold may be context dependent and provides lots of flexibility for modeling various instances of generalized quantifiers (like *most*, *almost all*).

Given two thresholds x_1 and x_2, with $\frac{1}{2} < x_2 < x_1 \leqslant 1$, we analyze the relations among the same sentence types in the two squares of opposition $\mathbf{S}(x_1)$ and $\mathbf{S}(x_2)$, with $\mathbf{S}(x_i) = (A(x_i), E(x_i), I(x_i), O(x_i))$, $i = 1, 2$. It can be easily proved that: $A(x_2)$ is a subaltern of $A(x_1)$, $E(x_2)$ is a subaltern of $E(x_1)$, $I(x_1)$ is a subaltern of $I(x_2)$, and $O(x_1)$ is a subaltern of $O(x_2)$. In the extreme case $x = 1$ we obtain the probabilistic interpretation under coherence of the basic

Sentence		Probability constraints	Assessment on $P\|S$
$A(x)$:	($Q_{\geqslant x}$ S are P)	$p(P\|S) \geqslant x$	$\mathcal{I}_{A(x)} = [x, 1]$
$E(x)$:	($Q_{\geqslant x}$ S are not P)	$p(\bar{P}\|S) \geqslant x$	$\mathcal{I}_{E(x)} = [0, 1-x]$
$I(x)$:	($Q_{>1-x}$ S are P)	$p(P\|S) > 1-x$	$\mathcal{I}_{I(x)} =]1-x, 1]$
$O(x)$:	($Q_{>1-x}$ S are not P)	$p(\bar{P}\|S) > 1-x$	$\mathcal{I}_{O(x)} = [0, x[$
$A(1)$:	(Every S is P)	$p(P\|S) = 1$	$\mathcal{I}_A = \{1\}$
$E(1)$:	(No S is P)	$p(\bar{P}\|S) = 1$	$\mathcal{I}_E = \{0\}$
$I(1)$:	(Some S is P)	$p(P\|S) > 0$	$\mathcal{I}_I =]0, 1]$
$O(1)$:	(Some S is not P)	$p(\bar{P}\|S) > 0$	$\mathcal{I}_O = [0, 1[$

Table 1: Probabilistic interpretation of the sentence types A, E, I, and O involving generalized quantifiers Q defined by a threshold x (with $x \in]\frac{1}{2}, 1]$) on the subject S and predicate P and the respective imprecise probabilistic assessments $\mathcal{I}_{A(x)}$, $\mathcal{I}_{E(x)}$, $\mathcal{I}_{I(x)}$, and $\mathcal{I}_{O(x)}$ on the conditional event $P|S$ (above). When $x = 1$, we obtain our probabilistic interpretation of the traditional sentence types A, E, I, and O (below).

sentence types involved in the traditional square of opposition (A, E, I, O), which is illustrated in Figure 1. Notice that this square of opposition coincides with the *default square of opposition* involving defaults and negated defaults given in [34].

Remark 6. *In agreement with De Morgan (as pointed out by [24]) by the quadruple (a, e, i, o) we denote the square of opposition obtained from (A, E, I, O) when the events P and S are replaced by $\bar{P}$ and $\bar{S}$, respectively. Specifically, $a : (\bar{P}|\bar{S}, \{1\})$, $e : (\bar{P}|\bar{S}, \{0\})$, $i : (\bar{P}|\bar{S},]0, 1])$, and $o : (\bar{P}|\bar{S}, [0, 1[)$. From a geometric point of view, one might ask whether it is possible to construct a "cube of opposition" such that (A, E, I, O) and (a, e, i, o) are opposing facing sides of such a cube (see, e.g., [24]) and where each edge represents a probabilistic constraint between the end points of the edge (i.e., the two vertexes). Then, we would require probabilistic constraints between some sentences s_1 and s_2, where $s_1 \in \{A, E, I, O\}$ and $s_2 \in \{a, e, i, o\}$. However, we observe that in the general case when P and S are logically independent, it can be proved that the set of all coherent assessments on $(P|S, \bar{P}|\bar{S})$ is the unit square $[0, 1]^2$ (see, e.g., [31, Proposition 12]; related theoretical results are given in [17, Proposition 1] and [18, Theorem 4]). Thus, in the general case there are no relations between any two sentences s_1 and s_2, where $s_1 \in \{A, E, I, O\}$ and $s_2 \in \{a, e, i, o\}$. Therefore, it does not make sense to construct a "cube of opposition" (with these two squares as opposite facing sides) in our context, as both squares of opposition (A, E, I, O) and (a, e, i, o) are "independent" of each other.*

Remark 7. *Given two thresholds x, y such that $0 \leqslant y \leqslant \frac{1}{2} < x \leqslant 1$, we set $\mathcal{B}_1(x, y) = [x, 1]$, $\mathcal{B}_2(x, y) = [0, y]$, and $\mathcal{B}_3(x, y) =]y, x[$. Then, $(\mathcal{B}_1(x, y), \mathcal{B}_2(x, y), \mathcal{B}_3(x, y))$ is a tripartition of $[0, 1]$. Consider now the four sentences $(P|S, [x, 1])$, $(P|S, [0, y])$, $(P|S,]y, 1])$, $(P|S, [0, x[)$, which represent the probability assessments $p(P|S) \geqslant x$, $p(P|S) \leqslant y$, $p(P|S) > y$, and $p(P|S) < x$, respectively. As $\mathcal{B}_1(x, y) = [x, 1]$, $\mathcal{B}_2(x, y) = [0, y]$, $\mathcal{B}_1(x, y) \cup \mathcal{B}_3(x, y) =]y, 1]$, and $\mathcal{B}_2(x, y) \cup \mathcal{B}_3(x, y) = [0, x[$, by applying Corollary 2 it follows that the quadruple of sentences $((P|S, [x, 1])$, $(P|S, [0, y])$, $(P|S,]y, 1])$, $(P|S, [0, x[))$ is a square of opposition.*

5. Hexagon of Opposition

Compared to the millennia long history of investigations on the square of opposition, the hexagon of opposition was discovered fairly recently, namely in the 1950ies. The hexagon generalizes the square by adding the disjunction of the top vertices of the square to build a new vertex at the top and by adding the conjunction of the bottom vertices of the square to build a new vertex at the bottom. According to Béziau ([3]), the hexagon of opposition was introduced by the French priest and logician Augustin Sesmat ([62]) and by the philosopher Robert Blanché ([10]), who worked out the full structure of the hexagon of opposition (for his main work on the hexagon of opposition see [11]). Jaspers and Seuren ([41]) trace the history of the hexagon back to the American philosopher Paul Jacoby ([40], see also [24]). In this section we will use the tools developed in Section 3, to construct a *hexagon of opposition* by starting from a square of opposition. More precisely, given a traditional square of opposition (A, E, I, O), by setting $U = A \vee E$, $Y = I \wedge O$, the tuple (A, E, I, O, U, Y) defines a hexagon of opposition. Accordingly, we define the (probabilistic) hexagon of opposition in our approach as follows:

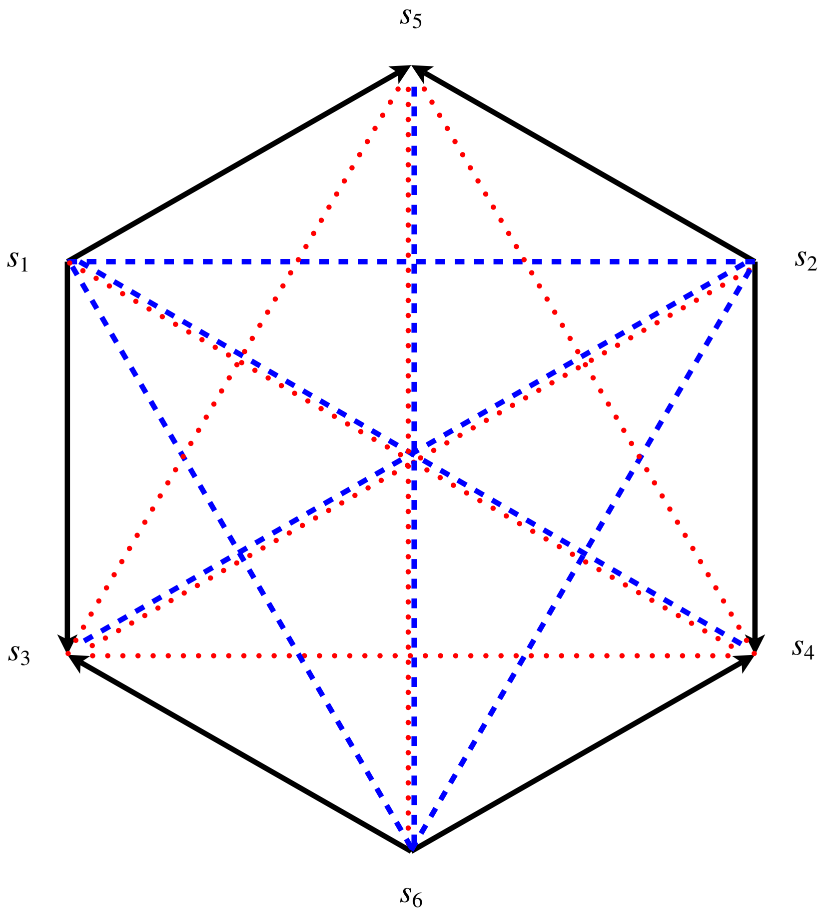

Figure 7: Probabilistic hexagon of opposition defined on the six sentence types $(s_1, s_2, s_3, s_4, s_5, s_6)$, where (s_1, s_2, s_3, s_4) is a square of opposition, $s_5 = s_1 \vee s_2$, and $s_6 = s_3 \wedge s_4$ (see Definition 12). The arrows indicate subalternation, dashed lines indicate contraries, and dotted lines indicate sub-contraries. Contradictories are indicated by combined dotted and dashed lines.

Definition 12 (Hexagon of opposition)**.** *Let* $s_k : (\mathcal{F}, \mathcal{I}_k)$*,* $k = 1, 2, 3, 4, 5, 6$*, be six sentences. We call the ordered tuple* $(s_1, s_2, s_3, s_4, s_5, s_6)$ *a* hexagon of opposition *(under coherence) if and only if the following relations among the six sentences hold:*

(i) (s_1, s_2, s_3, s_4) *is a square of opposition;*

(ii) $s_5 = s_1 \vee s_2$*;*

(iii) $s_6 = s_3 \wedge s_4$.

Figure 7 shows the probabilistic hexagon of opposition as given by Definition 12.

Theorem 3. *Let* $s_k : (\mathcal{F}, \mathcal{I}_k)$*,* $k = 1, 2, 3, 4, 5, 6$*, be six sentences. The tuple* $(s_1, s_2, s_3, s_4, s_5, s_6)$ *is a hexagon of opposition if and only if* $(s_1, s_2, s_3, s_4, s_5, s_6) = (s_1, s_2, \bar{s}_2, \bar{s}_1, s_1 \vee s_2, \bar{s}_1 \wedge \bar{s}_2)$*, with* s_1 *and* s_2 *being contraries.*

Proof. $(\Rightarrow)$. Let $(s_1, s_2, s_3, s_4, s_5, s_6)$ be a hexagon of opposition. Then, as (s_1, s_2, s_3, s_4) is a square of opposition, s_1 and s_2 are contraries. Moreover, by Corollary 3, it follows that $(s_1, s_2, s_3, s_4) = (s_1, s_2, \bar{s}_2, \bar{s}_1)$. Then, by Definition 12, $s_5 = s_1 \vee s_2$ and $s_6 = s_3 \wedge s_4 = \bar{s}_1 \wedge \bar{s}_2$. Therefore, $(s_1, s_2, s_3, s_4, s_5, s_6) = (s_1, s_2, \bar{s}_2, \bar{s}_1, s_1 \vee s_2, \bar{s}_1 \wedge \bar{s}_2)$.
$(\Leftarrow)$. Let $(s_1, s_2, s_3, s_4, s_5, s_6) = (s_1, s_2, \bar{s}_2, \bar{s}_1, s_1 \vee s_2, \bar{s}_1 \wedge \bar{s}_2)$, with s_1 and s_2 being contraries. From Corollary 3, it follows that (s_1, s_2, s_3, s_4) is a square of opposition. Then, by relations (ii) and (iii) in Definition 12, it follows that $(s_1, s_2, s_3, s_4, s_5, s_6)$ is a hexagon of opposition.

$\square$

Remark 8. *Assume that s_1 and s_2 are contraries. Then, by Corollary 3, the quadruple $(s_1, s_2, \bar{s}_2, \bar{s}_1)$ is a square of opposition, and by Definition 12, the tuple $(s_1, s_2, \bar{s}_2, \bar{s}_1, s_1 \vee s_2, \bar{s}_1 \wedge \bar{s}_2)$ is a hexagon of opposition.*

We now consider relations among a tripartition of the set of all coherent assessments Π and a hexagon of opposition.

Remark 9. *Given a hexagon of opposition $(s_1, s_2, s_3, s_4, s_5, s_6)$, we observe that the sentence $s_6 = s_3 \wedge s_4$ represents the pair $(\mathcal{F}, \mathcal{I}_6)$, where $\mathcal{I}_6 = \mathcal{I}_3 \cap \mathcal{I}_4$. Moreover, by Remark 1, $\pi(\mathcal{I}_6) = \pi(\mathcal{I}_3 \cap \mathcal{I}_4) = \pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4)$. Therefore, based on Theorem 1, the triple $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, where $\mathcal{D}_1 = \pi(\mathcal{I}_1)$, $\mathcal{D}_2 = \pi(\mathcal{I}_2)$, and $\mathcal{D}_3 = \pi(\mathcal{I}_6)$, is a tripartition of Π. Conversely, based on Corollary 1, given a tripartition $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ of Π, the sequence $(s_1, s_2, s_3, s_4, s_5, s_6)$ where $s_k : (\mathcal{F}, \mathcal{I}_k)$, $k = 1, \ldots, 6$, with $\pi(\mathcal{I}_1) = \mathcal{D}_1$, $\pi(\mathcal{I}_2) = \mathcal{D}_2$, $\pi(\mathcal{I}_3) = \mathcal{D}_1 \cup \mathcal{D}_3$, $\pi(\mathcal{I}_4) = \mathcal{D}_2 \cup \mathcal{D}_3$, $\pi(\mathcal{I}_5) = \mathcal{D}_1 \cup \mathcal{D}_2$, and $\pi(\mathcal{I}_6) = \mathcal{D}_3$, is a hexagon of opposition (see also [14, 24, 25]).*

Next, we consider relations among a tripartition of $[0, 1]^n$ and a hexagon of opposition.

Remark 10. *Based on Corollary 2, we can also construct a hexagon of opposition by starting from a tripartition of the whole set $[0, 1]^n$. Specifically, given a tripartition $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ of $[0, 1]^n$, let $\mathcal{I}_1 = \mathcal{B}_1$, $\mathcal{I}_2 = \mathcal{B}_2$, $\mathcal{I}_3 = \mathcal{B}_1 \cup \mathcal{B}_3$, $\mathcal{I}_4 = \mathcal{B}_2 \cup \mathcal{B}_3$, $\mathcal{I}_5 = \mathcal{B}_1 \cup \mathcal{B}_2$, and $\mathcal{I}_6 = \mathcal{B}_3$. For any sequence of n conditional events $\mathcal{F}$, the tuple $(s_1, s_2, s_3, s_4, s_5, s_6)$, where $s_k : (\mathcal{F}, \mathcal{I}_k)$, $k = 1, \ldots, 6$, is a hexagon of opposition.*

Theorem 4. *Given a hexagon of opposition $(s_1, s_2, s_3, s_4, s_5, s_6)$, by Definition 12 all relations among the basic sentence types in the square (s_1, s_2, s_3, s_4) hold. Moreover, by Theorem 3 (and also by Remark 3), the following relations hold:*

(i) *s_1 and s_6 are contraries (since $s_6 = \bar{s}_2 \wedge \bar{s}_1$ and $\pi(\mathcal{I}_1 \cap \bar{\mathcal{I}}_2 \cap \bar{\mathcal{I}}_1) = \emptyset$);*

(ii) *s_2 and s_6 are contraries (since $s_6 = \bar{s}_2 \wedge \bar{s}_1$ and $\pi(\mathcal{I}_2 \cap \bar{\mathcal{I}}_2 \cap \bar{\mathcal{I}}_1) = \emptyset$);*

(iii) *s_3 is a subaltern of s_6 (since $s_6 = s_3 \wedge s_4$);*

(iv) *s_4 is a subaltern of s_6 (since $s_6 = s_3 \wedge s_4$);*

(v) *s_5 is a subaltern of s_1 (since $s_5 = s_1 \vee s_2$);*

(vi) *s_5 is a subaltern of s_2 (since $s_5 = s_1 \vee s_2$);*

(vii) *s_5 and s_3 are subcontraries (as $s_5 = s_1 \vee s_2$ and $s_3 = \bar{s}_2$, hence $\pi(\mathcal{I}_1 \cup \mathcal{I}_2) \cup \pi(\bar{\mathcal{I}}_2) = \Pi$);*

(viii) *s_5 and s_4 are subcontraries (as $s_5 = s_1 \vee s_2$ and $s_4 = \bar{s}_1$, hence $\pi(\mathcal{I}_1 \cup \mathcal{I}_2) \cup \pi(\bar{\mathcal{I}}_1) = \Pi$);*

(ix) *s_5 and s_6 are contradictories (as $s_5 = s_1 \vee s_2$, $s_6 = s_3 \wedge s_4 = \bar{s}_2 \wedge \bar{s}_1$, hence $\pi((\mathcal{I}_1 \cup \mathcal{I}_2) \cap (\bar{\mathcal{I}}_1 \cap \bar{\mathcal{I}}_2)) = \emptyset$ and $\pi((\mathcal{I}_1 \cup \mathcal{I}_2) \cup (\bar{\mathcal{I}}_1 \cap \bar{\mathcal{I}}_2)) = \Pi$).*

Figure 7 illustrates all the relations in the hexagon of opposition described in Theorem 4. This figure also shows the two triangles $T_1 : (s_1, s_2, s_6)$ and $T_2 : (s_3, s_4, s_5)$. We note that the sides of T_1 consist of contrary relations, whereas the sides of T_2 consist of subcontrary relations. Moreover, the coherent part of the imprecise assessments defined by sentences in T_1 (i.e., $\mathcal{D}_1 = \pi(\mathcal{I}_1)$, $\mathcal{D}_2 = \pi(\mathcal{I}_2)$ and $\mathcal{D}_3 = \pi(\mathcal{I}_6)$) forms a tripartition $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ of Π. Whereas, the imprecise assessments defined by sentences in T_2 are such that $\pi(\mathcal{I}_3) = \mathcal{D}_1 \cup \mathcal{D}_3$, $\pi(\mathcal{I}_4) = \mathcal{D}_2 \cup \mathcal{D}_3$, and $\pi(\mathcal{I}_5) = \mathcal{D}_1 \cup \mathcal{D}_2$.

Remark 11. *We can construct a hexagon of opposition involving assessments on a conditional event $P|S$ by basing it on the square of opposition $(A(x), E(x), I(x), O(x))$ introduced in Section 4. By using Definition 12, we obtain the following hexagon of opposition: $(A(x), E(x), I(x), O(x), U(x), Y(x))$ with $x \in]1/2, 1]$, where $U(x)$ denotes $A(x) \vee E(x)$ and $Y(x)$ denotes $I(x) \wedge O(x)$ (see Table 2). Figure 8 illustrates the hexagon $(A(x), E(x), I(x), O(x), U(x), Y(x))$ with $x \in]1/2, 1]$.*

We now consider a generalization of the hexagon of opposition $(A(x), E(x), I(x), O(x), U(x), Y(x))$ described in Remark 11 by considering a suitable assessment on n conditional events $P_1|S_1, P_2|S_2, \ldots, P_n|S_n$.

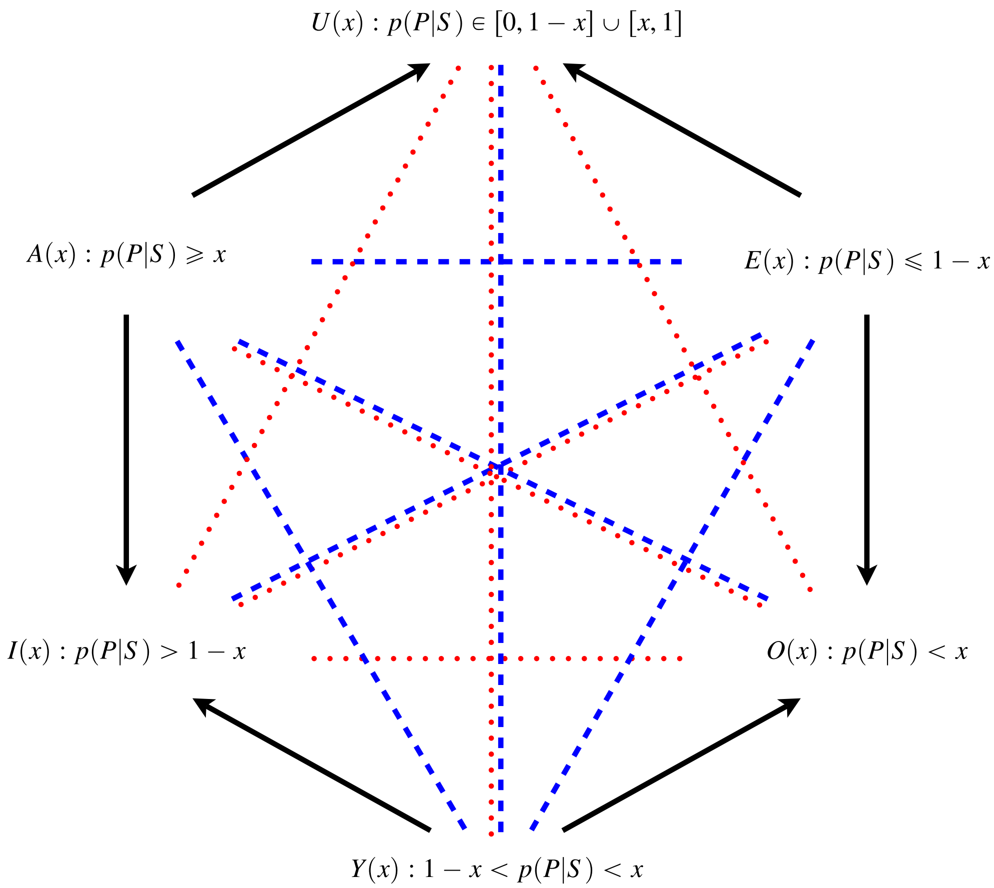

Figure 8: Probabilistic hexagon of opposition $(A(x), E(x), I(x), O(x), U(x), Y(x))$ involving generalized quantifiers defined on the six sentence types with the threshold $x \in]\frac{1}{2}, 1]$ (see Table 1 and Table 2). It provides a new interpretation of the hexagon of opposition, which we compose of the probabilistic square of opposition and the two additional vertices $U(x)$ (i.e., $A(x) \vee E(x)$; top) and $Y(x)$ (i.e., $I(x) \wedge O(x)$; bottom). In the extreme case when $x = 1$, we obtain our probabilistic version of the traditional hexagon of opposition (see also Figure 2).

Sentence		Probability constr.	Assessment on $P\|S$
$U(x)$:	$A(x) \vee E(x)$	$p(P\|S) \geqslant x$ or $p(\bar{P}\|S) \geqslant x$	$\mathcal{I}_{U(x)} = [0, 1-x] \cup [x, 1]$
$Y(x)$:	$I(x) \wedge O(x)$	$\begin{cases} p(P\|S) > 1-x \\ p(\bar{P}\|S) > 1-x \end{cases}$	$\mathcal{I}_{Y(x)} =]1-x, x[$
$U(1)$:	Every S is P or No S is P	$p(P\|S) = 1$ or $p(\bar{P}\|S) = 1$	$\mathcal{I}_U = \{0\} \cup \{1\}$
$Y(1)$:	Some S is P and Some S is $\bar{P}$	$\begin{cases} p(P\|S) > 0 \\ p(\bar{P}\|S) > 0 \end{cases}$	$\mathcal{I}_Y =]0, 1[$

Table 2: Probabilistic interpretation of the sentence types at the top (U) and at the bottom (Y) of the hexagon of opposition involving generalized quantifiers Q defined by a threshold x (with $x \in]\frac{1}{2}, 1]$) on the subject S and predicate P and the respective imprecise probabilistic assessments $\mathcal{I}_{U(x)}$, and $\mathcal{I}_{Y(x)}$ on the conditional event $P|S$ (above). When $x = 1$, we obtain our probabilistic interpretation of the traditional sentence types U, Y.

Remark 12. *Let $\mathcal{F} = (P_1|S_1, \ldots, P_n|S_n)$ be a sequence of n conditional events. Exploiting Remark 10, we construct a hexagon of opposition by considering the following tripartition of $[0,1]^n$: $(\mathcal{B}_1(x), \mathcal{B}_2(x), \mathcal{B}_3(x))$, with $x \in]1/2, 1]$, where*

$$\begin{array}{l} \mathcal{B}_1(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : \sum_{i=1}^n \frac{p_i}{n} \geqslant x\}, \\ \mathcal{B}_2(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : \sum_{i=1}^n \frac{p_i}{n} \leqslant 1-x\}, \\ \mathcal{B}_3(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : 1-x < \sum_{i=1}^n \frac{p_i}{n} < x\}. \end{array}$$

We obtain the following hexagon of opposition $(A(x), E(x), I(x), O(x), U(x), Y(x))$ involving the generalized quantifiers $A(x) : (\mathcal{F}, \mathcal{I}_{A(x)})$, $E(x) : (\mathcal{F}, \mathcal{I}_{E(x)})$, $I(x) : (\mathcal{F}, \mathcal{I}_{I(x)})$, $O(x) : (\mathcal{F}, \mathcal{I}_{O(x)})$, $U(x) : (\mathcal{F}, \mathcal{I}_{U(x)})$, $Y(x) : (\mathcal{F}, \mathcal{I}_{Y(x)})$, where

$$\begin{array}{l} \mathcal{I}_{A(x)} = \mathcal{B}_1(x),\ \mathcal{I}_{E(x)} = \mathcal{B}_2(x),\ \mathcal{I}_{Y(x)} = \mathcal{B}_3(x), \\ \mathcal{I}_{I(x)} = \mathcal{B}_1(x) \cup \mathcal{B}_3(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : \sum_{i=1}^n \frac{p_i}{n} > 1-x\}, \\ \mathcal{I}_{O(x)} = \mathcal{B}_2(x) \cup \mathcal{B}_3(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : \sum_{i=1}^n \frac{p_i}{n} < x\}, \\ \mathcal{I}_{U(x)} = \mathcal{B}_1(x) \cup \mathcal{B}_2(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : \sum_{i=1}^n \frac{p_i}{n} \geqslant x \text{ or } \sum_{i=1}^n \frac{p_i}{n} \leqslant 1-x\}. \end{array}$$

Finally, we generalize the hexagon of opposition involving n conditional events described in Remark 12

Remark 13. *Let $\mathcal{F} = (P_1|S_1, \ldots, P_n|S_n)$ be a sequence of n conditional events. Exploiting Remark 10, for any given vector $(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \geqslant 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^n \lambda_i = 1$, we construct a hexagon of opposition by considering the following tripartition of $[0,1]^n$: $(\mathcal{B}_1(x), \mathcal{B}_2(x), \mathcal{B}_3(x))$, with $x \in]1/2, 1]$, where*

$$\begin{array}{l} \mathcal{B}_1(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : \sum_{i=1}^n \lambda_i p_i \geqslant x\}, \\ \mathcal{B}_2(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : \sum_{i=1}^n \lambda_i p_i \leqslant 1-x\}, \\ \mathcal{B}_3(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : 1-x < \sum_{i=1}^n \lambda_i p_i < x\}. \end{array} \quad (1)$$

We obtain the following hexagon of opposition $(A(x), E(x), I(x), O(x), U(x), Y(x))$ involving the generalized quantifiers $A(x) : (\mathcal{F}, \mathcal{I}_{A(x)})$, $E(x) : (\mathcal{F}, \mathcal{I}_{E(x)})$, $I(x) : (\mathcal{F}, \mathcal{I}_{I(x)})$, $O(x) : (\mathcal{F}, \mathcal{I}_{O(x)})$, $U(x) : (\mathcal{F}, \mathcal{I}_{U(x)})$, $Y(x) : (\mathcal{F}, \mathcal{I}_{Y(x)})$, where

$$\begin{array}{l} \mathcal{I}_{A(x)} = \mathcal{B}_1(x),\ \mathcal{I}_{E(x)} = \mathcal{B}_2(x),\ \mathcal{I}_{Y(x)} = \mathcal{B}_3(x), \\ \mathcal{I}_{I(x)} = \mathcal{B}_1(x) \cup \mathcal{B}_3(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : \sum_{i=1}^n \lambda_i p_i > 1-x\}, \\ \mathcal{I}_{O(x)} = \mathcal{B}_2(x) \cup \mathcal{B}_3(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : \sum_{i=1}^n \lambda_i p_i < x\}, \\ \mathcal{I}_{U(x)} = \mathcal{B}_1(x) \cup \mathcal{B}_2(x) = \{(p_1, \ldots, p_n) \in [0,1]^n : \sum_{i=1}^n \lambda_i p_i \geqslant x \text{ or } \sum_{i=1}^n \lambda_i p_i \leqslant 1-x\}. \end{array}$$

Of course, we recover the hexagon of opposition described in Remark 12 by setting $\lambda_i = \frac{1}{n}$, $i = 1, \ldots, n$. If $n = 1$, we recover the hexagon of opposition described in Remark 11.

Example
Let $P|S$ be a conditional event. Moreover, let $S_1, \ldots, S_n$ be a partition of S, that is $S_1 \vee \cdots \vee S_n = S$ and $S_i \wedge S_j = \bot$, $i \neq j$. Coherence requires that

$$p(P|S) = p(P|S_1)p(S_1|S) + \ldots + p(P|S_n)p(S_n|S)$$

and that $\sum_{i=1}^{n} p(S_i|S) = 1$. Assume $p(S_i|S) = \lambda_i \geqslant 0$, $i = 1, \ldots, n$, with $\sum_{i=1}^{n} \lambda_i = 1$. Then, for any coherent assessment $(p_1, \ldots, p_n)$ on $(P|S_1, \ldots, P|S_n)$ it holds that $p(P|S) = \sum_{i=1}^{n} \lambda_i p_i$. Let $x \in]\frac{1}{2}, 1]$. We observe that, the susbsets $\mathcal{B}_1(x), \mathcal{B}_2(x)$, and $\mathcal{B}_3(x)$ of $[0, 1]^n$, which are defined as in (1), represents the sets of probability assessments $(p_1, \ldots, p_n)$ on $(P|S_1, \ldots, P|S_n)$ such that $p(P|S) \geqslant x$, $p(P|S) \leqslant 1 - x$, and $1 - x < p(P|S) < x$, respectively.

6. Concluding Remarks

Finally, we note that conditional probability interpretations of quantified statements were also proposed in psychology (see, e.g., [15, 16, 48, 51, 53, 56, 59]), since generalized quantifiers are psychologically much more plausible compared to the traditional logical quantifiers, as the latter are either too strict ($\forall$ does not allow for exceptions) or too weak ($\exists$ quantifies over at least one object) for formalizing everyday life sentences. Recent experimental data suggests that people negate conditionals and quantified statements mainly by building contraries (in the sense of inferring $p(\neg C|A) = 1 - x$ from the negated $p(C|A) = x$) but hardly ever by building contradictories (in the sense of inferring $p(C|A) < x$ from the negated $p(C|A) = x$; see [52, 59, 60]). However, this empirical result calls for further experiments. The square presented in Section 4 and the hexagon presented in Section 5 can serve as a new rationality framework for formal-normative and psychological investigations of basic relations among quantified statements.

Finally, we note that the generalized quantifiers can be used to study various forms of inferences. For instance, recall the probability propagation rules for the CUT rule of System P [30, p. 23]: from $x' \leqslant p(M|S) \leqslant x''$ and $y' \leqslant p(P|S \wedge M) \leqslant y''$ infer $x'y' \leqslant p(P|S) \leqslant x''y' + 1 - x''$. The CUT rule in terms of basic sentence types holds: From *every* S *is* M (interpreted by $p(M|S) = 1$), and *every* $S \wedge M$ *is* P (i.e., $p(P|S \wedge M) = 1$), infer *every* S *is* P (i.e. $p(P|S) = 1$). If the premises are interpreted by the generalized quantifiers *almost all* (by using the threshold $.9 = x' = y'$), the conclusion holds with the generalized quantifier *most*: In particular, the premises *almost all* S *are* M (interpreted by $.9 \leqslant p(M|S) \leqslant 1$) and *almost all* $S \wedge M$ *are* P (i.e., $.9 \leqslant p(P|S \wedge M) \leqslant 1$) imply the conclusion *most* S *are* P (i.e., $.81 \leqslant p(P|S) \leqslant 1$). We will devote future work on studying probabilistic versions of categorical syllogisms involving generalized quantifiers.

Acknowledgement

We are grateful to Angelo Gilio and to two anonymous referees for helpful comments on a previous version of this paper. We thank *Deutsche Forschungsgemeinschaft* (DFG), *Fondation Maison des Sciences de l'Homme* (FMSH), and *Villa Vigoni* for supporting joint meetings at Villa Vigoni where parts of this work originated (Project: "Human Rationality: Probabilistic Points of View"). Niki Pfeifer is supported by his DFG project PF 740/2-2 (within the SPP1516 "New Frameworks of Rationality"). Giuseppe Sanfilippo is supported by the INdAM–GNAMPA Project (2016 Grant U 2016/000391).

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