Cultural transposition: Italian didactic experiences inspired by Chinese and Russian perspectives on whole number arithmetic

Maria Mellone[1] · **Alessandro Ramploud**[2] · **Benedetto Di Paola**[3] · **Francesca Martignone**[4]

Abstract

The paper presents some reflections and activities developed by researchers and teachers involved in teacher education programs on cultural transposition. The construct of cultural transposition is presented as a condition for decentralizing the didactic practice of a specific cultural context through contact with other didactic practices of different cultural contexts. We discuss the background theoretical issues of this approach and also give an analysis of two examples of cultural transposition experiences carried out in Italy. In particular, by means of qualitative analysis of some excerpts, discussions, and interviews, we show that the contact with different perspectives coming from China and Russia fostered educational practices and reflections on whole number arithmetic education.

Keywords Cultural transposition · Whole number arithmetic · Early algebra

1 Introduction

Travelling, meeting, exploring, knowing—these are all terms whose meanings have changed in recent years, enriching new modes, times, and processes. In mathematics education, this enrichment translates into the possibility of coming into contact with educational practices in far-away countries more easily.

In this scenario, we propose the *cultural transposition* construct (Mellone and Ramploud 2015; Sun et al. 2015) as a condition for decentralizing the didactic practice of a specific cultural context through contact with the didactic practices of different cultural contexts. In fact, the contact with distant and hard to conceive didactic practices could represent an opportunity to consider what the French philosopher Jullien defines as *impensé* (Jullien 1993). With this term, '*unthought*' in English, Jullien refers to all the implicit assumptions in which a cultural paradigm is rooted. People remain unaware of these implicit assumptions while they stay in the same cultural paradigm; however, while moving into a different cultural paradigm, they can become aware of them.

Certainly, the experience of observing and considering the meanings embedded in the educational practices in other cultural contexts could represent the possibility of rethinking those rooted in our own educational practice. The Cultural Transposition construct is inspired by Skovsmose's (1994) theoretical background in which mathematics is seen as an invisible structure that plays an important role in related societies. Mathematics, in particular, can represent a powerful means of emancipation for learners. In our perspective, the contact with different mathematics school practices can represent an experience of emancipation for teachers.

We would like to address the following questions: Could the cultural transposition process allow teachers to become aware of the implicit, of the backgrounds, of the unthought that every educational choice entails? In the specific case of whole number arithmetic (WNA), could the contact with Chinese and Russian educational practices encourage Italian teachers' reflections about the features of specific proposals that focus on a structural approach to arithmetic?

✉ Maria Mellone
maria.mellone@unina.it

[1] Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli Federico II, Via Cintia, Monte S. Angelo, 80126 Naples, Italy

[2] Istituto Comprensivo J.F. Kennedy, Via Kennedy, 20, 42124 Reggio Emilia, Italy

[3] Dipartimento di Matematica e Informatica, Università di Palermo, Via Archirafi, 34, 90123 Palermo, Italy

[4] Dipartimento di Scienze e Innovazione Tecnologia, Università del Piemonte Orientale, Viale Teresa Michel, 11, 15121 Alessandria, Italy

This article is post-print version of https://doi.org/10.1007/s11858-018-0992-7

In our previous work, through qualitative and/or quantita-tive approaches (Bartolini Bussi et al. 2017; Di Paola 2016; Mellone and Ramploud 2015; Bartolini Bussi et al. 2014; Mellone et al. 2017; Spagnolo and Di Paola 2010), we point out how mathematics education researchers coming into contact with educational practices adopted in other cultural contexts are able to deconstruct them in order to reconsider the themes of educational intentionality, which is the back-ground of any educational practice. In view of the cultural transposition construct, researchers are able to elaborate new interpretative keys of the educational practice of their own cultural context. The results of these reflections have been employed in the design of some professional development (PD) paths for teachers realized in Italy. These educational programmes are obviously set in the Italian cultural context (Bartolini Bussi and Martignone 2013), although the con-tact with educational practices of other cultural contexts can stimulate reflections and even induce changes in teachers' educational practices.

In this paper, we examine the kinds of reflections and activities developed by teachers involved in PD courses cen-tered on cultural transposition. Therefore, we discuss the theoretical issues in the background of this approach and give also an analysis of those cultural transposition expe-riences reported by two teachers who have been involved in the PD course given by the authors of this paper. These reports deal with WNA in primary schools, a topic which was widely explored by researchers and is discussed as a central issue in international conferences, such as the ICMI Study 23 Conference (Sun et al. 2015;Bartolini Bussi and Sun 2018), and in surveys like the ICME13 Topical Survey (Nuñes et al. 2016).

The first example of educational activity presented here focuses on the "problems with variation", which is consid-ered as one of the most significant mathematics education tools in Chinese primary schools (Ramploud and Di Paola 2013; Di Paola et al. 2015; Bartolini Bussi et al. 2014; Sun 2011). The second example relates to the visionary mathe-matics curriculum for pupils attending the first grade, which is proposed by Davydov (1982). This curriculum is adopted by a few schools in Russia, as stated by Ivashova (2011): "Two other programs in elementary mathematical education appeared at the same time, created by L. V. Zankov and V. V. Davydov. They were used only in an experimental setting" (p. 58). Therefore Davydov's mathematics curriculum for primary school, unlike the problems with variation in China, cannot be considered as a representative methodology for the Russian WNA approach. Nevertheless, the study of this particular approach together with the related deconstruction process was an equally meaningful and insightful experience for us as researchers.

The analysis of these examples and the interviews with the teachers reveals the teachers'practices and their considerations about some important aspects of WNA. In detail, the contact with different approaches coming from other cultural contexts has been effective to foster discus-sion and reflection on WNA problems, to raise new aware-ness, and to develop new methods of teaching and learning addition and subtraction, according to unified approaches (Bartolini Bussi and Sun 2018).

2 Theoretical framework

The trend of studies set by the pioneering works of Bishop (1988) has highlighted the importance of recognizing math-ematics practices as social phenomena which are embedded in those cultures and those societies that generated them. Bishop's studies have brought to the attention of the sci-entific community of mathematics education some crucial issues, such as recognizing similarities and analogies in mathematics basic skills (in particular concerning numeracy and geometrical relationships) in different cultural contexts. From that time onwards, the awareness that it is crucial to take into account cultural and historical contextualization when studying mathematics practices, has been shared by the community of mathematics education researchers. Moreover, the ethnomathematics trend of studies (e.g., D'Ambrosio 2006) has proved that care and sensitivity about cultural and social issues, when inquiring into mathematical practices, contribute to the understanding of cultures and the understanding of mathematics itself.

Nowadays, there are several research approaches (e.g., Barton 2008; Gutierrez 2013) that, by using different foci, work from the crucial assumption that culture permeates all aspects of both mathematical practices and mathemat-ics education practices. Nevertheless, what is rather new is awareness of the effect of cultural diversity and the ways it may be beneficial in research on mathematics teacher educa-tion (e.g., Bartolini Bussi and Martignone 2013).

As previously stated, we propose the paradigm of Cul-tural Transposition with the aim of using the differences among mathematical education practices adopted in different cultures and societies to design professional development that aims to develop teachers' awareness and, eventually, change their mathematical education practices.

To clarify our perspective,, we first analyze the meaning of each individual term in the expression cultural transposi-tion; second, we point out the main features of this construct in its systemic complexity.

Strategically starting from the adjective *cultural*, in the Oxford Dictionary edited by A. S. Hornby, this entry has as a definition: "having to do with culture"; hence, the adjective cultural harkens back to the word culture. Surely, the redefi-nition of this conceptual structure goes beyond our goals, although the use of the term cultural needs the explanation of the theoretical perspective in which we place the construct of cultural transposition. The purpose of this condition is to reflect on and to rethink mathematics educational practices. In particular, it fosters reflection about the didactic practices of a specific cultural context through the contact with didactic practices of other cultural contexts. Our reflections are stimulated by the thought of Jurij Michajlovič Lotman, who defines culture as a complex semantic system made by different interlaced linguistic signs (Lotman and Uspenkij 1975). Therefore, transposition in a cultural perspective is strictly related to the signs and those linguistic systems in which it develops.

Focusing especially on the semiotic dimension, we want also to take into account those habits, those imaginations and that philosophy which feed and shape every culture (D'Ambrosio 2006).

By means of the *deconstruction* movement introduced by Jaques Derrida (1967), we consider the semiotic perspective in which every culture is fed; this movement entails an analysis of the different levels on which a culture becomes stratified. During an interview for an Italian newspaper, Derrida stated the following:[1]

> The word [deconstruction] comes from a Heidegger expression, 'Destruktion', meaning 'de-destruct' and not as 'destruction'. I use it in the sense of an analysis of the different layers in which it stratifies culture.

Therefore, Derrida intends deconstruction (destructuration) as a methodology, or rather a critical exercise on cultural stratifications. Throughout his philosophical activity, Derrida developed the idea of deconstruction as a process that arises as an attitude that serves to continually deconstruct a culture, that is, to put in place a radical critique. In this sense, we would like to accomplish a didactic deconstructionism in our research, through a reflection that handles the differences among the didactics of mathematics in different cultures. According to this view, which differs from Bishop's approach engaged in the search of equivalence among cultures, we are more concerned with the investigation of differences among mathematics education practices.

Furthermore, to clarify our use of the concept of *transposition*, the etymology of the word is a starting point; the word transposition comes from the Latin *transponere*, in which both the prefix *trans-* and the verb *ponere* are easily recognizable. The prefix indicates a passage, a transition, a change from one condition to another, while the verb means to place, to put. Therefore, the noun transposition is composed of two elements, the first identifying a passage, a transition, a change, while the latter provides a more static image suggested by the verb to place.

Accepting these premises, we use the term transposition exactly to describe something placed after a transition from some initial conditions. In the field of education, we define the paradigm of Cultural Transposition as a process to decentralize the educational practice of one's own cultural context through the contact with educational practices of other cultural contexts. Certainly, the experience of observing and considering the meanings embedded in the educational practices of other cultural contexts could represent the possibility of rethinking those rooted in our own educational practices. From this perspective, we believe that the process of cultural transposition works exactly to encourage a critical vision of mathematics education (Ernest et al. 2016). No culture could claim dominance or precedence for the creation of mathematics conceived as a pan-cultural activity characterized by playing, designing, locating, explaining, counting and measuring (Bishop 1988). In the same way, no mathematics education practice should claim dominance in terms of effectiveness or success and no international assessments should be read accordingly.

We place our thought about mathematics activities in the trend of studies that originate from Skovsmose (1994). In his opinion, mathematics could be regarded as an invisible structure that has a role in the process by which societies are shaped and evolve; further, mathematics is a tool and a means of emancipation. In other words, his research approach underlies a particular political perspective on the role played by mathematics in society. In his approach to different cultural contexts, for example, there is no attempt to understand how a particular mathematical tool is used for particular jobs in that social environment, but rather there is an attempt to understand in which ways it is possible to allow the children of that society to learn mathematical tools in order to potentially work in jobs different from the jobs of their parents. We embrace this perspective, in which learners are not seen as passive recipients for institutionalized knowledge, rather they are seen as actively part of an educational process in which they are those who question, challenge and even shape the nature of their own learning experience. Similarly, teachers should shift from being passive receptacles of institutionalized knowledge into determining the nature of the mathematics teaching experience which they offer. We wish to demonstrate to teachers that coming into contact with educational practices unlike their own could be a crucial experience to increase their own awareness when defining the nature of their mathematical education practices.

In this context, we propose the idea of cultural transposition as a process activated by researchers, educators, and teachers who deconstruct those educational practices adopted in other cultural contexts in order to reconsider the issues of educational intentionality, which is the background

[1] In La Repubblica, 2002: http://ricerca.repubblica.it/repubblica/archivio/repubblica/2002/07/03/calma-scienzia-io-vi-rispetto.html.

of any educational practice. In fact, the role of researchers and educators is crucial, since they can introduce new interpretative keys for the same educational practice related to their own cultural context through the deconstruction of those several levels in which an educational practice is stratified. The different cultural backgrounds generate different possibilities of meaning and different mathematics education perspectives that, in their turn, organize the contexts and the school mathematics practices in different ways (Mellone and Ramploud 2015). On these premises and in line with recent trends in mathematics teacher education research (e.g., Wood 2008), researchers design and realize professional development (PD) paths for teachers to induce a reconsideration of their usual educational practices. These PDs are not built as a comparative study of mathematics education; rather, they are conceived as a dialogue between different educational practices, during which practitioners can become more aware of each teaching choice, by knowing different one (Jullien 2005).

Moreover, our research adopts the perspective of Jullien (2005, 2006), who turned his gaze towards differences and beyond the borders of Western thought using deconstruction from the outside as a methodological tool. He traced a path which anyone could follow, thus successfully introducing one's own unthought theme; this meeting is between different philosophical perspectives and thoughts, but above all, it deals with the difference or the gap between cultures. The sense of this methodological process can be understood from the following emblematic quote:

> This is not about comparative philosophy, about paralleling different conceptions, but about a philosophical dialogue in which every thought, when coming towards the other, questions itself about its own *unthought*.[2] (Jullien 2006, p. 8, our translation)

Analyzing the passage by Jullien carefully, we note that it has a dual and complementary structure that is active and passive at the same time. In its active dimension, this implies the availability and the wish, as well as the opportunity, to meet the other going in his direction, with all the implications that this entails. In this activity, however, is already involved a passive dimension that entails accepting the difference; such a perspective implies the ability to play strings not yet, or no longer, touched.

Following this approach, the implemented PD courses aim at fostering teachers' observations concerning mathematics education with a consequent innovation of the related practices; further, this could mean doing the same things they did before, although with deeper consciousness.

3 Method

3.1 The development of the process of deconstruction

In the previous section, we described and contextualized, among the different aspects involved in the cultural transposition process, specific work developed by researchers and educators who operate from the gap between cultures, especially between different didactic practices implemented in different cultures.

First of all, the researchers come into contact with educational practices of other cultural contexts; this happens in many different ways, such as by reading textbooks used at school, through research papers or interviews with special witnesses such as other researchers, school deans, teachers, students, parents, and by watching classroom video scenes. After this first contact, the researchers start a process of deconstruction, which is meant as a radical critique of linguistic, philosophical, and political aspects as well as of value systems. Afterwards, the researchers realize PD practices that could be developed in different ways, as in seminars, research groups, educational design labs. In this sense, the cultural transposition potentially takes place twice (with some differences)—first on the part of the PD designers (researchers), and then again on the part of the participating teachers.

Indeed, the researchers start a process of deconstruction and reflection on the cultural differences underlying the mathematical educational practices through the study, for example, of linguistic factors, geographical factors, value system, philosophical approaches, and history of mathematics. In the light of these reflections, the researchers implement PDs that create possibilities of contamination. The teacher trained in the paradigm of cultural transposition promotes didactic innovation (or it can also result in making no changes but being more aware of one's educational intentionality).

In the study we are presenting in this paper, the research team was formed by several Italian researchers, including the four authors of this paper; each of them had different expertise and knowledge of math education practices in Chinese primary schools and in Russian primary schools inspired by the Davydov curriculum (Davydov 1982).

Specifically, the design of the PD activities was created by devoting special attention to the shift from arithmetic towards a structural vision of WNA. This shift becomes a reflection of contexts linked to different didactic methodological practices in the world, especially in the Eastern

[2] Il ne s'agit pas là de philosophie comparée, par mise en parallèle des conception; mais d'un dialogue philosophique, où chaque pensée, à la rencontre de l'autre, s'interroge sur son impensé.

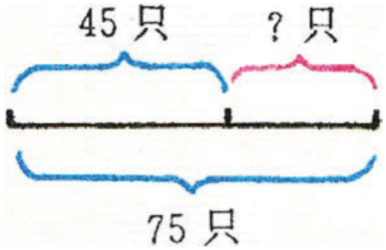


Fig. 1 In the pond, we have white and black ducks. There are 75 ducks in total, 45 of them are white. How many black ones are there?

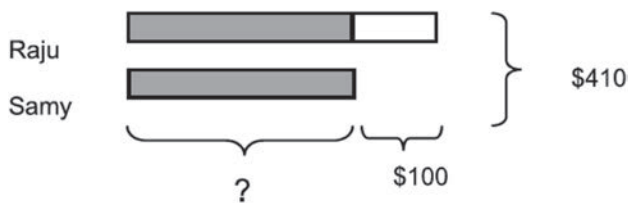


Fig. 2 Raju and Samy had $ 410 between them. Raju received $ 100 more than Samy. How much money did Samy receive?

Countries. In general, we can say that in the Italian curriculum, the study of Algebra starts from grade 7 as the study of formal algebraic language, consisting of activities of transforming algebraic expressions or solving equations (Di Paola et al. 2016). In this direction in Italian primary school no care is usually given to early algebra, except for some activities regarding patterns or regularities. One of the features that appear to emerge from the mathematics curriculum of primary schools in China and Singapore but also in Russia, within the schools that follow Davydov's curriculum, is a distinct focus on early algebra, which is essentially rooted in a different perspective on the didactics of Whole Numbers Arithmetic (WNA) (Cai and Knuth 2011). These mathematics curricula state explicitly that their main goal in teaching arithmetic is to develop students' understanding of quantitative relationships and their use of mathematical signs to represent them. In order to achieve this goal, both curricula encourage an extensive use of graphic models, known as pictorial equations (Fig. 1) or bar models (Fig. 2), to support students in understanding and examining quantitative relationships in depth (Cai and Knuth 2011). These curricula support students in the development of a particular habit and thinking practice that is to examine quantitative relationships from different perspectives and to represent quantitative relationships in different ways.

In our studies, we focus on the methodology known as "problems with variation" (Fig. 3), which is considered one of the most significant mathematics education tools used in Chinese primary school (Ramploud and Di Paola 2013; Bartolini Bussi et al. 2014). Typically, problems with variation

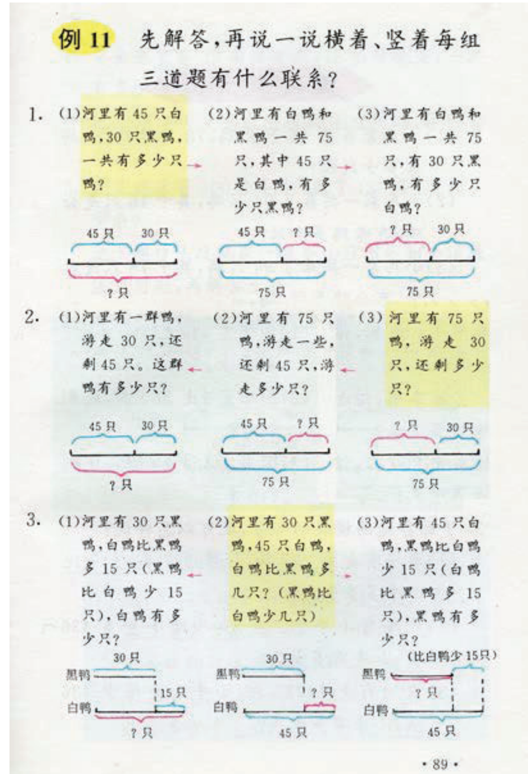


Fig. 3 Problem with variation (Beijing 1996, p. 88)

are word problems concerning the same reality contexts, a lake with 45 white ducks and 30 black ducks in this case, and with the same arithmetical structure presented on the same page one after the other. The assignment for the pupils is not only to solve the problems, but to recognize what these have in common. Below each problem statement, there is a pictorial equation that shows the common arithmetical structure underlying the different problems. In this kind of problem, what is essential is the shifting from a purely arithmetical to a more relational field, oriented towards "informal algebra" (Cai and Knuth 2011).

Analysing these tasks in terms of cultural transposition, the first point to highlight is that the Chinese writing is different from our Italian alphabetic language and from all the alphabetic Indo-European languages in general, which have a strong tendency towards abstraction (Jullien 1993; Spagnolo and Di Paola 2010). Conversely, the Chinese writing is built on an immanent background in which the descriptive features of the ideograms maintain strong links with material features and shapes of their semantic referents (Mellone and Ramploud 2015). In other words, Chinese writing is more oriented towards *rubrication* (Jullien 1993) rather than *categorization* and this feature is widely acknowledged in sinology studies (see for example Jullien 1993). The term rubrication refers to the particular way of working of the

Chinese language, where there is no use of terms to indicate general categories, rather the terms refer to the possible cases of a certain category. So, in Chinese language, generality is understood in a different way compared to how this occurs using an alphabetic language. In this sense, our language as an expression of our cultural approach, which is recognizable also in our educational perspective, appears to emphasize the transition from the concrete to the abstract field. In contrast, the Chinese language and its related cultural approach appear to express abstract and concrete levels simultaneously, and this can be also observed in the problems with variations. Indeed in the problems with variation one of the elements that pushes the child from the paradigm of arithmetic to that of informal algebra is the element of simultaneity (Ling Lo 2012). The presence of the nine problems together at the same time pushes the child towards the pursuit of relationships (Ramploud and Di Paola 2013), and of structure (Leung et al. 2013).

Furthermore, a similar perspective is provided by the approach to WNA proposed by the Russian psychologist Davydov (1982), in which the relationships between quantities emerge from a different approach to number. At this point it is worth noticing the direct influence of USSR on the Chinese reforms in education curricula. The work of the Russian psychologist Kairov is very influential in China (Shao et al. 2012), and Kairov and Davydov are often associated with their approaches.

According to such a theory, the genesis of the number concept is rooted in the experience of measuring continuous quantities (Davydov 1982). The notion of quantity comes from a comparison among elements of a given class, as lengths of segments, amounts of water, weights, etc., thus to measure means relating a given quantity to a part of it that has been assumed as a unit. In this perspective, counting itself may be conceived as the particular measuring process of discrete objects, whence the sequence of natural numbers appears as just an example of quantity. A core aspect of Davydov's approach is to start the instruction of whole numbers from experiences of continuous quantities (and not of discrete quantities, which is a more common approach also in Italy); this choice has the advantage of supporting a smooth transition to the domain of rational and real numbers. Indeed, in his perspective, a deeper acquaintance with quantities allows children to widen their knowledge of numbers to include, in turn, integers, rational, and real numbers. Davydov suggests that from an early age children should manipulate the properties of quantities and use algebraic language to treat them, well before practice with natural numbers. He presents activities where an order relationship between two physical quantities is first recognized and expressed, then attention is focused on the quantity to be added to the smaller quantity to obtain the larger one. In order to support children's progress towards a more abstract level, he suggests transforming qualitative aspects into physically segmental quantities, without focusing too close attention on their lengths:

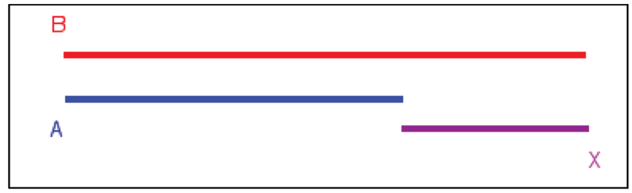


Fig. 4 The graphic representation of the segments—the *pictorial equation*

> To prepare children for this shift, an intermediate strategy of graphic representation is used. The children represent the physical quantities with two segments A and B. How the difference in measures of the quantities can be determined is then discussed. Line segment A is superimposed on the line segment B. The difference expressed in the form B–A is defined equal to x. (Davydov 1982, p. 234)

According to this perspective, the graphic representation of the segments (Fig. 4) guides children to recognize the relationship between the two quantities A and B and to use subtraction as a formal description of the process of comparison between A and B, rather than a decrease. It is noteworthy that the representation is seen as a bridge between the physical and the conceptual world. As Davydov (1982) wrote: "In a school subject, intermediate means of description have crucial significance because they mediate between a property of an object and a concept" (p. 237). In this sense, children are invited to use the graphic representation of the pictorial equation in Fig. 4 as an intermediate step between the concrete experience about physical quantities and the more formal representations of algebraic symbols.

Different from the analysis of the Chinese problems with variation, the analysis of Davydov's proposal in terms of cultural transposition is not to be focused on linguistic issues, since the Russian language, as Italian, is an Indo-European language. Therefore, our deconstruction process of Davydov's (1982) approach to WNA is focused on the philosophical-cultural context where this is grounded and developed. Davydov's approach is rooted in the sociocultural reflections of Vygotskij (1962) and, consequently, in the Marxism and dialectic materialism philosophical trend. The Marxist and Leninist perspective intends science as an active and actual work aimed at the transformation of society. As a result, proving a scientific theory is not limited to empirical research but requires immediate validation from the field of social relations, work, and school. In this sense, Davydov's mathematics curriculum could be regarded as an education proposal aimed directly at the education of a social class called on for the reconstruction of the society. This urgency probably leads to consideration of the ultimate intention of any action, even of educative actions: "the ultimate aim of instruction in mathematics should be clear from the very beginning" (Davydov 1982, p. 230). Also, he recognized the awareness of algebraic relations as the core of school mathematics.

3.2 The steps of the study

The first and essential step of this research work was the prior engagement of schools and teachers with the help of Regional Education Offices and school headmasters. We were able to involve in the PDs about 80 primary teachers in Reggio Emilia and about 30 teachers in Naples. The teachers were enrolled on a voluntary basis. There is no evaluation at the end of the PDs, but the teachers know that they will have a certification of attendance at the end of the PD.

Holding this as a premise, we summarise the methodological structure of the research; the key steps followed in our research are as follows:

- Deconstruction process developed by PDs designers
- Entry questionnaire on teachers' beliefs
- PDs focus on the Cultural Transposition paradigm
- Focus group
- Teaching examples
- Interviews with teachers.

Before the PD courses, a questionnaire was prepared and distributed to the teachers (for more details see Ramploud 2015); this instrument gave us a picture of teachers' ideas, especially those about mathematics, as a key to cultural encounter. Through the analysis of their answers to this entry questionnaire, we attempted to read their ideas and to design the PDs by matching their needs with our research interests. The main elements of the PDs were as follows:

1. outlining a shared vision of basic mathematical goals in early school grades from a multicultural perspective;
2. reflecting on the different school systems, in Italy, China, and Russia, as an opportunity to reflect on, and to examine in depth, each cultural system and each set of values as a frame of reference;
3. analyzing some tasks belonging to other cultures, such as problems with variation and activities of explorations of the relationship between quantities and use of algebraic language to register them, as an opportunity to reflect on mathematical activities of other countries;
4. reflections on the different mathematics education approaches, such as variation and the use of particular representations (e.g., the pictorial equation), to try to rethink our way of doing school in possibly new, unthought of ways.

The authors of this paper were involved in the PD courses both as plenary speakers and as working group leaders during the meetings, which lasted 3 days; the researchers, the educators, and the teachers remained in contact during their experimentations. After the PDs, a series of focus groups were carried out by one of the authors. The focus groups consisted of 7–9 teachers; some of whom were teachers and educators who had taken part in the PDs (volunteer–chosen), while the other part, named the control group, consisted of 7–9 people who did not participate in the PDs (Ramploud 2015).

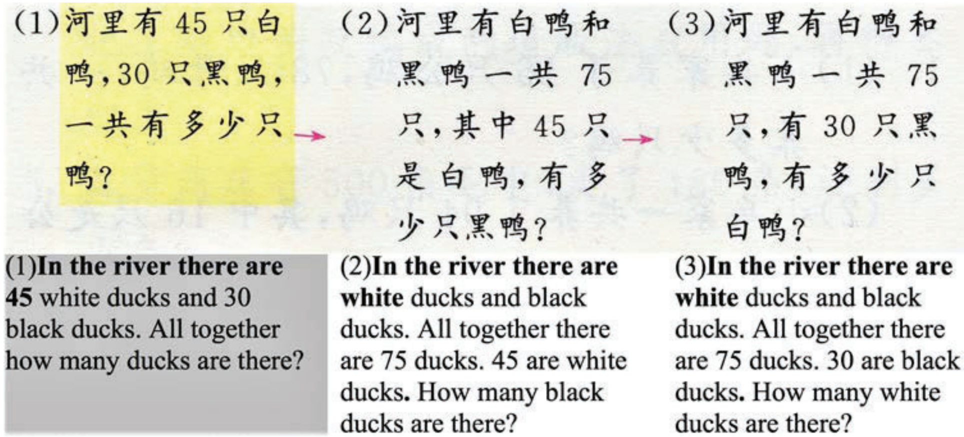
After the PDs, the teachers were invited to design and develop teaching experiments following the suggestions given within the PDs, and to keep the researchers informed about them. Many of the involved teachers designed interventions in schools and so educational projects were implemented in some primary school classes. In this paper, we analyze two teaching experiments that are representative, from the cultural transposition perspective, carried out by two of the teachers involved in the PDs, one in the second grade of primary school and one in the fifth grade of primary school. These experiments are presented as examples to testify the development of mathematics teaching that approaches WNA in a structural way and that is placed within the methodological construct of cultural transposition. Moreover, we present some excerpts from the interviews with these two teachers after their experiments.

4 Analysis of two examples of teaching experiments

In this section, we present two teaching experiments as paradigmatic examples of the cultural transposition process and some final considerations by the teachers involved. The analysis of their experiences shows how they worked in their classes after having attended the PD courses and, therefore, how their practices about WNA developed subsequently. The teachers autonomously designed the teaching experiments (with brief exchange of opinions between teachers and researchers when requested by the teachers). The researchers took part in the teaching experiments mostly as observers, taking notes and recording the class discussions.

4.1 The first teaching example

Loretta, one of the teachers involved in the PD of Reggio Emilia, decided to take inspiration from the Chinese education practice of the problem with variation to perform her experimentation. She was influenced by the observations


(1)河里有45只白鸭，30只黑鸭，一共有多少只鸭？

(2)河里有白鸭和黑鸭一共75只，其中45只是白鸭，有多少只黑鸭？

(3)河里有白鸭和黑鸭一共75只，有30只黑鸭，有多少只白鸭？

(1)**In the river there are 45** white ducks and 30 black ducks. All together how many ducks are there?

(2)**In the river there are white** ducks and black ducks. All together there are 75 ducks. 45 are white ducks. How many black ducks are there?

(3)**In the river there are white** ducks and black ducks. All together there are 75 ducks. 30 are black ducks. How many white ducks are there?

Fig. 5 First three variation problems, in Chinese and English translation

about this methodology and about the Chinese language made during the PD (cf. Sect. 3.1). Loretta reversed the problem with variation starting from Chinese writing and moving to possible representations of the arithmetical structure of these word problems as she worked with her second-grade pupils.

To begin with, she handed out the original Chinese text with the three problems without the Italian translation, corresponding to those in the first row of Fig. 1, but without the pictorial equations as in Fig. 5.

Next, she guided her pupils to the exploration of some syntactic and semantic aspects of the Chinese texts, such as, for example, the identification of parts of the texts, of the presence of numeral symbols and of the question marks. Afterwards, she provided the Italian translation of the texts on worksheets and asked her pupils to create a representation of these word problems. Therefore, the aim of the teacher was to achieve a common representation, to be chosen among those created by the pupils, a representation that could be used to describe all three problems. In order to fulfill this function, it had to highlight the common structure of the three problems, which is one of the key aspects of Chinese variation problems. This representation did not necessarily have to be the pictorial equation, and for this reason, it was removed from the texts.

Some examples of pupils' drawings are presented in Figs. 6 and 7. It is possible to identify two different ways of representing the problems: Fig. 6 shows a more descriptive and perceptual representation consisting in black and white ducks, the protagonists of the problems' story, whereas Fig. 7 shows a more conceptual and abstract representation using black and white balls. After a class debate orchestrated by the teacher (Bartolini Bussi 1996) about the differences of effectiveness and of usefulness in the representations created by the pupils, the whole class welcomed the idea of using a single representation for the additive situation of the three problems.

On a page from a pupil's notebook (Fig. 8), it is possible to observe the representation chosen by the class as the most effective to illustrate the three problems. Notice the selected representation is not the pictorial equation; in fact, it resembles an array in which the arrangement by rows of ten recalls the decimal number system representation. This choice is consistent with the work carried out by the pupils and also with the teacher's purpose, which was working on and discussing the possible representations presented by the pupils, in order to represent the problems with the same arithmetical structure. Even though the representation selected by the pupils does not correspond to the Chinese pictorial equation, it highlights the common structure of this kind of problems all the same.

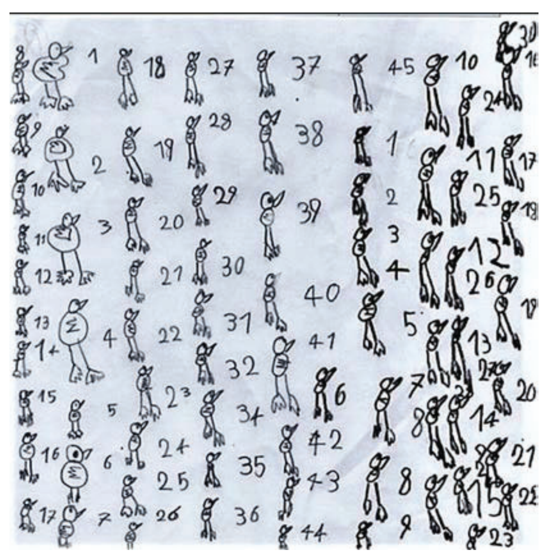


Fig. 6 Example of pupils' drawings

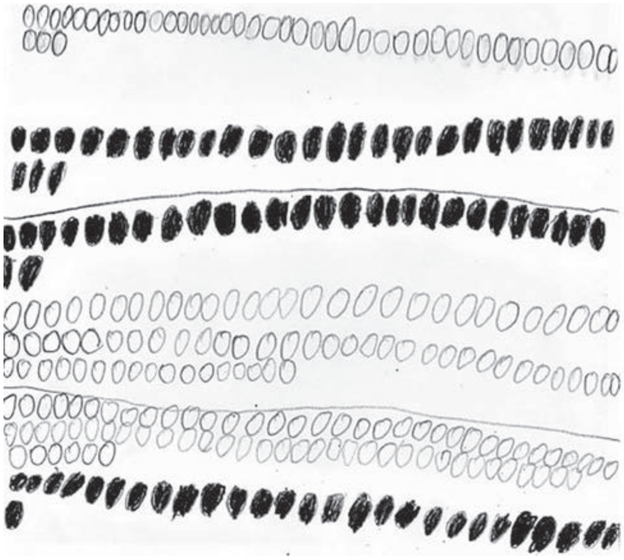


Fig. 7 Example of pupils' drawings

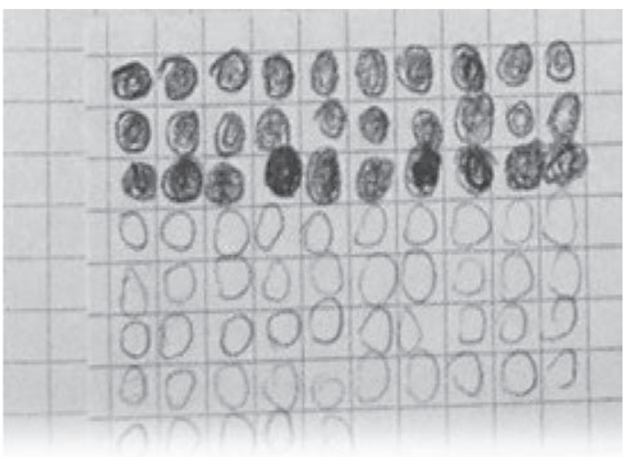


Fig. 8 Example of pupils' drawings

4.2 The second teaching example

Colomba, one of the teachers involved in the PD in Naples, decided to take inspiration from the curriculum of the Russian Psychologist Davydov to perform her experiment. During the PDs, different observations were made together with the presentation of some cultural and, particularly, philosophical aspects of Davydov's approach (cf. Sect. 3.1).

Colomba promoted the idea that working with physical quantities can support the development of algebraic skills before and even after the exploration with numbers and arithmetical algorithms. She thought that Davydov's theoretical reflections and related educational implications could be intertwined with a more traditional primary path,

Fig. 9 The containers experiment

as far as Italy is concerned, in which numbers are used from the beginning. So, she followed an experimental path in her fifth-grade class, which until that time had followed a traditional pattern. She engaged her pupils in some activities that in Davydov's curriculum are intended for a first-grade class (Davydov 1982).

We report some excerpts from the teaching experiments that lasted 1 year (for a more detailed report, see Mellone and Tortora 2017).

The teacher followed Davydov's approach, hence the pupils were given a task involving the description and the analysis of equalities and inequalities detectable from the observation of continuous quantities. Three identical containers were filled with different volumes of water and placed on the desk (Fig. 9).

The fact is noteworthy that she performed Davydov's activities in her cultural context, introducing some changes: in fact, the activity designed by Davydov for first-grade pupils (Davydov 1982) used only two containers with different amounts of water, whereas Colomba favored a more complex situation with three glasses; moreover, there was no scale on the containers, as in the original Davydov proposal, in order to avoid the use of numbers. Since the fifth-grade pupils were well familiarized with numbers, the teacher explicitly discouraged their use, telling them to use letters A, B, and C instead, in expressing the relations of equality and inequality.

The task is as follows: represent the equalities and inequalities that you observe.

Drawings and writings from all pupils' notebooks were reported on the blackboard and used as a prompt for a first discussion. In a page from Giulia's notebook (Fig. 10), we observe the drawing of the three containers, some inequalities and an equality, where the letters A, B, C refer to the quantities of water in the containers.

The discussion started with the teacher asking the class to observe the containers and the expressions on the board and to look for other relationships between the quantities

Fig. 10 Giulia's sketch

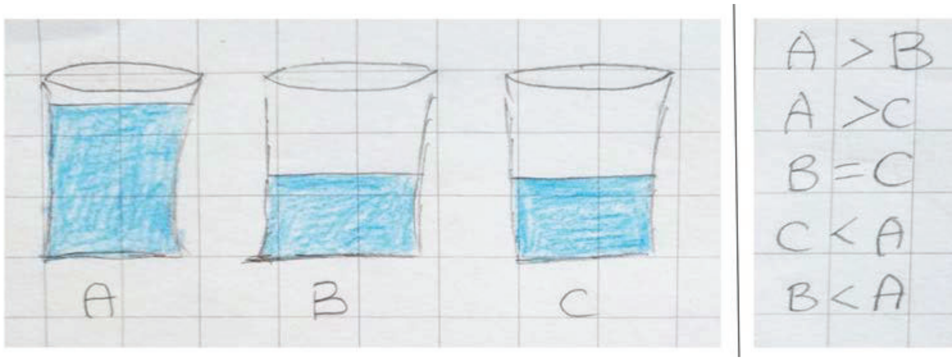
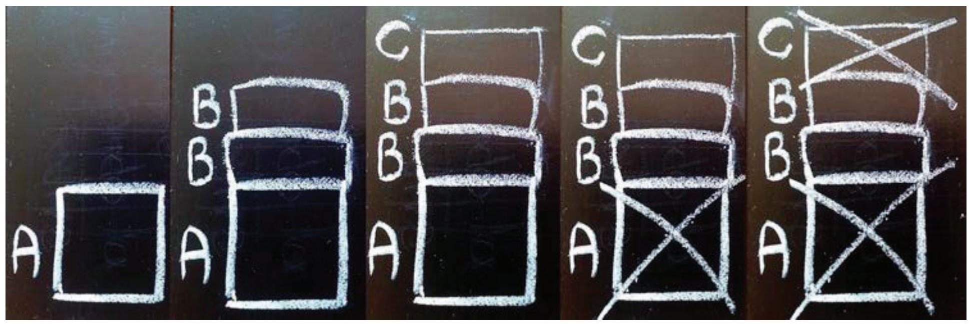


Fig. 11 Flora's sketch



of water. At first, the pupils used letters as labels, as we can notice from Giulia's intervention:

Giulia: Miss, we only know that those two, B and C, are filled to the same level with the same amount of water, whereas the first has more water.

It is useful to underline that this experimental path, being pursued in the fifth grade involving pupils who already knew the four operations and their properties, was followed by them as a new active process of sense-making of the additive structure in the context of quantities. The synergy between verbal language and symbolic language to describe the various relations stimulated a first refinement of the algebraic language, the substitution of *and* with the + in particular, and a purposeful use of brackets.

The discussion continued for a while, up to a very interesting intervention by Flora:

Flora: Miss, I have found another one. If we sum two Bs or two Cs, that are the same, then we add C once again and then subtract A and C, the result is two Bs or two Cs. May I write it on the board? [Flora writes: A + B + B + C − (A + C) = B × 2 or C × 2 (Fig. 11).]

In Flora's production, we can recognize how the context of relations between quantities can act by its own nature as a rich source of examples for algebraic relations.

In the following lesson, the pupils were asked to express the relationships and the transformations observed in the previous lessons by means of pictorial equations.

Subsequently, the teacher proposed to the pupils several word problems to be solved comparing quantities and expressing one of them in terms of other unknown quantities (for a more detailed account of this teaching experiment, see Mellone and Tortora 2017).

4.3 Interviews

At the end of the teaching experiments, we held some interviews to better understand the transposition processes accomplished by the teachers, and to share some reflections about cultural transposition. Here, we present some excerpts from the interviews with the teachers Loretta and Colomba who performed the teaching experiments that were briefly presented in the previous section of this paper. These interviews were audiotaped and then transcribed; here, we report only a few answers to reveal what Loretta told us about the way in which the PD path changed her practices, and to share Colomba's reflections about the transposition of Davydov activities in her class.

4.3.1 Interview with Loretta

Researcher: One of the fundamental words that characterized this path was 'variation'. How do you understand this expression? How has the concept of 'variation' been applied in the mathematics performed at school?

Loretta: I'll give you an example that seems important to me, to understand what I have experienced with the problems of variation. I immediately interpreted this approach as a 'trick' that allowed all my pupils, even the weakest of them, to solve some mathematical word problems. So, I liked the possibilities of inclusiveness that this approach allowed, although I had also some perplexity. Only later I became aware that variation is something more: it is the ability to make 'fluid' the arithmetic concepts, which means to be able to 'flow' in one direction or another. Pupils can manipulate numbers and mathematical meanings in many different ways, which has had great consequences in my classes: since first grade, children manipulate numbers easily, mastering the concept of quantity better than what used to happen in my previous classes.

Researcher: How did the experience of participating in this course influence your everyday mathematics teaching activities?

Loretta: In addition to all that I have already stated, it has influenced my teaching in general, leading me to an even greater reflection on choices to be made about the content but also about the method and especially about the 'artifacts' to propose to the children.

It would have been impossible for me to carry out these experiments without participating in the PD courses on mathematics education. It would not have been possible to give meaning and correct teaching intentionality without this study on Italian and Chinese teaching. My way of teaching, as well as the sense I give to what I do in my classes, has deeply changed.

In the first part of her interview, Loretta asserted that she decided to design teaching experiments using problems with variation because she considered them as a tool useful for including and supporting pupils, including the weak ones, to solve mathematical word problems. Therefore, in the first stage, she chose this educational tool because it was consistent and useful for her intention to include all pupils in the mathematical activity: which is different from the 'Chinese' use of variations for recognizing an invariant algebraic relationship. Loretta's care for inclusiveness is also related to the Italian educational context, since in Italy classes are formed by students of different levels of achievement.

Only later, probably during the teaching experiment, she realized that the problems with variation can be used as a tool to "make 'fluid' the arithmetic concepts" and "manipulate numbers easily mastering the concept of quantity". Loretta's expressions seem to refer to the Eastern idea of early algebra, but there is no explicit reference to this in her interview. However, it seems crucial to notice that it is the teaching experiment, carried out after the PD, that allowed her to become aware of it.

Moreover, all the reflections on the second question seem to reveal a perception of herself as more conscious: "leading me to an even greater reflection on choices to be made about the contents but also about the method and especially about the 'artifacts' to propose to the children". In this sense, this supports the idea expressed in the theoretical framework that the guided cultural transposition activities can represent an opportunity to reflect on teachers' educational choices and to increase their own awareness in defining the nature of their practice.

4.3.2 Interview with Colomba

Researcher: What was your thought about the first meeting with Davydov's math curriculum for primary school? What impact did his proposal have on your school practice?

Colomba: The encounter with Davydov's approach was so important to me that I thought to experiment with it even though my class was for fifth-graders. So, I presented the activities that Davydov suggests for a first-grade to my pupils as a sort of backward path, recognizing some properties of numbers and operations through the measurement of continuous quantities, the graphics, and then the algebraic representation. I planned measurements of quantities of water in containers of different shapes and sizes, thus stimulating comparison and reasoning on equalities and differences, I did the same thing for measurements of distances and paths urging pupils to identify the suitable unit of measurement each time. This led me to ponder about transformations and conservation of quantities, concepts expressed through representations, operations, and algebraic language.

> In this phase, I verified the effectiveness of the graphical representations, especially some of them, as a mediator for the comprehension of the algebraic language […]
> In particular, working with measure activities of continuous quantities with a different unit, I realized that the approach to multiplicative structure precedes or can be simultaneously presented with the additive structure.

Different from Loretta's, we found in Colomba's interview more explicit references to algebra. This also could be due to the fact the Davydov's curriculum explicitly refers to algebraic language, while the algebraic potentiality of problems with variation is less evident. She decided to try to engage herself and her pupils (even if fifth graders) in a Davydov-type mathematical environment. The algebraic thinking of her 10-year-old pupils, in particular Flora,, cannot be compared from the research point of view to the thinking of early learners.

However the PD and the teaching experiments with a measure of continuous quantities, helped her to realize something new for her: "the approach to multiplicative structure precedes or can be simultaneously presented with the additive structure". Even if here, due to space constraints, we showed excerpts from her teaching experiment relative only to the additive structure, in Colomba's comments we can read her awareness about Davydov's approach to mathematics: i.e., how starting from the context of measurement of magnitudes can provide, since early grades, an intertwined approach of additive and multiplicative structures.

5 Conclusions

The opening section of this paper considers the current significant opportunities of meeting with mathematics education methodologies implemented in faraway parts of the world. In this context, we proposed the construct of cultural transposition as a condition to decentralize the educational practice of one's own cultural context through contact with practices in other cultural contexts.

The paper deals with the particular deconstruction process which unfolds by coming into contact with particular educational practices related to different cultural backgrounds, such as the Chinese and the Russian backgrounds. Our analysis shows how the approach to WNA focuses on early algebra in primary schools in China and Singapore, but also in Russia within the schools that follow Davydov's curriculum. These mathematics curricula explicitly state that their main goal in teaching arithmetic is to develop students' understanding of quantitative relationships using mathematical models to represent them, such as the pictorial equation.

There are many different trends of research in mathematics education concerning the cultural component of mathematical activities: in our study, we refer to Skovsmose's (1994) critical view of mathematics as a tool of emancipation. From his perspective, learners are seen as an active part of the process, who challenge and shape the nature of their teaching–learning experience; from our perspective, teachers play the same active part. Coming into contact with educational practices unlike their own could be a crucial experience for teachers to increase their own awareness when defining the nature of their mathematical education practices. For these reasons, we propose cultural transposition processes as a tool to inspire and affect PD courses for in-service teachers.

Precisely, we investigated how these teacher educational programmes, which focus on cultural transposition, could support teachers in becoming aware of the implicit backgrounds that every educational choice entails; therefore, we designed and implemented PD courses with this purpose. The PD focused on the encounter with the particular Chinese education practice of problems with variation and with the visionary curriculum proposed by Davydov (1982) for the first grade. We designed the PDs and a set of tools including an entry questionnaire, focus groups, teaching experiments, and interviews, in order to collect data. These data answered the following questions: Is it possible to involve a teacher in the cultural transposition process? Can this process, which invests in teaching methodologies, allow a teacher to become aware of the backgrounds, of the *unthoughts* that every educational choice entails? In the particular case of WNA, can the contact with Chinese and Russian educational practices foster Italian teachers' reflections about the features of specific practices which focus on the arithmetical structure of word problems?

Through the analysis of collected data, we were able to detect in those teachers involved in the PD courses an openness to the previously unthinkable teaching in which the teaching practices related to WNA are oriented more on a vehicular awareness of the relations between mathematical objects (quantity-numbers) than of the object itself.

The teaching experiments of Loretta and Colomba show how the teachers implemented those suggestions shared during PD courses. As far as Loretta's teaching experiment is concerned, her idea of using the original Chinese texts to work on language aspects is interesting, as well as her choice not to converge on the figural equation, but to find a new solution as the array, showing great personal elaboration of the tool. Therefore, Loretta seemed to welcome the suggested decentralization of the educational practice through the contact with educational practices from other cultural contexts. In fact, she used problems with variation in order to allow her pupils to recognize the same underlying additive structure, even though her activities diverged from the original ones. We consider the teacher's free and imaginative vision of the Chinese tool as an example of emancipation (Skovsmose 1994) where the encounter with another cultural mathematics educational practice creates awareness and conditions to experiment with new previously un-thought-of paths.

As far as Colomba's teaching experiment is concerned, she decided to use the activities that Davydov imagined for first-grade classes, that is, before children start to use numbers, in her fifth-grade class. This first choice seemed to represent a free adaptation of Davydov's curriculum, in which she decided to break her previous routine in order to experiment with something new, with pupils working on quantities and developing algebraic thinking expressed in various ways. The problem-solving situation based on the context of quantities and the large use of collective discussion about the graphical representations were those ingredients which allowed 10-year-old pupils' recognition of algebraic relations. Even if, from one side the algebraic discoveries of Colomba's pupils, in particular Flora's, could not yet be labelled as 'early' algebraic thinking, from the other side it is interesting to report that the experience with this fifth-grade class was so crucial for Colomba that the next year she decided to engage her first grade class in new experimentation inspired by Davydov's curriculum (see Mellone et al. 2018).

It is important to highlight in both teaching examples that the methodology followed by the teachers in their usual work in the classroom, like the use of collective discussion orchestrated by the teachers, was flexibly intertwined with suggestions from the foreign methods of problems with variation and the work on quantities by Davydov. In addition, this shows how they managed to converge educational approaches coming from different cultural contexts, preserving their distinctive teaching features. In this sense, the two teachers, after their participation in the PDs and their teaching experiment-experiences, were intentionally oriented and free in the choice of their paths as well. This, of course, can also mean the possibility that teachers' enactments and rationales can differ from the purposes developed in the original cultural context. In fact, we demonstrate this scenario in the case of Loretta, who firstly recognizes in problems with variation a tool for inclusion more than a tool for recognizing algebraic structures. With regard to the teachers' reflections about their own experiences of the activities, we presented some excerpts of the interviews and argued that the cultural transposition approach itself determined Loretta and Colomba's free, imaginative, and critical vision (Ernest et al. 2016) of the Chinese tool and of Davydov's approach. However, we do not know to what extent this depended on each teacher's attitude towards our proposal.

Finally, we emphasize how the cultural transposition process has been even for us, as researchers and teacher educators, an experience to foster awareness both of mathematics education in our own cultural contexts, and of teachers' potential. Indeed, we were totally amazed by how these teachers used the suggestions coming from foreign mathematics educational practices to create new paths for their pupils without losing what they considered important from their previous work, such as the use of mathematical discussions or working on the representations proposed by pupils. Therefore, we would like to promote cultural transposition as a possible theoretical framework in which to catch and to use the complexity of the contact among different mathematics education cultures, for the purpose of fostering both teachers' and educators' awareness and effectiveness.

References

Bussi, M. G. B. (1996). Mathematical discussion and perspective drawing in primary school. *Educational studies in mathematics, 31*(1–2), 11–41.

Bussi, M. G. B., Bertolini, C., Ramploud, A., & Sun, X. (2017). Cultural transposition of Chinese lesson study to Italy. An exploratory study on fractions in a fourth-grade classroom. *International Journal for Lesson and Learning Studies, 6*(4), 1–17.

Bussi, M. G. B., & Martignone, F. (2013). Cultural issues in the communication of research on mathematics education. *For the Learning of Mathematics, 33*, 2–8.

Bartolini Bussi, M. G., Sun, X., & Ramploud, A. (2014). A dialogue between cultures about task design for primary school. In C. Margolinas (Ed.), *Proceedings of the ICMI Study 22* (pp. 549–558). Oxford.

Bussi, M. G. B., & Sun, X. H. (Eds.). (2018). *Building the foundation: Whole Numbers in the Primary Grades*. Cham: Springer International Publishing AG.

Barton, B. (2008). *The language of mathematics: Telling mathematical tales* (Vol. 44). New York: Springer.

Beijing Education Science Research Institute and Beijing Instruction Research Center for Basic Education (1996). *Shu Xue*, vol. 24. Beijing.

Bishop, A. J. (1988). *Mathematical enculturation. A cultural perspective on mathematics education*. Dordrecht: Kluwer Academic Publishers.

Cai, J., & Knuth, E. (Eds.). (2011). *Early algebraization: A global dialogue from multiple perspectives*. New York: Springer.

D'Ambrosio, U. (2006). *Ethnomathematics—Link between traditions and modernity*. Rotterdam: Sense Publishers.

Davydov, V. V. (1982). The psychological characteristics of the formation of elementary mathematical operations in children. In T. P. Carpenter et al. (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 224–238). Hillsdale: Lawrence Erlbaum.

Derrida, J. (1967). *De la grammatologie*. Paris: De Minuit.

Di Paola, B. (2016). Why Asian children outperform students from other countries? Linguistic and parental influences comparing Chinese and Italian children in Preschool Education. *International Electronic Journal of Mathematics Education, 11*(9), 3351–3359.

Di Paola, B., Battaglia, O. R., & Fazio, C. (2016). Non-hierarchical clustering to analyse an open- ended questionnaire on algebraic thinking. *South African Journal of Education, 36*, 1–13.

Di Paola, B., Mellone, M., Martignone, F., & Ramploud, A. (2015). Un'esperienza educativa di trasposizione culturale nella scuola primaria. *L'insegnamento della matematica e delle scienze integrate, 38*(3), 363–387.

Ernest, P., Sriraman, B., & Ernest, N. (Eds.). (2016). *Critical mathematics education: Theory, praxis, and reality*. Charlotte: Information Age Publishing.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education, 44*(1), 37–68.

Ivashova, O. (2011). The history and the present state of elementary mathematical education in Russia. In A. Karp & B. R. Vogel (Eds.), *Russian mathematics education. Programs and practices* (pp. 37–80). Singapore: World Scientific Publishing.

Jullien, F. (1993). *Figures de l'immanence. Pour une lecture philosophique du Yi king. Le Classique du changement*. Paris: Edition Grasset & Fasquelle.

Jullien, F. (2005). La decostruzione da fuori. Dalla Grecia alla Cina e ritorno. *Aut aut, 328*, 71–87.

Jullien, F. (2006). *Si parler va sans dire. Du logos et d'autres ressources*. Editions du Seuil.

Leung, A., Baccaglini-Frank, A., & Mariotti, M. A. (2013). Discernment of invariants in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460.

Ling Lo, M. (2012). *Variation theory and the improvement of teaching and learning*. Acta Universitatis: Gothoburgensis.

Lotman, J. M., & Uspenskij, B. A. (1975). *Tipologia della Cultura*. Milan: Bompiani.

Mellone, M. (2011). "Looking for tricks": a natural strategy, early forerunner of algebraic thinking… In M. Pytlak, T. Rowland, E. Swoboda (Eds.), *Proceedings of the 7th Conference of the European Society for Research in Mathematics Education* (pp. 1882–1890). Rzeszow: CERME.

Mellone, M., Carotenuto, G., Di Bernardo, R., & Punzo, C. (2018). Algebraic thinking among graphical representations and algebraci symbols. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of PME42*, Vol. 1 (pp. 247–254). Umeå, Sweden: PME.

Mellone, M., & Ramploud, A. (2015). Additive structure: An educational experience of cultural transposition. In Sun X., Kaur B. and Novotná N. (Eds.), *Proceedings of the ICMI Study 23* (pp. 567–574). China, Macau: University of Macau.

Mellone, M., Ramploud, A., Di Paola, B., & Martignone, F. (2017). Cultural transposition as a theoretical framework to foster teaching innovations. *Proceedings of PME41*, Vol.1 (p. 244). Singapore: PME.

Mellone, M., & Tortora, R. (2017). A design study for an Italian fifth-grade class following Davydov. *International Journal for Mathematics Teaching and Learning, 18*(2), 240–256.

Nuñes, T., Dorneles, B. V., Lin, P. J., & Rathgeb-Schnierer, E. (2016). Teaching and learning about whole numbers in primary school. In *Teaching and learning about whole numbers in primary school* (pp. 1–50). Cham: Springer International Publishing.

Ramploud, A. (2015). 数学 [shùxué] matematica, sguardi (d)alla Cina, Ph.D. Thesis, University of Modena e Reggio Emilia (Italy). https://morethesis.unimore.it/theses/browse/by_autore/r.html. Accessed 20 Dec 2017.

Ramploud, A., & Di Paola, B. (2013). The Chinese perspective of variation to rethink the Italian approach to word-problems from a pre-algebraic point of view. In Fazio, C. (Ed.), *Proceedings of CIEAEM 65* (pp. 525–535). Turin, Italy.

Shao, G., Fan, Y., Huang, R., Ding, E., & Li, Y. (2012). Mathematics classroom instruction in China viewed from a historical perspective. In Y. Li & R. Huang (Eds.), *How Chinese teach mathematics and improve teaching* (pp. 11–28). New York: Routledge.

Skovsmose, O. (1994). Towards a critical mathematics education. *Educational Studies in Mathematics, 27*(1), 35–57.

Spagnolo, F., & Di Paola, B. (2010). European and Chinese cognitive styles and their impact on teaching mathematics. *Springer,. Studies in Computational Intelligence, 277*, 1–267.

Sun, X. (2011). An insider's perspective: "Variation roblems" and their cultural grounds in Chinese curriculum practice. *Journal of Mathematics Education, 4*(1), 101–114.

Sun, X., Kaur, B., & Novotná, J. (Eds.). (2015). *Conference proceedings of the ICMI study 23: primary mathematics study on whole numbers*. http://www.umac.mo/fed/ICMI23/doc/Proceedings_ICMI_STUDY_23_final.pdf. Accessed Dec 2017.

Vygotskiĭ, L. S. (1962). *Thought and language*. Massachusetts Institute of Technology. Cambridge: MIT Press.

Wood, T. (Ed.). (2008). *The international handbook of mathematics teacher education*. Rotterdam: Sense Publishers.