Multidimensional dyadic Kurzweil–Henstock- and Perron-type integrals in the theory of Haar and Walsh series

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A B S T R A C T

The problem of recovering the coefficients of rectangular convergent multiple Haar and Walsh series from their sums, by generalized Fourier formulas, is reduced to the one of recovering a function (the primitive) from its derivative with respect to the appropriate derivation basis. Multidimensional dyadic Kurzweil–Henstock- and Perron-type integrals are compared and it is shown that a Perron-type integral, defined by major and minor functions having a special continuity property, solves the coefficients problem for series which are convergent everywhere outside some uniqueness sets.

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Coefficients problem
Kurzweil–Henstock dyadic integral Perron dyadic integral
Derivation bases
Saks continuity of interval functions

1. Introduction

In this paper we consider the problem of recovering the coefficients of multidimensional Haar and Walsh series from their sums by generalized Fourier formulas.

The theory of integrals which are related to the problem of recovering the coefficients of orthogonal series from their sums started with classical work of Denjoy [3] who defined an integration process so powerful that the sum of any everywhere convergent trigonometric series is integrable and the coefficients can be computed by Fourier formulas in which this integral is used. Later an easier approach based on Perron and Kurzweil–Henstock methods was developed by several authors (see [28,33] for details). For some other orthogonal series, including Walsh, Haar and Vilenkin series, the coefficients problem in the one-dimensional case was solved using various types of non-absolute generalization of the Lebesgue integral, including dyadic Denjoy (see [16,20,6]), dyadic Perron (see [17]) and dyadic Kurzweil–Henstock integrals

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(see [22–24]). In the multidimensional case a solution for the same series depends on the type of convergence. The so-called regular convergence was considered in [13,14]. For the rectangular convergence the coefficients problem was solved in [18,19] for Walsh and Haar series under assumption that the sum is Perron integrable functions.

The convergence of series everywhere in formulation of the coefficients problem can be replaced by convergence everywhere outside some particular exceptional sets, so-called sets of uniqueness or U-sets. We recall that a set E is said to be U-set for a system of functions if the convergence of a series with respect to this system to zero outside the set E implies that all coefficients of the series are zero. For references to a large body of literature on the theory of uniqueness of Walsh, Haar and Vilenkin series, including subtle theory of sets of uniqueness, see [1,4,15,30,31], whereas the classical trigonometric case is treated for example in [33,7].

In the present paper we are considering multiple Walsh and Haar series which are rectangular convergent outside some class of U-sets, without assuming a priori integrability of the sum in any prescribed sense, and we are solving the coefficients problem by finding an appropriate integral to be used in generalized Fourier formulas. As it was in the one-dimensional case and in the cases of some other series (see [21]), the method is based on reducing the coefficients problem to the one of recovering a function from its derivative with respect to the appropriate dyadic derivation basis. The difficulties which should be overcome in applying this method here are related to the fact that the primitive we want to recover is differentiable not everywhere but outside an exceptional set. Having in mind application to the coefficients problem we are interested in exceptional sets which are U-sets for multiple Walsh and Haar series. We investigate continuity assumptions which should be imposed on the primitive at the points of exceptional sets to garantee its uniqueness. It turns out that usual continuity with respect to the dyadic basis is not enough for this purpose and we introduce a stronger notion of continuity, which we call local Saks continuity with respect to the basis.

The most natural integration process to recover primitives is Kurzweil–Henstock integral (see [29]). In Section 2 we consider continuity properties of the dyadic Kurzweil–Henstock integral in a dimension greater then one and show that it has local Saks continuity. But it solves the problem of recovering a primitive only in the case of rather "thin" exceptional sets and fails to solve it in the case of the sets we are interested in. So we have to introduce in Section 3 a suitable Perron-type integral defined by major and minor functions having local Saks continuity property. We show in Section 4 that each two-dimensional Walsh series which converges everywhere outside a U-set of the type we consider here, is the Fourier series of its sum in the sense of this Perron-type integral. The same result, with some additional assumption on the behavior of the coefficients, is obtained for Haar series.

2. Kurzweil–Henstock integral with respect to the dyadic basis and its continuity properties

We consider here the dyadic derivation basis on the unit cube $K = [0,1]^m$. We denote by Q_d the set of all *dyadic-rational numbers in* $[0,1]$, i.e., the numbers of the form $\frac{j}{2^n}$ with $0 \le j \le 2^n,\ n = 0,1,2,\ldots$. The points $[0,1] \setminus Q_d$ constitute the set of *dyadic-irrational numbers* in $[0,1]$.

We denote one-dimensional dyadic intervals by

$$I_j^{(n)} := \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right], \quad 0 \le j \le 2^n - 1,$$

where $n = 0,1,2,\ldots$ is the *rank* of the interval. Let $\mathcal{I}$ be the family of all m-dimensional dyadic intervals

$$I_{\mathbf{j}}^{(\mathbf{n})} := I_{j_1}^{(n_1)} \times \cdots \times I_{j_m}^{(n_m)} \tag{1}$$

in K, where $\mathbf{n} = (n_1, \ldots, n_m)$ is the rank of $I_{\mathbf{j}}^{(\mathbf{n})}$ and $\mathbf{j} = (j_1, \ldots, j_m)$. We denote by $I^{(\mathbf{n})}$ an arbitrary interval of rank $\mathbf{n}$ and by $I_{\mathbf{x}}^{(\mathbf{n})}$, where $\mathbf{x} = \{x_1, ..., x_m\} \in K$, an interval of rank $\mathbf{n}$ containing $\mathbf{x}$.

An important role in this paper will be played by the set Z of points having at least one dyadic-rational coordinate, i.e.,

$$Z := \bigcup_{i=1}^{m} \big([0,1]^{i-1} \times Q_d \times [0,1]^{m-i}\big). \tag{2}$$

We shall use also a more general set

$$Y := \bigcup_{i=1}^{m} \big([0,1]^{i-1} \times Y_i \times [0,1]^{m-i}\big) \tag{3}$$

where Y_i, $i = 1, 2, ..., m$, is any countable set *containing* Q_d.

The *dyadic basis* $\mathcal{B}$ is defined as the collection of *basis sets*

$$\beta_\delta := \big\{(I, \mathbf{x}) : I \in \mathcal{I},\ \mathbf{x} \in I \subset U\big(\mathbf{x}, \delta(\mathbf{x})\big)\big\},$$

where δ is the so-called *gauge*, i.e., a positive function defined on K, and $U(\mathbf{x}, r)$ denotes the neighborhood of $\mathbf{x}$ of radius r. So we have

$$\mathcal{B} := \big\{\beta_\delta :\ \ \delta : K \to (0, \infty)\big\}.$$

If $(I, \mathbf{x}) \in \beta_\delta$, we say that $\mathbf{x}$ is the *tag* of I and that the pair $(I, \mathbf{x})$ is δ*-fine*.

Note that if $\mathcal{I}$ in the above definition is replaced with the family of *all* m-dimensional closed intervals, then we get the classical *full interval basis* (see [5,8,26]).

We refer to elements of $\mathcal{I}$ as $\mathcal{B}$*-intervals* and to a finite union of $\mathcal{B}$-intervals as $\mathcal{B}$*-figure*.

For each fixed $\mathbf{x} \in K$ there exists a sequence of $\mathcal{B}$-intervals $\{I_{\mathbf{x}}^{(\mathbf{n})}\}$ such that $\bigcap_{\mathbf{n}} I_{\mathbf{x}}^{(\mathbf{n})} = \{\mathbf{x}\}$. Note that if $\mathbf{x}$ is an interior point of K, the sequence $\{I_{\mathbf{x}}^{(\mathbf{n})}\}$ is constituted by 2^s subsequences of pair-wise overlapping $\mathcal{B}$-intervals with nested projections to coordinate axis, where s is the number of dyadic-rational coordinates of the point $\mathbf{x}$. In particular, if $\mathbf{x} \in K \setminus Z$ the sequence $\{I_{\mathbf{x}}^{(\mathbf{n})}\}$ cannot be splitted into non-overlapping subsequences and $\mathbf{x}$ is an interior point for any interval of this sequence.

We denote by $int(E)$ the interior of a set E and by $|E|$ the Lebesgue measure of E.

For a set $E \subset K$ and $\beta_\delta \in \mathcal{B}$ we write

$$\beta_\delta(E) := \big\{(I, \mathbf{x}) \in \beta_\delta : I \subset E\big\} \quad \text{and} \quad \beta_\delta[E] := \big\{(I, \mathbf{x}) \in \beta_\delta : \mathbf{x} \in E\big\}.$$

A β_δ*-partition* is a finite collection π of elements of β_δ, where the distinct elements $(I', \mathbf{x}')$ and $(I'', \mathbf{x}'')$ in π have I' and I'' non-overlapping, i.e., they have no common interior points. Let $E \in \mathcal{I}$. Then $\pi \subset \beta_\delta(E)$ is called β_δ-partition in E. If $\bigcup_{(I,\mathbf{x})\in\pi} I = E$ then π is called β_δ-partition *of* E. For a set E and a β_δ-partition π we denote $\pi[E] := \{(I, \mathbf{x}) \in \pi : (I, \mathbf{x}) \in \beta_\delta[E]\}$ and we call $\pi[E]$ β_δ*-partition with tags on* E.

The dyadic basis $\mathcal{B}$ has all the usual properties of a general derivation basis (see [11,27]). In particular it has the *partitioning property*: for each $\mathcal{B}$-interval I and for any $\beta_\delta \in \mathcal{B}$ there exists a β_δ-partition of I.

Let F be an additive set function on $\mathcal{I}$ and E an arbitrary subset of K. For a fixed $\beta_\delta \in \mathcal{B}$, we set

$$Var(E, F, \beta_\delta) := \sup_{\pi \subset \beta_\delta[E]} \sum \big|F(I)\big|.$$

We put also

$$V_F(E) = V(E, F, \mathcal{B}) := \inf_{\beta_\delta \in \mathcal{B}} Var(E, F, \beta_\delta).$$

The extended real-valued set function $V_F(\cdot)$ is called *variational measure generated by* F, with respect to the basis $\mathcal{B}$.

Definition 1. Given a $\mathcal{B}$-interval function F, the *upper* and the *lower* $\mathcal{B}$*-derivatives* of F at a point $\mathbf{x}$, with respect to the basis $\mathcal{B}$, are defined as

$$\overline{D}_{\mathcal{B}}F(\mathbf{x}) := \inf_{\beta_\delta \in \mathcal{B}} \sup_{(I,\mathbf{x})\in\beta_\delta} \frac{F(I)}{|I|} \quad \text{and} \quad \underline{D}_{\mathcal{B}}F(\mathbf{x}) := \sup_{\beta_\delta \in \mathcal{B}} \inf_{(I,\mathbf{x})\in\beta_\delta} \frac{F(I)}{|I|}, \tag{4}$$

respectively. If $\overline{D}_{\mathcal{B}}F(\mathbf{x}) = \underline{D}_{\mathcal{B}}F(\mathbf{x})$ we call this common value the $\mathcal{B}$*-derivative* $D_{\mathcal{B}}F(\mathbf{x})$ at $\mathbf{x}$. We say that F is $\mathcal{B}$*-differentiable* at $\mathbf{x}$ if the $\mathcal{B}$-derivative at this point exists and is finite.

Below, in application to series, we shall need theorems on recovering a primitive from its derivative when the derivative is defined everywhere outside some exceptional set. In most cases the role of these sets will be played by the set (2) or (3). It is clear that in order to garantee that a primitive is uniquely defined by the derivative we have to impose some continuity assumptions on the primitive at the points of the exceptional set.

In the case of a general basis a set-function F is said to be $\mathcal{B}$*-continuous* at a point $\mathbf{x}$, with respect to the basis $\mathcal{B}$, if $V_F(\{\mathbf{x}\}) = 0$. For our dyadic basis it means that for the sequence of $\mathcal{B}$-intervals $I_{\mathbf{x}}^{(\mathbf{n})}$, with a fixed $\mathbf{x}$, the value of function F on these intervals tends to zero together with diameter of the intervals.

It is not difficult to see that if the exceptional set E is countable and a primitive is $\mathcal{B}$-continuous on E, then it is defined uniquely. But it is not true if we take Z as an exceptional set.

This is seen by the following simple example of $\mathcal{B}$-interval function F defined on two-dimensional intervals $I \in \mathcal{I}$. Put

$$F(A) = \begin{cases} 1 & \text{if } A = K, \\ 0 & \text{if } A = I \text{ with } I \cap (\{0\} \times [0,1]) = \emptyset, \\ \frac{1}{2^n} & \text{if } A = I_0^0 \times I_j^{(n)} \end{cases} \tag{5}$$

and extend the definition of F on any other $I \in \mathcal{I}$ using additivity. This function is obviously $\mathcal{B}$-continuous. Besides, not being trivial, it has derivative equal zero everywhere on $K \setminus (\{0\} \times [0,1])$.

So we need a stronger notion of continuity to guarantee uniqueness of a primitive. We recall that an interval function F is said to be *continuous in the sense of Saks* if $\lim_{|I|\to 0,} F(I) = 0$. We define a local version of this type of continuity adjusted to $\mathcal{B}$-interval functions.

Definition 2. We say that a $\mathcal{B}$-interval function F is *locally* $\mathcal{B}$*-continuous in the sense of Saks*, or briefly $\mathcal{B}S$*-continuous, at a point* $\mathbf{x}$ if

$$\lim_{|I|\to 0,\, \mathbf{x}\in I} F(I) = 0 \tag{6}$$

In the two-dimensional case the last equality can be rewritten in terms of ranks of $\mathcal{B}$-intervals in the following way:

$$\lim_{k+l\to\infty} F\big(I_{\mathbf{x}}^{(k,l)}\big) = 0. \tag{7}$$

We shall see in the next section that the assumption of $\mathcal{B}S$-continuity guarantee the desired uniqueness.

Now we recall the definition of the Kurzweil–Henstock-type integral with respect to the dyadic basis (see for example [21]). All the point- and set-functions below are supposed to be real-valued.

Definition 3. A point-function f on $L \in \mathcal{I}$ is said to be *$H_{\mathcal{B}}$-integrable* on L, with *$H_{\mathcal{B}}$-integral* A, if for every $\varepsilon > 0$, there exists a gauge δ such that for any β_δ-partition π of L we have:

$$\left| \sum_{(I,\mathbf{x})\in\pi} f(\mathbf{x})|I| - A \right| < \varepsilon.$$

We denote the integral value A by $(H_{\mathcal{B}}) \int_L f$.

In an obvious way we can extend the definition of the $H_{\mathcal{B}}$-integral to the case of integration over any $\mathcal{B}$-figure.

We say that a function f is *$H_{\mathcal{B}}$-integrable on a set* $E \subset K$ if the function $f \cdot \chi_E$ is $H_{\mathcal{B}}$-integrable on K and $\int_E f = \int_K f \cdot \chi_E$.

Similarly to the case of full interval basis in the dimension one (see [9,26]) it can be proved that a function which is equal to zero almost everywhere on a $\mathcal{B}$-interval L, is $H_{\mathcal{B}}$-integrable on L with integral value zero. This implies that $H_{\mathcal{B}}$-integrability of a function and the value of the integral do not depend on the value of the function on a set of measure zero. This also justifies an extension of the previous definition to the case of a function defined almost everywhere: it is just enough to define the function to be zero on the set where it is not defined and to consider $H_{\mathcal{B}}$-integrability of the extended function.

If a function f is $H_{\mathcal{B}}$-integrable on L then it is $H_{\mathcal{B}}$-integrable on any $\mathcal{B}$-figure $I \subset L$. So we can define the indefinite $H_{\mathcal{B}}$-integral which is an additive $\mathcal{B}$-interval function.

We need the following version of Saks–Henstock lemma for our basis that can be proved as in the case of the full interval basis (see [8,9,26]).

Lemma 1. *If a point-function f on $L \in \mathcal{I}$ is $H_{\mathcal{B}}$-integrable, with indefinite integral F, then for any $\varepsilon > 0$, there exists a gauge δ such that for any β_δ-partition π in L we have:*

$$\sum_{(I,\mathbf{x})\in\pi} \left| f(\mathbf{x})|I| - F(I) \right| < \varepsilon.$$

Definition 4. Given a closed set $T \subset K$ and a gauge δ on T, we say that a $\mathcal{B}$-figure O_T is a *β_δ-halo* of T if $O_T = \bigcup_{j=1}^k I_j$, where $\{(x_j, I_j)\}_{j=1}^k$ is a β_δ-partition tagged on T and $T \subset int(O_T)$.

It is easy to check that for any gauge δ on K and for any closed set T a β_δ-halo of T exists.

In the next section we shall use the following lemma which is a particular case of a Hake-type result for Kurzweil–Henstock-type integral in a more general setting (see [25, Th. 2]).

Lemma 2. *If a function f is $H_{\mathcal{B}}$-integrable on K and a closed set $T \subset K$ is of measure zero, then for any $\varepsilon > 0$ there exists a gauge δ so that for any two β_δ-halos O'_T and O''_T we have*

$$\left| \int_{K\setminus O'_T} f - \int_{K\setminus O''_T} f \right| < \varepsilon.$$

Using Lemma 1, applied to the partition $\pi[\{\mathbf{x}\}]$, we can prove that the indefinite $H_{\mathcal{B}}$-integral is $\mathcal{B}$-continuous at each point of K. But in fact $H_{\mathcal{B}}$-integral has a stronger continuity property.

Theorem 1. *Let f be an $H_{\mathcal{B}}$-integrable function on K. Then the indefinite $H_{\mathcal{B}}$-integral F is $\mathcal{B}S$-continuous everywhere on K.*

Proof. We give the proof for $m = 2$ but the same type of argument can be used for any m. Having fixed a point (x, y) consider a double sequence $\{I_{(x,y)}^{(k,l)}\}$ of $\mathcal{B}$-intervals containing this point. If both rank components

k and l tend to infinity, then (7) is implied by $\mathcal{B}$-continuity of F. So without loss of generality, we can assume that one of the component, say k, is fixed and the other tends to infinity. In this case we have to prove that

$$\lim_{l\to\infty} F\big(I^{(k,l)}_{(x,y)}\big) = 0. \tag{8}$$

Note that

$$\lim_{l\to\infty} I^{(k,l)}_{(x,y)} = I^{(k)}_x \times \{y\}. \tag{9}$$

We can assume that $f(x) = 0$ for $x \in I^{(k)}_x \times \{y\}$. Fix $\varepsilon > 0$ and take a gauge δ corresponding to ε according to Lemma 1. Considering $\delta_y(x) := \delta(x,y)/2$ as a gauge on $I^{(k)}_x$ we take a finite (one-dimensional) β_{δ_y}-partition $\pi = \{(T_i, x_i)\}_i$ of $I^{(k)}_x$. Then for all $l > l_0$ where l_0 is big enough, the interval $I^{(k,l)}_{(x,y)}$ can be represented as the union $\cup_i(T_i \times I^{(l)})$ and the family $\{(T_i \times I^{(l)}, (x_i, y))\}_i$ forms a two-dimensional β_δ-partition with tags on $I^{(k)}_x \times \{y\}$. By additivity of F and by Lemma 1 we get $|F(I^{(k,l)}_{(x,y)})| \le \sum_i |F(T_i \times I^{(l)})| < \varepsilon$ for all $l > l_0$ and this proves (8). $\square$

The following theorem, known in fact for a wide class of bases (see [11]), characterizes a class of exceptional sets for which $H_{\mathcal{B}}$-integral solves the problem of recovering the primitive not differentiable on those sets.

Theorem 2. *If an additive $\mathcal{B}$-interval function F is $\mathcal{B}$-differentiable with $D_{\mathcal{B}}F(\mathbf{x}) = f(\mathbf{x})$ everywhere on K, outside a set E such that $V_F(E) = 0$, then the function f is $H_{\mathcal{B}}$-integrable and F is its indefinite $H_{\mathcal{B}}$-integral.*

It is easy to see that $\mathcal{B}$-continuity of a function F on any countable set E implies the equality $V_F(E) = 0$. Together with this observation the above theorem implies the following result on recovering the primitive.

Corollary 1. *If an additive $\mathcal{B}$-interval function F is $\mathcal{B}$-differentiable with $D_{\mathcal{B}}F(\mathbf{x}) = f(\mathbf{x})$ everywhere on K outside a countable set where F is $\mathcal{B}$-continuous, then the function f is $H_{\mathcal{B}}$-integrable on K and F is its indefinite $H_{\mathcal{B}}$-integral.*

Unfortunately, as we shall see in the next section (see Theorem 4 below), $H_{\mathcal{B}}$-integral is not strong enough to solve the problem of recovering the primitive in the case of more massive exceptional sets which are dictated by the coefficients problem for multiple series.

Because of this we shall have to introduce another dyadic integral, namely a version of Perron-type integral, which will serve our purpose.

3. Saks continuous Perron-type integral and the problem of recovering a primitive

The Kurzweil–Henstock integral with respect to a basis is known to be equivalent to the Perron integral with respect to the same basis (see [11]). In particular it is true for the dyadic basis. Moreover this Perron dyadic integral, $P_{\mathcal{B}}$-integral, can be defined by $\mathcal{B}$-continuous major and minor functions (see [2] for the case of full interval basis; a proof for the dyadic case is similar). We need not recall here the definition of $P_{\mathcal{B}}$-integral and we pass directly to constructing another Perron-type integral defined by $\mathcal{B}S$-continuous major and minor functions, which will be used to solve the coefficients problem. We start with a few lemmas which we need to justify our definition. In these lemmas ψ denotes any *additive* $\mathcal{B}$-interval function.

Lemma 3. *Let a $\mathcal{B}$-interval I be represented as union of two non-overlap-ping $\mathcal{B}$-intervals I_1 and I_2 of smaller rank. Suppose that $\psi(I) \ge A$ and $\psi(I_1) \le C|I_1|$ for some numbers A and C. Then $\psi(I_2) \ge A - \frac{C}{2}|I|$.*

Proof. Note that $|I_1| = |I_2| = \frac{1}{2}|I|$. Then having in mind the additivity of ψ we can write

$$\psi(I_2) = \psi(I) - \psi(I_1) \geq A - C|I_1| = A - \frac{C}{2}|I| \qquad \square$$

Lemma 4. *Let ψ be $\mathcal{BS}$-continuous everywhere in K and let $\psi(I^{(\mathbf{n})}) > C|I^{(\mathbf{n})}|$ for some $I^{(\mathbf{n})}$. Then there exist at least two $\mathcal{B}$-intervals $I^{(\mathbf{n}')}$ and $I^{(\mathbf{n}'')}$ contained in $I^{(\mathbf{n})}$ whose projections onto each of the coordinate axes are disjoint and for which $\psi(I^{(\mathbf{n}')}) > C|I^{(\mathbf{n}')}|$ and $\psi(I^{(\mathbf{n}'')}) > C|I^{(\mathbf{n}'')}|$.*

Proof. Take C_1 such that $\frac{\psi(I^{(\mathbf{n})})}{|I^{(\mathbf{n})}|} \geq C_1 > C$. Split the interval $I^{(\mathbf{n})}$ into two intervals of rank $\mathbf{n} = (n_1 + 1, n_2, ..., n_m)$ and using the additivity of ψ choose one of them, say $I_+^{(n_1+1,n_2,\ldots,n_m)}$, for which $\psi(I_+^{(n_1+1,n_2,\ldots,n_m)}) \geq \frac{\psi(I^{(\mathbf{n})})}{2}$. Then for this interval we have

$$\psi\big(I_+^{(n_1+1,n_2,\ldots,n_m)}\big) \geq \frac{\psi(I^{(\mathbf{n})})}{2} > \frac{1}{2}C\big|I^{(\mathbf{n})}\big| = C\big|I_+^{(n_1+1,n_2,\ldots,n_m)}\big|.$$

Now repeating the same argument for the interval $I_+^{(n_1+1,n_2,\ldots,n_m)}$ we choose interval $I_+^{(n_1+2,n_2,\ldots,n_m)} \subset I_+^{(n_1+1,n_2,\ldots,n_m)}$ such that

$$\psi\big(I_+^{(n_1+2,n_2,\ldots,n_m)}\big) > C\big|I_+^{(n_1+2,n_2,\ldots,n_m)}\big|.$$

In this way we obtain a sequence $\{I_+^{(n_1+j,n_2,\ldots,n_m)}\}_j$ of nested interval of rank $(n_1 + j, n_2, ..., n_m)$ for which

$$\psi\big(I_+^{(n_1+j,n_2,\ldots,n_m)}\big) > C\big|I_+^{(n_1+j,n_2,\ldots,n_m)}\big|. \tag{10}$$

We denote by $I_-^{(n_1+j,n_2,\ldots,n_m)}$ the interval of rank $(n_1 + j, n_2, ..., n_m)$ complementing the interval $I_+^{(n_1+j,n_2,\ldots,n_m)}$ to the $I_+^{(n_1+j-1,n_2,\ldots,n_m)}$.

If we have $\psi(I_-^{(n_1+j,n_2,\ldots,n_m)}) \leq C|I_-^{(n_1+j,n_2,\ldots,n_m)}|$ for all j then applying Lemma 3 with $I = I^{(\mathbf{n})}$, $I_1 = I_-^{(n_1+1,n_2,\ldots,n_m)}$ and $A = C_1|I^{(\mathbf{n})}|$ we get

$$\psi\big(I_+^{(n_1+1,n_2,\ldots,n_m)}\big) \geq C_1\big|I^{(\mathbf{n})}\big| - \frac{C}{2}\big|I^{(\mathbf{n})}\big|.$$

Now we apply Lemma 3 with $I = I_+^{(n_1+1,n_2,\ldots,n_m)}$, $I_1 = I_-^{(n_1+2,n_2,\ldots,n_m)}$ and $A = C_1|I^{(\mathbf{n})}| - \frac{C}{2}|I^{(\mathbf{n})}|$ getting

$$\psi\big(I_+^{(n_1+2,n_2,\ldots,n_m)}\big) \geq C_1\big|I^{(\mathbf{n})}\big| - \frac{C}{2}\big|I^{(\mathbf{n})}\big| - \frac{C}{2}\big|I_+^{(n_1+1,n_2,\ldots,n_m)}\big| = \big|I^{(\mathbf{n})}\big|\left(C_1 - \frac{C}{2} - \frac{C}{4}\right).$$

Proceeding by induction we get for any j

$$\psi\big(I^{(n_1+j,n_2,\ldots,n_m)}\big) \geq \big|I^{(\mathbf{n})}\big|\left(C_1 - \frac{C}{2} - .... - \frac{C}{2^j}\right)$$

and so

$$\liminf_{j\to\infty} \psi\big(I^{(n_1+j,n_2,\ldots,n_m)}\big) \geq \big|I^{(\mathbf{n})}\big|(C_1 - C) > 0.$$

This inequality is obviously in contradiction with $\mathcal{BS}$-continuity of ψ at the points of the set $\bigcap_j I_+^{(n_1+j,n_2,\ldots,n_m)}$. Hence for some j, $\psi(I_-^{(n_1+j,n_2,\ldots,n_m)}) > C|I_-^{(n_1+j,n_2,\ldots,n_m)}|$. Since the same estimation is also true for $I_+^{(n_1+j,n_2,\ldots,n_m)}$, see (10), we obtain two intervals $I_+^{(n_1+j,n_2,\ldots,n_m)}$ and $I_-^{(n_1+j,n_2,\ldots,n_m)}$ for which the desired inequality holds. For each of them we can repeat the preceding argument having fixed $n_1 + j, n_3, ..., n_m$ and varying the second index n_2. We thereby obtain four intervals for each of which the corresponding estimation holds. By geometric considerations we can choose two of them whose projections onto each of the first two axis do not overlap. If the projections have common end-point we can repeat the previous construction for each of the obtained intervals getting, once again by geometric consideration, two intervals with disjoint projections on two axis. Then we can proceed varying the third index and keeping fixed the others, and so on. We obtain after m steps two desired intervals. $\square$

Lemma 5. *Let ψ be $\mathcal{B}S$-continuous in K and let $\psi(I^{(\mathbf{n})}) > C|I^{(\mathbf{n})}|$ for some $I^{(\mathbf{n})}$. Then there exists a perfect set $P \subset I^{(\mathbf{n})}$, any two points of which have pair-wise distinct coordinates and*

$$\overline{D}_{\mathcal{B}}\psi(\mathbf{x}) \geq C$$

for all $\mathbf{x} \in P$.

Proof. The statement can be obtained by the repeated application of Lemma 4. $\square$

Applying the previous lemma to the function $-\psi$ instead of ψ we can formulate the following version of it.

Lemma 6. *Let ψ be $\mathcal{B}S$-continuous in K and let $\psi(I^{(\mathbf{n})}) < C|I^{(\mathbf{n})}|$ for some $I^{(\mathbf{n})}$. Then there exists a perfect set $P \subset I^{(\mathbf{n})}$, any two points of which have pair-wise distinct coordinates and*

$$\underline{D}_{\mathcal{B}}\psi(\mathbf{x}) \leq C$$

for all $\mathbf{x} \in P$.

Definition 5. Let f be a point-function defined at least on $K \setminus Z$. An additive $\mathcal{B}S$-continuous on K $\mathcal{B}$-interval function M (resp., m) is called a $\mathcal{B}S$-*major* (resp., $\mathcal{B}S$-*minor*) function of f if the lower (resp., the upper) $\mathcal{B}$-derivative satisfies the inequality

$$\underline{D}_{\mathcal{B}}M(\mathbf{x}) \geq f(\mathbf{x}) \quad \big(\text{resp. } \overline{D}_{\mathcal{B}}m(\mathbf{x}) \leq f(\mathbf{x})\big) \quad \text{for all } \mathbf{x} \in K \setminus Z. \tag{11}$$

Lemma 7. *Let an additive $\mathcal{B}$-interval function R be $\mathcal{B}S$-continuous on K and satisfy the inequality $\underline{D}_{\mathcal{B}}R(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in K \setminus Y$ with Y defined by (3). Then $R(I) \geq 0$ for any $\mathcal{B}$-interval I.*

Proof. Suppose that there exists a $\mathcal{B}$-interval I such that $R(I) < 0$. Then for some negative C we have $R(I) < C|I|$. By Lemma 6 we have a perfect set $P \subset I$ any two points of which have pair-wise distinct coordinates and

$$\underline{D}_{\mathcal{B}}R(\mathbf{x}) \leq C < 0 \tag{12}$$

for all $\mathbf{x} \in P$. The properties of P and Y imply that the sets $P \cap ([0,1]^{i-1} \times Y_i \times [0,1]^{m-i})$ for each i are countable, therefore $P \setminus Y$ is not empty and hence there exists at least one point $\mathbf{x}$ in P such that $\underline{D}_{\mathcal{B}}R(\mathbf{x}) \geq 0$ giving a contradiction with (12). $\square$

Lemma 8. *Let M and m be a $\mathcal{B}S$-major and a $\mathcal{B}S$-minor function for a point-function f on K. Then for each $\mathcal{B}$-interval I we have $M(I) \geq m(I)$.*

Proof. Consider $R = M - m$. Then $\underline{D}_{\mathcal{B}}(R(\mathbf{x})) \geq \underline{D}_{\mathcal{B}}(M(\mathbf{x})) - \overline{D}_{\mathcal{B}}(m(\mathbf{x})) \geq f(\mathbf{x}) - f(\mathbf{x}) = 0$ for all $\mathbf{x} \in K \setminus Z$ and it is enough to apply Lemma 7. $\square$

This lemma implies that for any function f we have

$$\inf_M \{M(K)\} \geq \sup_m \{m(K)\}$$

where "inf" and "sup" are taken over all $\mathcal{B}S$-major and $\mathcal{B}S$-minor function of f, respectively. This justifies the following definition.

Definition 6. A point-function f defined at least on $K \setminus Z$ is said to be *$P_{\mathcal{B}}S$-integrable* on K, if there exists at least one $\mathcal{B}S$-major function and at least one $\mathcal{B}S$-minor function of f and

$$-\infty < \inf_M \{M(K)\} = \sup_m \{m(K)\} < +\infty$$

where "inf" and "sup" are taken as above. The common value is called *$P_{\mathcal{B}}S$-integral* of f on K and is denoted by $(P_{\mathcal{B}}S)\int_K f$.

In the same way we can define $P_{\mathcal{B}}S$-integral on any $\mathcal{B}$-interval I.

Directly from the definitions we get the following result which shows that the $P_{\mathcal{B}}S$-integral solves the problem of recovering the primitive from its $\mathcal{B}$-derivative in the form we need.

Theorem 3. *If an additive $\mathcal{B}S$-continuous $\mathcal{B}$-interval function F is $\mathcal{B}$-differentiable with $D_{\mathcal{B}}F(\mathbf{x}) = f(\mathbf{x})$ everywhere on $K \setminus Z$ then the function f is $P_{\mathcal{B}}S$-integrable on K and F is its indefinite $P_{\mathcal{B}}S$-integral.*

Remark 1. Note that we have justified the well-definedness of $P_{\mathcal{B}}S$-integral using only a partial case of Lemma 7 where Y coincides with Z. If we use Lemma 7 in its full generality we can extend the previous definition of $P_{\mathcal{B}}S$-integral to the case when the inequalities (11) related to major and minor function hold outside a fixed set Y defined by (3). Such an integral for function f, defined at least on $K \setminus Y$, depends on the chosen exceptional set Y and we call it $P_{\mathcal{B}}^Y S$-integral. As Y contains Z, $P_{\mathcal{B}}^Y S$-integral includes $P_{\mathcal{B}}S$-integral. Theorem 3, with Z replaced by Y, holds true for this integral.

Remark 2. It follows from example (5) that the assumption of $\mathcal{B}S$-continuity of F in the above theorem (and in Lemma 7) cannot be weakened to the one of $\mathcal{B}$-continuity.

The next theorem shows in particular that the $H_{\mathcal{B}}$-integral (and so also $P_{\mathcal{B}}$-integral) fails to solve the problem of recovering the primitive under assumption of Theorem 3.

Theorem 4. *There exists a $P_{\mathcal{B}}S$-integrable function f on $K := [0,1]^2$ which is not $H_{\mathcal{B}}$-integrable. Moreover if Φ is the indefinite $P_{\mathcal{B}}S$-integral of f, then $D_{\mathcal{B}}\Phi(\mathbf{x}) = f(\mathbf{x})$ everywhere on $K \setminus Z$.*

Proof. In this proof we use the notation $\widetilde{I}_j^{(k)}$ for the dyadic half-open intervals $[\frac{j}{2^k}, \frac{j+1}{2^k})$. For $k \geq 0$ and $0 \leq j \leq 2^k - 1$ we put

$$J_{1,2j}^{(k)} = \widetilde{I}_1^{(2k+1)} \times \widetilde{I}_{2j}^{(2k+1)}, \qquad J_{1,2j+1}^{(k)} = \widetilde{I}_1^{(2k+1)} \times \widetilde{I}_{2j+1}^{(2k+1)}$$

$$J_{2,2j}^{(k)} = \widetilde{I}_1^{(2k+2)} \times \widetilde{I}_{2j}^{(2k+1)}, \qquad J_{2,2j+1}^{(k)} = \widetilde{I}_1^{(2k+2)} \times \widetilde{I}_{2j+1}^{(2k+1)}.$$

We note that $|J^{(k)}_{1,2j}| = |J^{(k)}_{1,2j+1}| = 2|J^{(k)}_{2,2j}| = 2|J^{(k)}_{2,2j+1}| = \frac{1}{2^{4k+2}}$. We define the function $f : [0,1]^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2^{3k+2} & \text{if } (x,y) \in J^{(k)}_{1,2j} \\ -2^{3k+2} & \text{if } (x,y) \in J^{(k)}_{1,2j+1} \\ -2^{3k+3} & \text{if } (x,y) \in J^{(k)}_{2,2j} \\ 2^{3k+3} & \text{if } (x,y) \in J^{(k)}_{2,2j+1} \end{cases}$$

We put $f(x) = 0$ on the boundary of the unit square where it is not yet defined.

We use Lemma 2 to prove that f is not H_B-integrable on $[0,1]^2$ taking as T of this lemma the set $\{0\} \times [0,1]$. Note that the function f is obviously integrable in the usual Riemann sense on each dyadic interval that is not intersecting with T, so it is also $H_{\mathcal{B}}$-integrable on such an interval. We have

$$F\big(J^{(k)}_{p,q}\big) = \int\limits_{J^{(k)}_{p,q}} f = \begin{cases} \frac{1}{2^k} & \text{if } p=1,\ q=2j \text{ or } p=2,\ q=2j+1, \\ -\frac{1}{2^k} & \text{if } p=1,\ q=2j+1 \text{ or } p=2,\ q=2j \end{cases}$$

Having in mind the chess board structure of signs in the definition of function f we get

$$\int\limits_{\frac{1}{2^{2s}}}^{1} dx \int\limits_{I^{(m)}_j} f\ dy = 0, \tag{13}$$

$$\int\limits_{\frac{1}{2^{2s+1}}}^{1} dx \int\limits_{I^{(2s+1)}_{2j}} f\ dy = \int\limits_{J^{(s)}_{1,2j}} f = \frac{1}{2^s} \tag{14}$$

for any j, m and s. We note that each interval $I^{(s)}_i$ contains 2^s intervals $I^{(2s+1)}_{2j}$ with $2^s i \le j \le 2^s(i+1)-1$. So for any natural number s and any $i = 0, ..., 2^s - 1$ we have, by summing up (14),

$$\sum_{j=2^s i}^{2^s(i+1)-1} \int\limits_{I^{(2s+1)}_1} dx \int\limits_{I^{(2s+1)}_{2j}} f\ dy = 2^s \cdot \frac{1}{2^s} = 1. \tag{15}$$

We construct, for any gauge δ on K, two β_δ-halos of T for which the inequality of Lemma 2 fails to be true with $\varepsilon = 1$. Having fixed an arbitrary gauge δ on K we put $T_n = \{y \in [0,1] : \delta(0,y) \ge \frac{1}{2^n}\}$. By Baire category theorem, we can find an interval $[a,b] \subset [0,1]$ such that for some n, the set T_n is dense on $[a,b]$. We can choose $s > n$ and i so that $I^{(s)}_i \subset [a,b]$. Then the intervals $I^{(2s)}_0 \times I^{(2s+1)}_{2j}$ with j such that $I^{(2s+1)}_{2j} \subset I^{(s)}_i$, together with tags in $\{0\} \times (T_n \cap I^{(2s+1)}_{2j})$, are δ-fine and we include them in β_δ-halo O'_T. We complete this β_δ-halo by any δ-fine intervals with tags in $T \setminus \{0\} \times I^{(s)}_j$. For the same δ we can construct another β_δ-halo O''_T replacing the chosen above intervals $I^{(2s)}_0 \times I^{(2s+1)}_{2j}$ with $I^{(2s+1)}_0 \times I^{(2s+1)}_{2j}$ and keeping all the other intervals constituting O'_T. Then by (15) we obtain

$$\Bigg| \int\limits_{K \setminus O'_T} f - \int\limits_{K \setminus O''_T} f \Bigg| = \sum_{j=2^s i}^{2^s(i+1)-1} \int\limits_{I^{(2s+1)}_1} dx \int\limits_{I^{(2s+1)}_{2j}} f\ dy = 1$$

This implies that f is not $H_{\mathcal{B}}$-integrable.

Now we define an additive $\mathcal{B}$-interval function Φ to be the indefinite $P_{\mathcal{B}}S$-integral of f. Let $\Phi(I) = \int_I f$ for any $\mathcal{B}$-interval I which have no intersection with $T = \{0\} \times [0,1]$ and $\Phi(I^{(0)}_0 \times I^{(k)}_i) = 0$ for any $k \ge 0$

and $0 \le i \le 2^k - 1$. The value of Φ on any other $\mathcal{B}$-intervals can be computed using the assumption of additivity of Φ.

Take any $(x,y) \in K \setminus Z$ where Z is defined by (2). Then (x,y) is an interior point of some $J^{(k)}_{p,q}$, where the function f is constant. If $(x,y) \in I \subset J^{(k)}_{p,q}$, then

$$\frac{\Phi(I)}{|I|} = \frac{1}{|I|} \int\limits_I f = f(x,y).$$

Therefore $D\Phi(x,y) = f(x,y)$. It means that both the inequalities (11) hold for Φ.

Now we proof that Φ is $\mathcal{B}S$-continuous everywhere on K. Notice that for any $\delta > 0$, f is bounded and so Lebesgue integrable on $[\delta,1] \times [0,1]$. Then if $(x,y) \in K \setminus T$ and the intervals $I^{(k,m)}_{(x,y)}$ do not intersect T, then $\mathcal{B}S$-continuity (see (7)) follows from the absolute continuity of the Lebesgue integral. Now, if $I^{(k,m)}_{(x,y)} \cap T \neq \emptyset$ then we can write $I^{(k,m)}_{(x,y)} = I^{(k)}_0 \times I^{(m)}_j = (I^{(0)}_0 \times I^{(m)}_j) \setminus [(I^{(0)}_0 \setminus I^{(k)}_0) \times I^{(m)}_j]$. As $\Phi(I^{(0)}_0 \times I^{(m)}_j) = 0$ by definition then we have to estimate the value of Φ on $(I^{(0)}_0 \setminus I^{(k)}_0) \times I^{(m)}_j$. In the case in which k is fixed and m goes to infinity we can use once again the absolute continuity of the Lebesgue integral. If m and j are fixed and k goes to infinity, then $(x,y) \in T$, and for $k = 2s$ we have $\Phi(I^0_0 \setminus I^{(2s)}_0 \times I^{(m)}_j) = 0$ according to (13). If $k = 2s+1$ then $I^{(0)}_0 \setminus I^{(2s+1)}_0 = I^{(2s+1)}_1 \cup (I^{(0)}_0 \setminus I^{(2s)}_0)$ and so $\Phi(I^{(0)}_0 \setminus I^{(2s+1)}_0 \times I^{(m)}_j) = \Phi(I^{(2s+1)}_1 \times I^{(m)}_j) = \int_{I^{(2s+1)}_1 \times I^{(m)}_j} f$. If $2s+1 > m$ then $I^{(m)}_j$ can be represented as a union of even number of intervals of rank $2s+1$ and the construction of the function f implies $\int_{I^{(2s+1)}_1 \times I^{(m)}_j} f = 0$. So (7) is always fulfilled. Therefore the function Φ is both $\mathcal{B}S$-major and $\mathcal{B}S$-minor function for f and consequently f is $\mathcal{B}S$-integrable according to Definitions 5 and 6. This completes the proof of the theorem. $\square$

According to Theorem 1 $H_{\mathcal{B}}$-integral is $\mathcal{B}S$-continuous. But we do not know whether it can be defined by Perron method using $\mathcal{B}S$-continuous major and minor functions. In this connection we are leaving open the following problem:

Problem 1. *Is any $H_{\mathcal{B}}$-integrable function $P_{\mathcal{B}}S$-integrable?*

4. Application to Walsh and Haar series

We apply now the $P_{\mathcal{B}}S$-integral to solves the coefficients problem for multidimensional Walsh and Haar series which are convergent outside some U-sets. We recall the definitions (see [4,15]).

First we define the *Rademacher functions* r_n, $n = 0,1,\ldots,$ on $[0,1]$ putting

$$r_n(x) = sign \sin\big(2^{n+1}\pi x\big) \quad \text{if } x \in (0,1),$$

$r_n(0) = 1$ and $r_n(1) = -1$. Note that the function r_n is constant on interior of each dyadic interval of rank $n+1$.

The *Walsh functions* are defined as products of Rademacher functions. We use the dyadic representation for $n \ge 0$:

$$n = \sum_{i=0}^{\infty} n_i 2^i,$$

where $n_i = 0$ or 1 and the sum is in fact finite, and we put

$$w_n(x) := \prod_{i=0}^{\infty} \big(r_i(x)\big)^{n_i}.$$

In particular $w_0 \equiv 1$.

Now we define the *Haar functions* on $[0,1]$. Put $\chi_0(x) \equiv 1$. If $n = 2^k + i$, $k = 0, 1, \dots$, $i = 0, \dots, 2^k - 1$, we put

$$\chi_n(x) := \begin{cases} 2^{k/2}, & \text{if } x \in (\frac{2i-2}{2^{k+1}}, \frac{2i-1}{2^{k+1}}), \\ -2^{k/2}, & \text{if } x \in (\frac{2i-1}{2^{k+1}}, \frac{2i}{2^{k+1}}), \\ 0, & \text{if } x \in (0,1) \setminus [\frac{2i-2}{2^{k+1}}, \frac{2i}{2^{k+1}}], \end{cases}$$

and we agree that at each point of discontinuity $\chi_n(x) = \frac{1}{2}(\chi_n(x+0) + \chi_n(x-0))$ and that at $x = 0$ and $x = 1$ Haar functions are continuous from the right and from the left, respectively.

An *m-dimensional Walsh* and *Haar series* are defined by

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}} w_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \dots \sum_{n_m=0}^{\infty} a_{n_1,\dots,n_m} \prod_{i=1}^{m} w_{n_i}(x_i) \tag{16}$$

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \dots \sum_{n_m=0}^{\infty} b_{n_1,\dots,n_m} \prod_{i=1}^{m} \chi_{n_i}(x_i) \tag{17}$$

where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are real numbers. It follows from the above definitions that for $\mathbf{n} = (n_1, ..., n_m)$ with $2^{k_j - 1} \le n_j < 2^{k_j}$, $j = 1, ..., m$, the functions $\chi_{\mathbf{n}}$ and $w_{\mathbf{n}}$ are constant in the interior of each dyadic interval of rank $\mathbf{k} = (k_1, ..., k_m)$. Moreover, with the same notation, the functions $\chi_{\mathbf{n}}$ are supported by some intervals of rank $\mathbf{k} - \mathbf{1} = (k_1 - 1, ..., k_m - 1)$.

If $\mathbf{N} = (N_1, \dots, N_m)$, then the $\mathbf{N}$*th rectangular partial sum* $S_{\mathbf{N}}$ of series (16) (resp., (17)) at a point $\mathbf{x} = (x_1, \dots, x_m)$ is

$$S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \dots \sum_{n_m=0}^{N_m-1} a_{\mathbf{n}} w_{\mathbf{n}}(\mathbf{x}) \quad \left(\text{resp., } S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \dots \sum_{n_m=0}^{N_m-1} b_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x})\right).$$

The series (16) (or (17)) *rectangularly converges* to sum $S(\mathbf{x})$ at a point $\mathbf{x}$ and we write $\lim_{\mathbf{N}\to\infty} S_{\mathbf{N}}(\mathbf{x}) = S(\mathbf{x})$ if

$$S_{\mathbf{N}}(\mathbf{x}) \to S(\mathbf{x}) \quad \text{as } \min_i \{N_i\} \to \infty.$$

In the rest of the paper, to simplify calculation, we shall formulate and proof most of the results for the two-dimensional case, but all of them are true for any dimension.

The following propositions were proved in [19,18], respectively.

Proposition 1. *If a two-dimensional series* (16) *is rectangular convergent everywhere on the "cross"* $\{a \times [0,1]\} \cap \{[0,1] \times b\}$*, where* $(a,b) \in K$*,* $a, b \notin Q_d$*, except a countable set then for this series*

$$\lim_{i+j\to\infty} a_{i,j} = 0. \tag{18}$$

Proposition 2. *If a two-dimensional series* (17) *is rectangular convergent on the "cross"* $\{a \times [0,1]\} \cap \{[0,1] \times b\}$*,* $(a,b) \in K$*, everywhere except a countable set* E *and at each point of* E *we have*

$$\lim_{k,l\to\infty} \frac{b_{n_k,m_l} \chi_{n_k,m_l}(x,y)}{2^k 2^l} = 0, \tag{19}$$

then for this series

$$\lim_{k+l\to\infty} \frac{b_{n_k,m_l}\chi_{n_k,m_l}(a,b)}{2^k 2^l} = 0 \tag{20}$$

where $2^{k-1} \le n_k < 2^k$, $2^{l-1} \le m_l < 2^l$.

Note that (20) and (19) are in fact meaningful only for those indexes n_k, m_l for which the support of function χ_{n_k,m_l} contains the point (x,y).

On the basis of these propositions it was proved in fact in [19] that Z (and also Y) is U-set for rectangular convergent multiple Walsh series (see also [10]). So it makes sense to state a problem of recovering the coefficients of those series from their sums defined outside of these U-sets. As for Haar series, non-empty U-sets exist only under additional assumptions of the type (19) or (20). Namely, Z is U-sets for Haar series under condition, that (20) holds everywhere. Under weaker assumption (19) on the exceptional set only countable sets are U-sets for rectangular convergent Haar series. Note that for ρ-regular convergent Haar series, with ρ close to 1, even the empty set is not U set (see [12]).

A standard method (see [21]) of application of the dyadic derivative and the dyadic integral to the theory of Walsh and Haar series is based on the fact that for the partial sums $S_{2^{\mathbf{k}}}$ of those series (here $2^{\mathbf{k}}$ stand for $(2^{k_1},\ldots,2^{k_m})$), the integral $\int_{I_{\mathbf{j}}^{(\mathbf{k})}} S_{2^{\mathbf{k}}}$ defines an additive $\mathcal{B}$-interval function $\psi(I)$ on the family $\mathcal{I}$ of all dyadic intervals. In dyadic analysis the function ψ is referred to as the *quasi-measure generated by the series* (see [15,32]). Since the sum $S_{2^{\mathbf{k}}}$ is constant on interior of each $I_{\mathbf{j}}^{(\mathbf{k})}$ we get

$$S_{2^{\mathbf{k}}}(\mathbf{x}) = \frac{1}{|I_{\mathbf{j}}^{(\mathbf{k})}|}\int\limits_{I_{\mathbf{j}}^{(\mathbf{k})}} S_{2^{\mathbf{k}}} = \frac{\psi(I_{\mathbf{j}}^{(\mathbf{k})})}{|I_{\mathbf{j}}^{(\mathbf{k})}|} \tag{21}$$

for any point $\mathbf{x} \in int(I_{\mathbf{j}}^{(\mathbf{k})})$.

In fact any additive $\mathcal{B}$-interval function ψ defines Walsh or Haar series for which it is a quasi-measure and (21) holds. So we have one-one correspondence between family of additive $\mathcal{B}$-interval functions and family of Walsh or Haar series.

The equality (21) obviously gives a relation between $\mathcal{B}$-differentiability of ψ at $\mathbf{x}$ and convergence of the series. In particular, at least at the points $\mathbf{x} \in \mathbf{K} \setminus \mathbf{Z}$, we get

$$\lim_{\mathbf{k}\to\infty} S_{2^{\mathbf{k}}}(\mathbf{x}) = D_{\mathcal{B}}\psi(\mathbf{x}), \tag{22}$$

and therefore the convergence of the series (16) (or (17)) at such points $\mathbf{x}$ to a sum $f(\mathbf{x})$ implies $\mathcal{B}$-differentiability of the function ψ at $\mathbf{x}$ with $f(\mathbf{x})$ being the value of $\mathcal{B}$-derivative.

Now we consider continuity properties of the quasi-measure. We use below the following generalization of Toeplitz method of summation for two-dimensional sequences.

Definition 7. Let a four-dimensional sequence $A = \{\alpha_{k,l,i,j}\}$ be given such that

1) $\lim_{k+l\to\infty} \alpha_{k,l,i,j} = 0$ for all i,j,
2) $N = \sup_{k,l} N_{k,l} < \infty$,
3) $\lim_{k+l\to\infty} A_{k,l} = 1$,

where $N_{k,l} = \sum_{i,j=0}^{\infty} |\alpha_{k,l,i,j}|$, $A_{k,l} = \sum_{i,j=0}^{\infty} \alpha_{k,l,i,j}$. We say that a double sequence $\{s_{i,j}\}$ is *A-summable* to σ if the sequence

$$\sigma_{k,l} := \lim_{\nu\to\infty} \sum_{i+j\le\nu} \alpha_{k,l,i,j} s_{i,j}$$

converges to σ when $k+l\to\infty$.

Lemma 9. *Let a double sequence $s_{i,j}$ be such that $\lim_{i+j\to\infty} s_{i,j} = 0$. Then $\lim_{k+l\to\infty} \sigma_{k,l} = 0$, where $\sigma_{k,l}$ is given by Definition 7 with a sequence $\alpha_{k,l.i.j}$ satisfying* 1) *and* 2).

Proof. Take $\eta > 0$ and choose ν such that $|s_{i,j}| < \frac{\eta}{2N}$ if $i+j > \nu$. Having fixed such a ν and using property 1) of $\{\alpha_{k,l,i,j}\}$, we can choose p such that for $k+l > p$ we have $\sum_{i+j\le\nu} |\alpha_{k,l,i,j}s_{i,j}| < \frac{\eta}{2}$. So, having in mind also property 2), we get

$$\left|\sum_{i,j=0}^{\infty} \alpha_{k,l,i,j}s_{i,j}\right| \le \sum_{i+j\le\nu} |\alpha_{k,l,i,j}s_{i,j}| + \left(\sum_{i+j>\nu}^{\infty} |\alpha_{k,l,i,j}|\right)\frac{\eta}{2N} \le \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

for all $k+l > p$. $\square$

Lemma 10. *If the coefficients of two-dimensional series* (16) *satisfy the condition* (18)*, then at each point $(x,y)\in K$ the quasi-measure ψ is $\mathcal{BS}$-continuous, i.e.,* (7) *holds everywhere on K.*

Proof. We have for $(x,y)\in I^{(k,l)}_{(x,y)}$

$$\psi\big(I^{(k,l)}_{(x,y)}\big) = \int\limits_{I^{(k,l)}_{(x,y)}} S_{2^k,2^l} = \sum_{i,j=0}^{2^k-1,2^l-1} \int\limits_{I^{(k,l)}_{(x,y)}} a_{i,j}w_{i,j}(t,s)dtds = \frac{\sum_{i,j=0}^{2^k-1,2^l-1} a_{i,j}w_{i,j}(x,y)}{2^k2^l}$$

Now we put $s_{i,j} = a_{i,j}w_{i,j}(x,y)$ and using Proposition 1 we get $\lim_{i+j\to\infty} s_{i,j} = 0$. Put also $\alpha_{k,l,i,j} = \frac{1}{2^k2^l}$ if $0\le i\le 2^k-1$, $0\le j\le 2^l-1$ and $\alpha_{k,l,i,j} = 0$ if $i\ge 2^k$ or $j\ge 2^l$. We get $N_{k,l} = \sum_{i,j=0}^{\infty} \frac{1}{2^k2^l} = \sum_{i,j=0}^{2^k-1,2^l-1} \frac{1}{2^k2^l} = 1$. So the conditions of Lemma 9 are fulfilled and this implies that (7) holds for the function ψ. $\square$

Lemma 11. *If the coefficients of two-dimensional series* (17) *satisfy the condition* (20) *at a point $(x,y)\in K$, then at this point the quasi-measure ψ is $\mathcal{BS}$-continuous, i.e.,* (7) *holds at (x,y).*

Proof. We have for $(x,y)\in I^{(k,l)}_{(x,y)}$

$$\psi\big(I^{(k,l)}_{(x,y)}\big) = \int\limits_{I^{(k,l)}_{(x,y)}} S_{2^k,2^l} = \sum_{i,j=0}^{k,l} \int\limits_{I^{(k,l)}_{(x,y)}} b_{n_j,m_j}\chi_{n_j,m_j}(t,s)dtds = \sum_{i,j=0}^{k,l} b_{n_i,m_j}\chi_{n_i,m_j}(x,y)|I^{(i,j)}|\cdot\frac{|I^{(k,l)}|}{|I^{(i,j)}|}$$

Now we put $s_{i,j} = b_{n_i,m_j}\chi_{n_i,m_j}(x,y)|I^{(i,j)}|$. According to Proposition 2 we have $\lim_{i+j\to\infty} s_{i,j} = 0$. Put also $\alpha_{k,l,i,j} = \frac{|I^{(k,l)}|}{|I^{(i,j)}|} = 2^{i+j-k-l}$ if $0\le i\le k$, $0\le j\le l$ and $\alpha_{k,l,i,j} = 0$ if $i\ge k$ or $j\ge l$. We get $N_{k,l} = \sum_{i,j=0}^{\infty} \alpha_{k,l,i,j} = \sum_{i,j}^{k,l} 2^{i+j-k-l} \le 4$. So all the conditions of Lemma 9 are fulfilled and (7) holds for the function ψ. $\square$

Note that the above statement is not true for Walsh series which are convergent with respect to regular rectangulars, for example with respect to cubes, even under assumption of convergence everywhere on K (see [14]).

The following statement is essential for establishing that a given Walsh or Haar series is the Fourier series in the sense of some general integral (see for example [21]; a proof, in the one-dimensional version, can be found in [4, Th. 3.1.8]).

Proposition 3. *Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{I}$. Assume a series of the form (16) or (17) is given. Let a $\mathcal{B}$-interval function ψ be the quasi-measure generated by this series and (21) holds. Then this series is the Fourier series of an $\mathcal{A}$-integrable function f if and only if $\psi(I) = (\mathcal{A}) \int_I f$ for any $\mathcal{B}$-interval I.*

In view of (22) and the above proposition, in order to solve the coefficient problem it is enough to show that the quasi-measure ψ generated by Haar or Walsh series is the indefinite integral of its $\mathcal{B}$-derivative which exists at least on $K \setminus Z$. By this we reduce the problem of recovering the coefficients to the corresponding theorem on recovering the primitive with appropriate continuity assumptions which can be obtained either from a convergence condition or from some additional growth assumptions imposed on the coefficients of the series.

In the one-dimensional case $Z = Q_d$, that is the exceptional set Z (and Y) is in fact countable. Moreover $\mathcal{B}$-continuity everywhere on $[0,1]$ follows from the condition $\lim_{n\to\infty} a_n = 0$ (which in turn is a consequence of the convergence of the series at least at one dyadic-irrational point). So we can apply Corollary 1 to get the following known result (see [26, Th. 14.10]):

Theorem 5. *If the series (16) (in one dimension) is convergent to a sum f outside a countable set, then f is $H_{\mathcal{B}}$-integrable and (16) is the Fourier–Walsh series of f, i.e.,*

$$a_n = (H_{\mathcal{B}}) \int\limits_{[0,1]} f w_n.$$

In multidimensional case we have to use Theorem 3 to get

Theorem 6. *If a two-dimensional series (16) is rectangular convergent to a sum f everywhere in $K \setminus Z$, then f is $P_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P_{\mathcal{B}}S$-Fourier coefficients of f.*

Proof. Take any $(a, b) \in K \setminus Z$. Then intersection of the cross $\{a \times [0,1]\} \cap \{[0,1] \times b\}$ with Z is countable, and by Proposition 1 condition (18) holds. Then by Lemma 10 the quasi-measure ψ generated by the series (16) is $\mathcal{B}S$-continuous everywhere in K. Moreover, the equality (22) implies

$$\lim_{\mathbf{k}\to\infty} S_{2^{\mathbf{k}}}(\mathbf{x}) = D_{\mathcal{B}}\psi(\mathbf{x}) = f(\mathbf{x})$$

everywhere on $K \setminus Z$. Then application of Theorem 3 and Proposition 3 completes the proof. □

We can enlarge the exceptional set Z here by replacing it by the set Y defined in (3) (see Remark 1). So we get

Theorem 7. *If the series (16) is rectangular convergent to a finite function f everywhere in $K \setminus Y$, then f is $P^Y_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P^Y_{\mathcal{B}}S$-Fourier coefficients of f.*

In the same way using Proposition 2 and Lemma 11 we obtain

Theorem 8. *If a two-dimensional series* (17) *is rectangular convergent to a sum f everywhere in K outside a countable set E and* (19) *holds everywhere on E then f is $P_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P_{\mathcal{B}}S$-Fourier coefficients of f.*

Note that in the above theorem we can omit condition (19) if we assume that the series (17) is convergent everywhere on K.

Analyzing the proof of the above theorem and the one of Lemma 10 we note that the convergence everywhere of the series has been used in order to obtain the condition (20) on coefficients of the series which in turn imply $\mathcal{B}S$-continuity everywhere. So we can weaken the assumption of convergence in the formulation of Theorem 8 by supposing a priori that the condition (20) are fulfilled. In this way we can obtain the following version of Theorem 8.

Theorem 9. *If the series* (17) *is rectangular convergent to a sum f everywhere in $K \setminus Z$ and the coefficients of the series satisfy everywhere the condition* (20)*, then f is $P_{\mathcal{B}}S$-integrable on K and the coefficients of the series are $P_{\mathcal{B}}S$-Fourier coefficients of f.*

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