# NONLINEAR ELLIPTIC EQUATIONS WITH ASYMMETRIC ASYMPTOTIC BEHAVIOR AT $\pm\infty$

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ABSTRACT. We consider a nonlinear, nonhomogeneous Dirichlet problem with re-action which is asymptotically superlinear at $+\infty$ and sublinear at $-\infty$. Using minimax methods together with suitable truncation techniques and Morse theory, we show that the problem has at least three nontrivial solutions one of which is negative.

## 1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^2-$boundary $\partial\Omega$ and $2 < p < +\infty$. In this paper, we study the following nonlinear nonhomogeneous Dirichlet problem

(P)
$$\begin{cases} -\Delta_p u - \Delta u = f(z,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, $\Delta_p$ denotes the $p-$Laplace differential operator defined by

$$\Delta_p u := \operatorname{div}(\|\nabla u\|^{p-2}\nabla u),$$

for all $u \in W_0^{1,p}(\Omega)$. The reaction $f(z,x)$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}^N$, $f(\cdot,x)$ is measurable and for a.a. $z \in \Omega$, $f(z,\cdot)$ is continuous). The aim of this work is to prove a multiplicity theorem when the nonlinearity $f(z,\cdot)$ exhibits an asymmetric asymptotic behavior as $x \to \pm\infty$. More precisely, we assume that for a.a. $z \in \Omega$ $f(z,\cdot)$ is $(p-1)-$sublinear near $-\infty$ while $f(z,\cdot)$ is $(p-1)-$superlinear near $+\infty$. Two additional special features of our work are that asymptotically at $-\infty$ we allow for resonance to occur with respect to the principal eigenvalue $\widehat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$, while in the positive semiaxis, in order to express the $(p-1)-$superlinearity of $f(z,\cdot)$, we do not employ the usual in such cases Ambrosetti-Rabinowitz condition and instead we use a weaker condition which incorporates in our framework superlinear nonlinearities with "slower" growth near $+\infty$.

Semilinear problems driven by the Laplace differential operator and with an asymmetric nonlinearity of this kind, were investigated by Arcoya-Villegas [1], de Figueiredo-Ruf [2], Motreanu-Motreanu-Papageorgiou [3] and Perera [4]. Extensions to a class

of parametric nonlinear elliptic equations driven by the $p$−Laplacian can be found in the work of Motreanu-Motreanu-Papageorgiou [5]. In de Figueiredo-Ruf [2] and Perera [4], $N = 1$, that is the problem studied is an ordinary differential equation. In Arcoya-Villegas [1] and de Figueiredo-Ruf [2] the authors prove only existence theorems. Multiplicity theorems can be found in the works of Motreanu-Motreanu-Papageorgiou [3, 5] and Perera [4]. In [3, 4] the authors produce two nontrivial solutions and their hypotheses on the reaction do not allow for resonance to occur at $−\infty$. In [5], the problem is parametric and the authors show that for all small values of the parameter $\lambda > 0$, the problem has at least four nontrivial solutions. Again no resonance is permitted at $−\infty$ and the conditions on $f(z, \cdot)$ are more restrictive (they impose a sign condition on $f(z, \cdot)$).

In this paper, using variational methods based on the critical point theory, combined with suitable truncation and comparison techniques and Morse theory (critical groups), we show the existence of three nontrivial solutions. At $−\infty$, we permit resonance with respect to the principal eigenvalue $\widehat{\lambda}_1(p) > 0$.

In the next section for the convenience of the reader, we fix our notation and terminology and recall the main mathematical tools which we use in the sequel. In Section 3, using primarily variational arguments, we produce two nontrivial solutions, one of which is negative. In Section 4, by strengthening the regularity of $f(z, \cdot)$ and the condition near zero and by using also Morse theory, we produce a third nontrivial solution, for a three solutions multiplicity theorem.

## 2. Mathematical Background

Let $X$ be a Banach space and $X^*$ its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(X^*, X)$. Given $\varphi \in C^1(X)$, we say that $\varphi$ satisfies the "Cerami condition" (the "C-condition" for short), if the following holds: "Every sequence $\{x_n\}_{n\geq 1} \subseteq X$ s.t.$\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|x_n\|)\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This type of compactness-type condition on $\varphi$ leads to a deformation theorem from which we can derive the minimax theory of the critical values of $\varphi$. A basic result in this theory is the mountain pass theorem of Ambrosetti-Rabinowitz [6]. Here we state this result in a slightly more general form.

**Theorem 2.1.** *If $\varphi \in C^1(X)$ satisfies the C-condition, $x_0$, $x_1 \in X$, with $\|x_0 − x_1\| > \rho > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x − x_0\| = \rho\} = \eta_\rho,$$
$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t))$$

*where $\Gamma = \{\gamma \in C^([0, 1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$ and $\varphi$ satisfies the C-condition, then $c \geq \eta_\rho$ and $c$ is a critical value of $\varphi$.*

In the study of problem (P) we will use the Sobolev spaces $W_0^{1,p}(\Omega)$, $H_0^1(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0 \right\}.$$

This latter space is an ordered Banach space with positive cone

$$C_+ = \{ u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \}.$$

This cone has nonempty interior given by

$$intC_+ = \{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega \},$$

where $n(\cdot)$ denotes the outward unit normal to $\partial\Omega$.

Suppose that $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function with $p$−subcritical growth in the $x$−variable, that is there exist $\alpha_0 \in L^\infty(\Omega)_+$ and $1 < r < p^*$, s.t.

$$|f_0(z,x)| \leq \alpha_0(z)(1 + |x|^{r-1}) \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R},$$

where $p^* = \frac{pN}{N-p}$, if $p < N$ and $p^* = +\infty$, if $p \geq N$. We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the $C^1$ functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) := \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0(z, u(z))dz, \text{ for all } u \in W_0^{1,p}(\Omega).$$

The following is a particular case of a more general result of Gasinski-Papageorgiou [7]. The result is essentially a consequence of the nonlinear regularity theory due to Lieberman [8]. In the sequel, by $\|\cdot\|$ we denote the norm on the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p, \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

**Proposition 2.1.** *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$−minimizer of $\varphi_0$, i.e., there exists $\rho_0 > 0$ s.t.*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h), \quad \text{for all } h \in C_0^1(\overline{\Omega}) \text{ with } \|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0,$$

*then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$ and it is a local $W_0^{1,p}(\Omega)$−minimizer of $\varphi_0$, i.e.,there exists $\rho_1 > 0$ s.t.*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h), \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \|h\|_{W_0^{1,p}(\Omega)} \leq \rho_1.$$

Let $r \in (0, +\infty)$ and consider the map $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$ $\left(\frac{1}{r} + \frac{1}{r'} = 1\right)$ defined by

$$(1) \qquad \langle A_r(u), v \rangle = \int_\Omega \|Du\|^{r-2}(Du, Dv)_{\mathbb{R}^N}dz, \quad u, v \in W_0^{1,p}(\Omega).$$

If $r = 2$, then we set $A_2 = A$. Clearly $A \in \mathfrak{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$. We have the following well-known result (see, for example, Papageorgiou-Kyritsi [9, p.314]),

**Proposition 2.2.** *If $r \in (1, +\infty)$ and $A_r : W_0^{1,r}(\Omega) \to W_0^{1,r}(\Omega)^*$ is defined by (1) then $A_r$ is bounded ( that is maps bounded sets to bounded sets), demicontinuous, monotone (hence maximal monotone too) and of type $(S_+)$, i.e., if $x_n \rightharpoonup x$ in $W_0^{1,r}(\Omega)$ and*

$$\limsup_{n \to +\infty} \langle A(x_n), x_n - x \rangle \leq 0,$$

*then $x_n \to x$ in $W_0^{1,r}(\Omega)$.*

Next we recall some basic facts about the spectrum of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$. So we consider the following nonlinear eigenvalue problem

$$(2) \qquad \begin{cases} -\Delta_r u = \widehat{\lambda} |u|^{r-2} u & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue of $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$, if problem (2) admits a nontrivial solution $\widehat{u} \in W_0^{1,r}(\Omega)$, known as an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$. There exists a smallest eigenvalue for problem (2) denoted by $\widehat{\lambda}_1(r)$. We know that $\widehat{\lambda}_1(r)$ is positive $(\widehat{\lambda}_1(r) > 0)$, isolated (that is, there exists $\epsilon > 0$ such that the interval $\left[\widehat{\lambda}_1(r), \widehat{\lambda}_1(r) + \epsilon\right)$ does not contain any eigenvalue) and simple (if $\hat{u}, \hat{v}$ are two eigenfunctions corresponding to $\widehat{\lambda}_1(r)$, then $\hat{u} = \xi\hat{v}$ for some $\xi \in \mathbb{R} \setminus \{0\}$). The eigenvalue $\widehat{\lambda}_1(r)$ has the following variational characterization

$$(3) \qquad \widehat{\lambda}_1(r) = \inf\left\{ \frac{\|Du\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0 \right\}.$$

The infimum in (3) is realized on the corresponding one-dimensional eigenspace. Also, from (3) it follows that the elements of this eigenspace do not change sign. Let $\hat{u}_1(r)$ denote the $L^r$−normalized (that is $\|\hat{u}_1(r)\|_r = 1$) nonnegative eigenfunction corresponding to $\widehat{\lambda}_1(r) > 0$. The nonlinear regularity theory, (see [10, pp.737-738]) implies that $\hat{u}_1(r) \in int C_+$. We mention that any eigenfunction $\hat{u}$ corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1$ is nodal (that is, sign-changing) and belongs to $C_0^1(\overline{\Omega})$ (nonlinear regularity theory).

An easy consequence of the aforementioned properties of $\widehat{\lambda}_1(r) > 0$ and $\hat{u}_1(r) \in int C_+$, is the following lemma

**Lemma 2.1.** *If $\Theta \in L^\infty(\Omega)$ with $\Theta(z) \leq \hat{\lambda}_1(r)$ a.e. in $\Omega$ and this inequality is strict on a set of positive measure, then there exists $\xi^* > 0$ s.t.*

$$\|Du\|_r^r - \int_\Omega \Theta(z) |u(z)|^r dz \geq \xi^* \|Du\|_r^r \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

The Lusternik-Schnirelmann minimax theory implies that $\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$ admits a whole sequence $\{\widehat{\lambda}_k(r)\}_{k \geq 1}$ of distinct eigenvalues s.t. $\widehat{\lambda}_k(r) \uparrow +\infty$ as $k \to +\infty$. In general we do not know if this sequence exhausts the spectrum $\hat{\sigma}(r)$ of

$\left(-\Delta_r, W_0^{1,r}(\Omega)\right)$. This is the case, if $N = 1$ (ordinary differential equations) or if $r = 2$ (linear eigenvalue problem). In this latter case, we also consider the following weighted linear eigenvalue problem

$$(4) \qquad \begin{cases} -\Delta u = \widehat{\lambda} m(z) u & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $m \in L^\infty(\Omega)_+ \setminus \{0\}$. Such a problem has a sequence $\{\hat{\lambda}_k(2,m)\}_{k \geq 1}$ of distinct eigenvalues such that $\hat{\lambda}_k(2,m) \to +\infty$ as $k \to +\infty$. If $m \equiv 1$, then $\hat{\lambda}_k(2,m) = \hat{\lambda}_k(2)$. Let $E(\widehat{\lambda}_k(2,m))$ be the eigenspace corresponding to $\widehat{\lambda}_k(2,m)$. We know that each such eigenspace is finite dimensional and exhibits the "unique continuation property" (UCP, for short), i.e., if $u \in E(\widehat{\lambda}_k(2,m))$ and vanishes on a set of positive measure, then $u = 0$. Also, we have $E(\widehat{\lambda}_k(2,m)) \subseteq C_0^1(\overline{\Omega})$ and there are variational characterizations for all the eigenvalues, namely

$$(5) \qquad \hat{\lambda}_1(2,m) = \inf\left[\frac{\|Du\|_2^2}{\int_\Omega m u^2 dz} : u \in H_0^1(\Omega), u \neq 0\right]$$

and for $k \geq 2$, we have

$$\begin{aligned} \hat{\lambda}_k(2,m) &= \inf\left[\frac{\|Du\|_2^2}{\int_\Omega m u^2 dz} : u \in \overline{\bigoplus_{i \geq k+1} E(\widehat{\lambda}_i(2,m))}\right] \\ (6) \qquad &= \sup\left[\frac{\|Du\|_2^2}{\int_\Omega m u^2 dz} : u \in \bigoplus_{i=1}^{k} E(\widehat{\lambda}_i(2,m))\right]. \end{aligned}$$

Again in (5) the infimum is realized on $E(\widehat{\lambda}_1(2,m))$, while in (6) both the infimum and the supremum are realized on $E(\widehat{\lambda}_k(2,m))$. The map $m \to \widehat{\lambda}_k(2,m)$ is continuous from $L^\infty(\Omega)_+ \setminus \{0\}$ into $(0, +\infty)$ and the UCP together with (5) and (6) lead to the following monotonicity property "If $m(z) \leq \hat{m}(z)$ a.e. in $\Omega$ with strict inequality on a set of positive measure, then $\hat{\lambda}_k(2, \hat{m}) < \hat{\lambda}_k(2,m)$ for all $k \geq 1$."

Next let us recall some basic definitions and facts from Morse theory (critical groups). So, let $X$ be a Banach space and let $(Y_1, Y_2)$ be a topological pair s.t. $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the $k^{th}$ relative singular homology group with integer coefficients. For $k < 0$, we have $H_k(Y_1, Y_2) = 0$. Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets

$$\varphi^c := \{x \in X : \varphi(x) \leq c\}, \ K_\varphi := \{x \in X : \varphi'(x) = 0\}, \ K_\varphi^c := \{x \in K_\varphi : \varphi(x) = c\}.$$

Let $u \in X$ be an isolated critical point of $\varphi$ with $\varphi(u) = c$ (i.e. $u \in K_\varphi^c$). The critical groups of $\varphi$ at $u$ are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \geq 0,$$

where $U$ is a neighborhood of $u$ s.t. $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that this definition of critical groups is independent of the particular choice of the neighborhood $U$.

Suppose that $\varphi \in C^1(X)$ satisfies the "$C-$condition" and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of $\varphi$ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geq 0.$$

The second deformation theorem (see, for example, Gasinski-Papageorgiou [10, p. 628]), implies that the above definition of critical groups at infinity is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that $K_\varphi$ is finite. We set

$$M(t, u) = \sum_{k \geq 0} rank C_k(\varphi, u) t^k, \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi,$$

and

$$P(t, \infty) = \sum_{k \geq 0} rank C_k(\varphi, \infty) t^k, \quad \text{for all } t \in \mathbb{R}.$$

The Morse relation says that

(7) $$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t),$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

For more details see Mawhin-Willem [11] and Motreanu-Motreanu-Papageorgiou [12].

We conclude this section by fixing our notation. So, given $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then given $u \in W_0^{1,r}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$ and we have $u^\pm \in W_0^{1,r}(\Omega)$, $u = u^+ - u^-$ and $|u| = u^+ + u^-$.

By $|\cdot|_\mathbb{N}$ we denote the Lebesgue measure on $\mathbb{R}^N$. Also, if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Caratheodory function), then we define the Nemytskii operator corresponding to the function $h(\cdot, \cdot)$ by setting

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)), \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

## 3. Two nontrivial solution

In this section, using primarily variational methods, we produce two non-trivial solutions for problem (P), one of which is negative. To this end, we introduce the following hypotheses on the reaction $f(z, x)$:

$H_1 :$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function s.t. for a.a. $z \in \Omega$, $f(z, 0) = 0$ and
$(i)$ $|f(z, x)| \leq \alpha(z)(1 + |x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $\alpha \in L^\infty(\Omega)_+$, $p < r < p^*$;

($ii$) if $F(z,x) = \int_0^x f(z,s)ds$, then
$$\lim_{x \to +\infty} \frac{F(z,x)}{x^p} = +\infty, \text{uniformly for a.a. } z \in \Omega,$$

and there exist $\tau \in \left(\max\{1,(r-p)\frac{N}{p}\}, p^*\right)$ and $\beta_0 > 0$ s.t.

$$0 < \beta_0 < \liminf_{x \to +\infty} \frac{f(z,x)x - pF(z,x)}{x^\tau} \text{ uniformly for a.a. } z \in \Omega;$$

($iii$) there exist $\xi_0,\ \xi_1 > 0$ s.t.
$$-\xi_0 \le \liminf_{x \to -\infty} \frac{f(z,x)}{|x|^{p-2}x} \le \limsup_{x \to -\infty} \frac{f(z,x)}{|x|^{p-2}x} \le \widehat{\lambda}_1(p) \text{ uniformly for a.a. } z \in \Omega;$$
$-\xi_1 \le f(z,x)x - pF(z,x)$ for a.a.$z \in \Omega$, all $x \le 0$,
and for a.a. $z \in \Omega$, $f(z,\cdot)$ is lower locally Lipschitz, that is for each $K \subset (-\infty, 0]$ compact, we can find $\xi_K > 0$ s.t. $f(z,u) - f(z,y) \ge -\xi_K(u-y)$ for all $u,y \in K$ with $u > y$;

($iv$) there exist $\beta_1 \in L^\infty(\Omega)_+$, $\beta_1(z) \ge \widehat{\lambda}_1(2)$ a.e. in $\Omega$ with strict inequality on a set of positive measure and $\beta_2 > 0$ s.t.
$$\beta_1(z) \le \liminf_{x \to 0} \frac{f(z,x)}{x} \le \limsup_{x \to 0} \frac{f(z,x)}{x} \le \beta_2, \text{uniformly for a.a. } z \in \Omega.$$

**Remark 3.1.** Hypotheses $H_1(ii),(iii)$ express the asymmetric features of the reaction $f(z,\cdot)$. Indeed, according to hypothesis $H_1(ii)$ $f(z,\cdot)$ is $(p-1)-$superlinear near $+\infty$, while hypothesis $H_1(iii)$ implies that $f(z,\cdot)$ is $(p-1)-$sublinear near $-\infty$. Note that for the superlinearity condition, we do not employ the usual in such cases Ambrosetti-Rabinowitz condition. We recall that the Ambrosetti-Rabinowitz condition (unilateral version, since we require "superlinear" growth only in the positive direction), says that there exist $q > p$ and $M > 0$ s.t.

(a) $0 < qF(z,x) \le f(z,x)x$, for a.a. $z \in \Omega$, all $x \ge M$;
(b) $essinf_\Omega F(\cdot, M) > 0$, (see also [13]).

Integrating $(a)$ and using $(b)$, we obtain the following weaker condition

(8) $\qquad c_0 x^q \le F(z,x)$ for a.a. $z \in \Omega$, all $x \ge M$ with $c_0 > 0$.

Evidently (8) implies the much weaker condition

$$\lim_{x \to +\infty} \frac{F(z,x)}{x^p} = +\infty, \text{uniformly for a.a. } z \in \Omega.$$

Hypothesis $H_1(ii)$ is weaker than the AR-condition. Indeed, suppose that $(a)$ and $(b)$ hold. We may assume that $q > \max\{1,(r-p)\frac{N}{p}\}$. Then

$$\frac{f(z,x)x - pF(z,x)}{x^q} = \frac{f(z,x)x - qF(z,x)}{x^q} + \frac{(q-p)F(z,x)}{x^q} \ge c_0,$$

for a.a. $z \in \Omega$, all $x \ge M$. So, hypothesis $H_1(ii)$ is satisfied.
Hypothesis $H_1(iii)$ permits resonance to occur at $-\infty$ with respect to the principal

eigenvalue $\hat{\lambda}_1(p) > 0$. In fact, as we will see in the process of the proof of Proposition 3.1, this resonance occurs from the left of $\hat{\lambda}_1(p)$ and this makes the negative truncation of the energy functional coercive. This fact via the direct method will lead to the existence of a negative solution. The possibility of resonance at $-\infty$, necessitates the second condition in hypothesis $H_1(iii)$ in order to be able to prove the compactness condition (the C-condition) for the energy functional. In the literature, for similar situations of resonance, we encounter the condition

$$(9) \qquad \lim_{x \to -\infty} [f(z,x)x - pF(z,x)] = +\infty, \text{ uniformly for a.a. } z \in \Omega,$$

see, for example, [12]. Therefore, it is clear that our hypothesis

$$-\xi_1 \leq f(z,x)x - pF(z,x), \text{for a.a. } z \in \Omega, \text{ all } x \leq 0,$$

is much weaker than (9).

Finally, we explicitly observe that hypotheses $H_1(i),(iv)$ imply that

$$(10) \qquad f(z,x)x \geq c_1 x^2 - c_2|x|^r, \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } c_1, c_2 > 0.$$

**Example 3.1.** The following function satisfies hypotheses $H_1$. For the sake of simplicity we drop the $z-$dependence:

$$f(x) = \begin{cases} \hat{\lambda}_1(p)|x|^{p-2}x + \xi_-, & \text{if } x < -1; \\ \beta x, & \text{if } x \in [0,1]; \\ \xi_+ x^{p-1}(\ln x + \frac{1}{p}), & \text{if } 1 < x, \end{cases}$$

with $\xi_- = \hat{\lambda}_1(p) - \beta$, $\beta > \hat{\lambda}_1(2)$ and $\xi_+ = \beta p$. Note that $f$ does not satisfy the AR-condition (see $(a),(b)$ above).

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (P) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u(z))dz, \text{ for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\varphi \in C^1(W_0^{1,p}(\Omega))$.

**Proposition 3.1.** *If hypotheses $H_1$ hold, then the functional $\varphi$ satisfies the $C-$condition.*

*Proof.* Let $\{u_n\}_{n \geq 1} \subset W_0^{1,p}(\Omega)$ be a sequence s.t.

$$(11) \qquad |\varphi(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \geq 1,$$

$$(12) \qquad (1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W_0^{1,p}(\Omega)^* \text{ as } n \to \infty.$$

From (12) we have

$$(13) \qquad \left| \langle A_p(u_n), h \rangle + \langle A(u_n), h \rangle - \int_\Omega f(z,u_n)h\,dz \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|},$$

for all $h \in W_0^{1,p}(\Omega)$, with $\varepsilon_n \to 0^+$. In (13), we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$(14) \qquad \|Du_n^-\|_p^p + \|Du_n^-\|_2^2 + \int_\Omega f(z, -u_n^-)u_n^- dz \leq \varepsilon_n, \text{ for all } n \geq 1.$$

By contradiction, suppose that $\{u_n^-\} \subset W_0^{1,p}(\Omega)$ is unbounded. By passing to a subsequence if necessary, we may assume that $\|u_n^-\| \to +\infty$. Let $y_n = \dfrac{u_n^-}{\|u_n^-\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$(15) \qquad y_n \rightharpoonup y \quad \text{in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \quad \text{in } L^p(\Omega).$$

From (14) we have

$$(16) \qquad \|Dy_n\|_p^p + \frac{1}{\|u_n^-\|^{p-2}}\|Dy_n\|_2^2 \leq \frac{\varepsilon_n}{\|u_n^-\|^p} - \int_\Omega \frac{f(z, -u_n^-)}{\|u_n^-\|^{p-1}}(y_n)dz, \text{ for all } n \geq 1.$$

Hypotheses $H_1(i)(iii)$ imply that

$$(17) \qquad |f(z, x)| \leq c_3(1 + |x|^{p-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \leq 0, \text{ with } c_3 > 0.$$

From (15) and (17) it follows that $\left\{ \dfrac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}} \right\}_{n \geq 1} \subset L^{p'}(\Omega)$ is bounded and so we may assume that

$$(18) \qquad \frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}} \rightharpoonup g \in L^{p'}(\Omega).$$

Reasoning as in Aizicovici-Papageorgiou-Staicu [14] (see the proof of Proposition 16) and using hypothesis $H_1(iii)$, we obtain

$$(19) \qquad g = \Theta y^{p-1}, \quad \text{with } -\xi_0 \leq \Theta(z) \leq \hat{\lambda}_1(p) \text{ a.e. in } \Omega.$$

Returning to (16), passing to the limit as $n \to \infty$ and using (15), (18), (19), we obtain

$$(20) \qquad \|Dy\|_p^p \leq \int_\Omega \Theta y^p dz.$$

First suppose that $\Theta \not\equiv \hat{\lambda}_1(p)$. Then from (20) and Lemma 2.1, we have $\xi^* \|Dy\|_p^p \leq 0$, hence $y = 0$. From (16), it follows that $Dy_n \to 0$ in $L^p(\Omega, \mathbb{R}^N)$, which implies that $y_n \to 0$ in $W_0^{1,p}(\Omega)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$.

Next suppose that $\Theta(z) \equiv \hat{\lambda}_1(p)$ a.e. in $\Omega$. Then from (20) and (3), we have that $\|Dy\|_p^p = \hat{\lambda}_1(p)\|y\|_p^p$ and this means that there exists some $\hat{\xi} \geq 0$ such that $y = \hat{\xi}\hat{u}_1(p)$ (recall that $y \geq 0$). If $\hat{\xi} = 0$, then $y = 0$ and as above we have $y_n \to 0$ in $W_0^{1,p}(\Omega)$, a contradiction. So, $y \in intC_+$ and we have

$$(21) \qquad u_n^-(z) \to \infty \text{ for a.a. } z \in \Omega.$$

For a.a. $z \in \Omega$ and all $s < 0$, we have

$$
\begin{aligned}
\frac{d}{ds} \frac{F(z,s)}{|s|^p} &= \frac{f(z,s)|s|^p - p|s|^{p-2}sF(z,s)}{|s|^{2p}} \\
&= \frac{|s|^{p-2}s[f(z,s)s - pF(z,s)]}{|s|^{2p}} \\
&= \frac{[f(z,s)s - pF(z,s)]}{|s|^p s} \\
&\leq \frac{-\xi_1}{|s|^p s} \quad (\text{see hypothesis } H_1(iii) \text{ and recall } s < 0).
\end{aligned}
$$

So, for a.a. $z \in \Omega$, all $x \leq y \leq 0$, we have that

$$
\text{(22)} \qquad \frac{F(z,y)}{|y|^p} - \frac{F(z,x)}{|x|^p} \leq -\frac{\xi_1}{p}\left[\frac{1}{|x|^p} - \frac{1}{|y|^p}\right].
$$

Note that hypothesis $H_1(iii)$ implies that

$$
\text{(23)} \qquad \limsup_{x \to -\infty} \frac{F(z,x)}{|x|^p} \leq \frac{\hat{\lambda}_1(p)}{p}, \quad \text{uniformly for a.a. } z \in \Omega.
$$

Hence, if in (22) we pass to the limit as $x \to -\infty$ and use (23), then one has

$$
\frac{F(z,y)}{|y|^p} - \frac{\hat{\lambda}_1(p)}{p} \leq \frac{\xi_1}{p}\frac{1}{|y|^p}, \quad \text{for a.a. } z \in \Omega, \text{ all } y < 0,
$$

that is

$$
\text{(24)} \qquad -\xi_1 \leq \hat{\lambda}_1(p)|y|^p - pF(z,y), \quad \text{for a.a. } z \in \Omega, \text{ all } y < 0.
$$

Also, by virtue of hypotheses $H_1(i)(ii)$, we can find $\xi_2 > \xi_1 > 0$ s.t.

$$
\text{(25)} \qquad f(z,y)y - pF(z,y) \geq -\xi_2, \text{for a.a. } z \in \Omega, \text{ all } y \geq 0.
$$

Then, in (13) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ and we obtain

$$
\text{(26)} \qquad -\|Du_n^+\|_p^p - \|Du_n^+\|_2^2 + \int_\Omega f(z,u_n^+)u_n^+ dz \leq \varepsilon_n \text{ for all } n \geq 1.
$$

Also from (11), we have

$$
\|Du_n^+\|_p^p + \|Du_n^-\|_p^p + \frac{p}{2}\|Du_n^+\|_2^2 + \frac{p}{2}\|Du_n^-\|_2^2 - \int_\Omega pF(z,u_n)dz \leq pM_1 \text{ for all } n \geq 1,
$$

which by (3) gives,

$$
\|Du_n^+\|_p^p + \frac{p}{2}\|Du_n^+\|_2^2 + \int_\Omega [\hat{\lambda}_1(p)(u_n^-)^p - pF(z,-u_n^-)]dz - \int_\Omega pF(z,u_n^+)dz + \frac{p}{2}\|Du_n^-\|_2^2 \leq pM_1.
$$

Therefore, by (24), for all $n \geq 1$, we get

$$
\text{(27)} \qquad \|Du_n^+\|_p^p + \frac{p}{2}\|Du_n^+\|_2^2 - \int_\Omega pF(z,u_n^+)dz + \frac{p}{2}\|Du_n^-\|_2^2 \leq M_2, \text{some } M_2 > 0.
$$

Adding (26) and (27), we obtain

$$\left(\frac{p}{2} - 1\right)\|Du_n^+\|_2^2 + \frac{p}{2}\|Du_n^-\|_2^2 + \int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]dz \leq M_3,$$

for some $M_3 > 0$ and all $n \geq 1$. Since $p > 2$, bearing in mind (3) and (25), for some $M_4 > 0$ and all $n \geq 1$ we have

$$(28) \qquad \int_\Omega (u_n^-)^2 dz \leq M_4.$$

From (21) and Fatou's Lemma, we have

$$\int_\Omega (u_n^-)^2 dz \to +\infty, \text{ as } n \to +\infty,$$

which contradicts (28). This proves that

$$(29) \qquad \{u_n^-\}_{n\geq 1} \subset W_0^{1,p}(\Omega) \text{ is bounded.}$$

From (11) and (29), we have

$$(30) \qquad \|Du_n^+\|_p^p + \frac{p}{2}\|Du_n^+\|_2^2 - \int_\Omega pF(z, u_n^+)dz \leq M_5, \text{for some } M_5 > 0, \text{ all } n \geq 1.$$

Adding (26) and (30) and since $p > 2$, we obtain

$$(31) \qquad \int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]dz \leq M_6, \text{for some } M_6 > 0, \text{ all } n \geq 1.$$

By virtue of hypotheses $H_1(i), (ii)$ we can find $\beta_1 \in (0, \beta_0)$ and $c_4 = c_4(\beta_1) > 0$ s.t.

$$\beta_1 x^\tau - c_4 \leq f(z, x)x - pF(z, x), \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0,$$

which clearly insures that, see (31),

$$(32) \qquad \{u_n^+\}_{n\geq 1} \subset L^\tau(\Omega) \text{ is bounded.}$$

From hypothesis $H_1(i), (ii)$ it is clear that we may assume that $\tau < r < p^*$. First suppose that $N \neq p$. We can find $t \in (0, 1)$ such that

$$(33) \qquad \frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}.$$

The interpolation inequality (see, for example, Gasinski-Papageorgiou [10, p.905]) implies

$$\|u_n^+\|_r \leq \|u_n^+\|_\tau^{1-t}\|u_n^+\|_{p^*}^t \text{ for all } n \geq 1.$$

Thus, from (32) and the Sobolev embedding theorem

$$(34) \qquad \|u_n^+\|_r^r \leq M_7\|u_n^+\|^{tr} \text{ for some } M_7 > 0, \text{ all } n \geq 1.$$

Hypothesis $H(f)(i)$ implies that we can find $c_5 > 0$ such that

$$(35) \qquad |f(z, x)x| \leq c_5(1 + |x|^r) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

In (13) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then, for all $n \geq 1$, one has

$$\|Du_n^+\|_p^p + \|Du_n^+\|_2^2 \leq c_6 + c_6 \int_\Omega f(z, u_n^+) u_n^+ dz,$$

$$\leq c_7(1 + \|u_n^+\|_r^r),$$

$$(36) \qquad\qquad\qquad \leq c_8(1 + \|u_n^+\|^{tr}),$$

where $c_6$, $c_7$ and $c_8$ are suitable positive constants. The hypothesis on $\tau$ implies that $tr < p$. Then, from (36) it follows that

$$(37) \qquad\qquad \{u_n^+\} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

If $N = p$, then by definition $p^* = +\infty$, while by the Sobolev embedding theorem the space $W_0^{1,p}(\Omega)$ is embedded (compactly) into $L^q(\Omega)$ for all $q \in [1, \infty)$. So, for the previous arguments to work, we need to replace $p^*$ by $q > r$ large in such a way that

$$tr = \frac{q(r - \tau)}{q - \tau}.$$

Then as above, we obtain that (37) holds. From (29) and (37) it follows that $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ is bounded and so we may assume that there exists $u \in W_0^{1,p}(\Omega)$ such that

$$(38) \qquad\qquad u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega).$$

We return to (13) and choose $h = u_n - u \in W_0^{1,p}(\Omega)$. Passing to the limit as $n \to \infty$ and using (38), we obtain

$$\lim_{n\to\infty} [\langle A_p(u_n), u_n - u \rangle + \langle A(u_n), u_n - u \rangle] = 0,$$

and since $A$ is monotone we have

$$\limsup_{n\to\infty} \langle A_p(u_n), u_n - u \rangle \leq 0,$$

so $u_n \to u$ in $W_0^{1,p}(\Omega)$ (see (38) and Proposition 2.2). This proves that the functional $\varphi$ satisfies the C-condition. $\qquad\square$

To produce a negative solution, we introduce the negative truncation of $f(z, \cdot)$. So, we define the following Caratheodory function

$$f_-(z, x) = f(z, -x^-), \text{ for all } (z, x) \in \Omega \times \mathbb{R},$$

we set $F_-(z, x) = \int_0^x f_-(z, s) ds$ and consider the $C^1$-functional $\varphi_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_-(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_-(z, u(z)) dz, \text{ for all } u \in W_0^{1,p}(\Omega).$$

Inspired by some of the arguments in the proof of Proposition 3.1, we can show the following result.

**Proposition 3.2.** *If hypotheses $H_1$ hold, then the functional $\varphi_-$ is coercive.*

*Proof.* We argue indirectly. So, suppose that $\varphi_-$ is not coercive. Then, we can find $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ and $M_8 > 0$ s.t.

$$(39) \qquad \|u_n\| \to +\infty \quad \text{and} \quad \varphi_-(u_n) \leq M_8, \text{ for all } n \geq 1.$$

We have

$$(40) \qquad \varphi_-(u_n) = \frac{1}{p}\|Du_n\|_p^p + \frac{1}{2}\|Du_n\|_2^2 - \int_\Omega F_-(z, u_n(z))dz \leq M_8, \text{ for all } n \geq 1.$$

Let $y_n = \dfrac{u_n}{\|u_n\|}$ for all $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$(41) \qquad y_n \rightharpoonup y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega).$$

From (40) we have

$$(42) \qquad \frac{1}{p}\|Dy_n\|_p^p + \frac{1}{2\|u_n\|^{p-2}}\|Dy_n\|_2^2 - \int_\Omega \frac{F_-(z, u_n(z))}{\|u_n\|^p}dz \leq \frac{M_8}{\|u_n\|^p}, \text{ for all } n \geq 1.$$

Hypotheses $H_1(i), (iii)$ imply that

$$|F_-(z, x)| \leq c_9(1 + |x|^p), \text{ for a.a. } z \in \Omega, \text{ all } x \leq 0 \text{ and some } c_9 > 0.$$

It follows that

$$\left\{\frac{F_-(\cdot, u_n(\cdot))}{\|u_n\|^p}\right\}_{n \geq 1} \subset L^1(\Omega) \text{ is uniformly integrable.}$$

Invoking the Dunford-Pettis theorem, we may assume (at least for a subsequence) that

$$(43) \qquad \frac{F_-(\cdot, u_n(\cdot))}{\|u_n\|^p} \rightharpoonup g \text{ in } L^1(\Omega).$$

Bearing in mind $H_1(iii)$ one has that there exists $\xi_3 > 0$ s.t.

$$(44) \quad -\xi_3 \leq \liminf_{x \to -\infty} \frac{F_-(z, x)}{|x|^p} \leq \limsup_{x \to -\infty} \frac{F_-(z, x)}{|x|^p} \leq \frac{\widehat{\lambda}_1(p)}{p} \text{ uniformly for a.a. } z \in \Omega.$$

From (43) and reasoning as in Aizicovici-Papageorgiou-Staicu [14] (see the proof of Proposition 16), we have

$$(45) \qquad g = \frac{1}{p}\Theta(y^-)^p, \text{with } -p\xi_3 \leq \Theta(z) \leq \hat{\lambda}_1(p) \text{ a.e. } z \in \Omega.$$

We return to (42), pass to the limit as $n \to +\infty$ and use (41), (43) and (45). We obtain

$$(46) \qquad \|Dy^-\|_p^p \leq \int_\Omega \Theta(y^-)^p dz \text{ (recall } 2 < p).$$

If $\Theta \not\equiv \hat{\lambda}_1(p)$, then from (46) and Lemma 2.1, we have $\xi^*\|Dy^-\|_p^p \leq 0$, hence $y \geq 0$. From (42), we have $\|Dy_n\|_p^p \leq \dfrac{pM_8}{\|u_n\|^p}$ for all $n \geq 1$. Therefore, we get that $y_n \to 0$ in

$W_0^{1,p}(\Omega)$ as $n \to +\infty$ which contradicts the fact that $\|y_n\| = 1$ for all $n \geq 1$.
Next suppose that $\Theta \equiv \hat{\lambda}_1(p)$. From (46) and (3), we have that

$$\|Dy^-\|_p^p = \hat{\lambda}_1(p)\|y^-\|_p^p,$$

and this produces some $\hat{\xi} \geq 0$ such that $y^- = \hat{\xi}\hat{u}_1(p)$. If $\hat{\xi} = 0$, then $y^- = 0$ and as
above we have $y_n \to 0$ in $W_0^{1,p}(\Omega)$, a contradiction. If $\xi > 0$, then $y^- \in intC_+$ and
we have

(47)                    $u^-(z) \to \infty$ for a.a. $z \in \Omega$ as $n \to +\infty$.

Using hypothesis $H_1(iii)$ as in the proof of Proposition 3.1 (see (24)), for a.a. $z \in \Omega$
and all $x < 0$, we obtain

(48)                    $\hat{\lambda}_1(p)|x|^p - pF(z,x) \geq -\xi_1$.

From (40) we have

$$\|Du_n\|_p^p + \frac{p}{2}\|Du_n\|_2^2 - \int_\Omega pF_-(z,u_n)dz \leq pM_8 \text{ for all } n \geq 1,$$

which implies

$$\|Du_n^-\|_p^p - \int_\Omega pF(z,-u_n^-)dz + \frac{p}{2}\|Du_n^-\|_2^2 \leq pM_8 \text{ for all } n \geq 1.$$

From this, taking into account (3), it follows that

$$\int_\Omega [\hat{\lambda}_1(p)(u_n^-)^p - pF(z,-u_n^-)]dz + \frac{p}{2}\hat{\lambda}_1(2)\|u_n^-\|_2^2 \leq pM_8,$$

Therefore, for some $M_9 > 0$ and all $n \geq 1$, we have

(49)                    $$\int_\Omega (u_n^-)^2 dz \leq M_9.$$

But from (47) and Fatou's Lemma, we have

$$\int_\Omega (u_n^-)^2 dz \to +\infty, \text{ as } n \to +\infty,$$

which contradicts (49). This proves the coercivity of $\varphi_-$.                    $\square$

   Using this proposition and the direct method, we can produce a negative solution
for problem (P).

**Proposition 3.3.** *If hypotheses $H_1$ hold, then problem (P) admits a negative solution
$u_0 \in intC_+$ which is a local minimizer of the energy functional $\varphi$.*

*Proof.* From Proposition 3.2 we know that $\varphi_-$ is coercive. Also, exploiting the Sobolev
embedding theorem, we see that $\varphi_-$ is sequentially weakly lower semicontinuous. So,
by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ s.t.

(50)                    $\varphi_-(u_0) = \inf \left[ \varphi_-(u) : u \in W_0^{1,p}(\Omega) \right].$

By hypothesis $H_1(iv)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ s.t.

$$(51) \qquad \frac{1}{2}\left[\beta_1(z) - \varepsilon\right] x^2 \le F(z,x), \text{ for a.a. } z \in \Omega, \ |x| \le \delta.$$

Let $t \in (0,1)$ be small s.t. $t\hat{u}_1(2)(z) \in [0,\delta]$ for all $z \in \overline{\Omega}$ (recall that $\hat{u}_1(2) \in intC_+$). We have

$$(52) \qquad \varphi_-(-t\hat{u}_1(2)) \le \frac{t^p}{p}\|D\hat{u}_1(2)\|_p^p + \frac{t^2}{2}\left[\int_\Omega \left(\hat{\lambda}_1(2) - \beta_1(z)\right)\hat{u}_1(2)^2 dz + \varepsilon\right],$$

(see (51) and recall that $\|\hat{u}_1(2)\|_2 = 1$). Since $\hat{u}_1(2) \in intC_+$, (see also $H_1(iv)$), one has

$$\varepsilon_4 = \int_\Omega \left(\beta_1(z) - \hat{\lambda}_1(2)\right)\hat{u}_1(2)^2 dz > 0.$$

Hence, if we choose $\varepsilon \in (0, \varepsilon_4)$, then from (52) we have

$$(53) \qquad \varphi_-(-t\hat{u}_1(2)) \le \frac{t^p}{p}\|D\hat{u}_1(2)\|_p^p - \frac{t^2}{2}c_{10}, \text{ for some } c_{10} > 0.$$

Since $2 < p$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\varphi_-(u_0) \le \varphi_-(-t\hat{u}_1(2)) < 0,$$

hence $u_0 \ne 0$. From (50), we have $\varphi'_-(u_0) = 0$, that is

$$(54) \qquad A_p(u_0) + A(u_0) = N_{f_-}(u_0).$$

On (54) we act with $u_0^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\|Du_0^+\|_p^p + \|Du_0^+\|_2^2 = 0,$$

hence $u_0 \le 0$ and $u_0 \ne 0$. Then from (54) it follows that

$$(55) \qquad -\Delta_p u_0(z) - \Delta u_0(z) = f(z, u_0(z)), \text{ a.e. in } \Omega, \ u|_{\partial\Omega} = 0.$$

From Ladyzhenskaya-Uraltseva [15, p.286], we infer that $u_0 \in L^\infty(\Omega)$. Then we can use Lieberman [8, Theorem 1] and conclude that $u_0 \in (-C_+) \setminus \{0\}$.

Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be the map defined by

$$a(y) = |y|^{p-2}y + y \text{ for all } y \in \mathbb{R}^N.$$

Evidently $a \in C^1\left(\mathbb{R}^N, \mathbb{R}^N\right)$ (recall $p > 2$) and

$$\nabla a(y) = |y|^{p-2}\left[I + (p-2)\frac{y \bigotimes y}{|y|^2}\right] + I \text{ for all } y \in \mathbb{R}^N.$$

Hence we have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge |\xi|^2 \quad \text{for all } y, \xi \in \mathbb{R}^N.$$

Therefore, invoking the tangency principle of Pucci-Serrin [16, p.35] (see also hypothesis $H_1(iii)$) we infer that $u_0(z) < 0$ for all $z \in \Omega$. From (10) we have

$$(56) \qquad f(z,x)x \ge c_1 x^2 - c_2|x|^r, \text{ for a.a. } z \in \Omega, \text{ all } x \le 0 \text{ with } c_1, c_2 > 0.$$

Then from (55), for a.a. $z \in \Omega$, we have

$$-\Delta(-u_0)(z) - \Delta(-u_0)(z) + c_2|u_0(z)|^{r-2}u_0(z) \quad = \quad f(z, u_0(z)) + c_2|u_0(z)|^{r-2}u_0(z)$$
$$\leq \quad c_1 u_0(z) \leq 0,$$

from which we deduce

(57) $$\Delta_p(-u_0)(z) + \Delta(-u_0)(z) \leq c_2\|u_0\|_\infty^{r-p}(-u_0(z))^{p-1} \text{ a.e. in } \Omega.$$

From (57) we infer that $-u_0 \in intC_+$. Moreover, we have that $\varphi_{-|-C_+} = \varphi_{|-C_+}$. Since $u_0 \in -intC_+$, it follows that $u_0$ is a local $C_0^1(\overline{\Omega})-$minimizer of $\varphi$. Proposition 2.1 implies that $u_0$ is a local $W_0^{1,p}(\Omega)-$minimizer of $\varphi$. $\qquad\square$

Using this negative solution $u_0 \in -intC_+$ and minimax techniques combined with Morse theory, we will produce a second nontrivial solution for problem (P). To do this, we need to strengthen a little our hypothesis on $f(z, \cdot)$ near zero.

The new hypotheses on the reaction $f(z, x)$ are the following:

$H_2: f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Caratheodory function s.t.* $f(z, 0) = 0$ *for a.a.* $z \in \Omega$, *hypotheses* $H_2(i), (ii), (iii)$ *are the same as the corresponding hypotheses* $H_1(i), (ii), (iii)$ *and*
$(iv)$ *for a.a.* $z \in \Omega$, $f(z, \cdot)$ *is differentiable at* $x = 0$,

$$f_x'(z, 0) = \lim_{x \to 0} \frac{f(z, x)}{x} \text{ uniformly for a.a. } z \in \Omega$$

*and there exists an integer* $m \geq 2$ *s.t*

$$f_x'(z, 0) \in \left[\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)\right], \quad a.e. \text{ in } \Omega,$$

$$f_x'(\cdot, 0) \not\equiv \hat{\lambda}_m(2), \quad f_x'(\cdot, 0) \not\equiv \hat{\lambda}_{m+1}(2).$$

**Proposition 3.4.** *If hypotheses* $H_2$ *hold, then problem* (P) *admits a second solution* $\hat{u} \in C_0^1(\overline{\Omega})$ *with* $\hat{u} \neq u_0$.

*Proof.* Let $u_0 \in intC_+$ be the negative solution of problem (P) ensured by Proposition 3.3. We know that $u_0$ is a local minimizer of the energy functional $\varphi$. Then assuming that $K_\varphi$ is finite (otherwise we already have an infinity of solutions for problem (P) and so we are done) we can find $\rho \in (0, 1)$ small s.t.

(58) $$\varphi(u_0) < \inf\left[\varphi(u) : \|u - u_0\| = \rho\right] = \eta_\rho.$$

Hypothesis $H_2(ii)$ implies that for every $u \in intC_+$

(59) $$\varphi(tu) \to -\infty \text{ as } \to +\infty.$$

Recall that $\varphi$ satisfies the $C-$condition (see Proposition 3.1). This fact together with (58) and (59), permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ s.t.

(60)
$$\hat{u} \in K_\varphi \quad \text{and} \quad \varphi(u_0) < \eta_\rho \leq \varphi(\hat{u}).$$

From (60) it follows that $\hat{u} \neq u_0$ and $\hat{u}$ is a solution of problem (P). As before, the nonlinear regularity theory (see [15], [8]) implies that $\hat{u} \in C_0^1(\overline{\Omega})$. Since $\hat{u}$ is a critical point of $\varphi$ of mountain pass type, we have

(61)
$$C_1(\varphi, \hat{u}) \neq 0,$$

see for example, Motreanu-Motreanu-Papageorgiou [12, p.176]. We show that $\hat{u} \neq 0$. To this end, let $\tau : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the $C^2-$ functional defined by

$$\tau(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{1}{2}\int_\Omega f_x'(z,0)u(z)^2 dz, \text{ for all } u \in W_0^{1,p}(\Omega).$$

We consider the affine homotopy $\left\{\hat{h}_t(\cdot)\right\}$ defined by

$$\hat{h}_t(u) = (1-t)\varphi(u) + t\tau(u), \text{for all } (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Suppose that we can find $\{t_n\} \subset [0,1]$ and $\{u_n\} \subset W_0^{1,p}(\Omega)$ s.t.

(62)     $t_n \to t \in [0,1], \ u_n \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ and } h_{t_n}'(u_n) = 0, \text{ for all } n \geq 1.$

From the equality in (62), we have

(63)     $A_p(u_n) + A(u_n) = (1-t)N_f(u_n) + t_n f_x'(\cdot,0)u_n, \text{ for all } n \geq 1.$

Let $v_n = \dfrac{u_n}{\|u_n\|}$ for all $n \geq 1$. Then $\|v_n\| = 1$ for all $n \geq 1$ and so we may assume that

(64)
$$v_n \rightharpoonup v \text{ in } W_0^{1,p}(\Omega) \text{ and } v_n \to v \text{ in } L^r(\Omega).$$

From (63), for all $n \geq 1$, we have

(65)
$$\|u_n\|^{p-2}A_p(v_n) + A(v_n) = (1-t_n)\frac{N_f(u_n)}{\|u_n\|} + t_n f_x'(\cdot,0)v_n.$$

Hypotheses $H_2(i)(iv)$ imply that

$$|f(z,x)| \leq c_{11}(|x| + |x|^{r-1}), \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } c_{11} > 0.$$

Note that from (63) and Ladyzhenskaya-Uraltseva [15, p.286], we have that $\{u_n\} \subset L^\infty(\Omega)$ is bounded. So, it follows that

(66)
$$\left\{\frac{N_f(u_n)}{\|u_n\|}\right\}_{n \geq 1} \subset L^r(\Omega) \text{ is bounded.}$$

Because of (66), by passing to a subsequence if necessary and using hypothesis $H_2(iv)$, we have

$$(67) \qquad \left\{ \frac{N_f(u_n)}{\|u_n\|} \right\}_{n \geq 1} \rightharpoonup f_x'(\cdot, 0)v \text{ in } L^r(\Omega).$$

Since $\{A_p(v_n)\}_{n \geq 1} \subset W^{-1,p'}(\Omega)$ is bounded (see (64) and Proposition 2.2) and $\|u_n\|^{p-2} \to 0$, we have

$$(68) \qquad \|u_n\|^{p-2} A_p(v_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to +\infty.$$

So, if in (65) we pass to the limit as $n \to +\infty$ and use (64), (67) and (68), we obtain $A(v) = f_x'(\cdot, 0)v$, that is

$$(69) \qquad -\Delta v(z) = f_x'(z, 0)v(z) \text{ a.e. in } \Omega, \quad v|_{\partial\Omega} = 0.$$

From the strict monotonicity of the weighted eigenvalues (see Section 2), we have

$$(70) \qquad \hat{\lambda}_m(2, f_x'(\cdot, 0)) < \hat{\lambda}_m(2, \hat{\lambda}_m(2)) = 1 = \hat{\lambda}_{m+1}(2, \hat{\lambda}_{m+1}(2)) < \hat{\lambda}_{m+1}(2, f_x'(\cdot, 0)).$$

Combining (69) and (70), we deduce that $v = 0$. From (65), for a.a. $z \in \Omega$ and $n \geq 1$, we have

$$(71) \qquad \begin{cases} -\|u_n\|^{p-2}\Delta_p(v_n) - \Delta v_n(z) = \frac{1-t_n}{\|u_n\|}f(z, v_n(z)) + t_n f_x'(z, 0)v_n(z) \text{ a.e. in } \Omega, \\ v_n|_{\partial\Omega} = 0. \end{cases}$$

As before from Ladyzhenskaya-Uraltseva [15, p.286], we know that there exists $M_{10} > 0$ s.t. $\|v_n\|_\infty \leq M_{10}$ for all $n \geq 1$. Since $\|v_n\|^{p-2} \to 0$, from (71) and Theorem 1 of Lieberman [8] we can find $\alpha \in (0,1)$ and $M_{11} > 0$ s.t.

$$v_n \in C_0^{1,\alpha}(\overline{\Omega}), \quad \text{and} \quad \|v_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq M_{11}.$$

By virtue of the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and because of (64), we have $v_n \to 0$ in $C_0^1(\overline{\Omega})$, hence in $W_0^{1,p}(\Omega)$ too, which contradicts the fact that $\|v_n\| = 1$ for all $n \geq 1$. This shows that (62) can not occur. Then the homotopy invariance of critical groups implies that

$$(72) \qquad C_k(\varphi, 0) = C_k(\tau, 0), \text{ for all } k \geq 0.$$

From Cingolani-Vannella [17, Theorem 1], we have $C_k(\tau, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \geq 0$ with $d_m = \dim \bigoplus_{i=1}^m E(\hat{\lambda}_i(2)) \geq 2$ and this owing to (72) implies

$$(73) \qquad C_1(\varphi, 0) = 0.$$

Comparing (61) and (73), we conclude that $\hat{u} \neq 0$. Therefore $\hat{u} \in C_0^1(\overline{\Omega})$ is the second nontrivial solution of problem (P) distinct from $u_0$. $\qquad \square$

So, we can state our first multiplicity theorem from problem (P). We stress that in this theorem $f(z, \cdot)$ need not to be $C^1$.

**Theorem 3.1.** *If hypotheses $H_2$ hold, then problem* (P) *has at least two nontrivial solutions*

$$u_0 \in -int C_+ \ and \ \hat{u} \in C_0^1(\overline{\Omega}) \setminus \{0\}, \ u_0 \neq \hat{u}$$

*and $u_0$ is a local minimizer of the energy functional $\varphi$.*

## 4. A THREE SOLUTIONS THEOREM

In this section, we produce a third nontrivial solution for problem (P). To do this, we improve the regularity of $f(z, \cdot)$ (now $f(z, \cdot) \in C^1(\mathbb{R})$) and we use tools from Morse theory (critical groups). So, the new hypotheses on the reaction $f(z, x)$ are the following:

$H_3$ : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function s.t. for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f_x'(z, x)| \leq \alpha(z)(1 + |x|^{r-2})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $\alpha \in L^\infty(\Omega)_+$, $p < r < p^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then

$$\lim_{x \to +\infty} \frac{F(z, x)}{x^p} = +\infty \text{uniformly for a.a. } z \in \Omega,$$

and there exist $\tau \in \left(\max\{1, (r-p)\frac{N}{p}\}, p^*\right)$ and $\beta_0 > 0$ s.t.

$$0 < \beta_0 < \liminf_{x \to +\infty} \frac{f(z, x)x - pF(z, x)}{x^\tau} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exist $\xi_0$, $\xi_1 > 0$ s.t.

$$-\xi_0 \leq \liminf_{x \to -\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \limsup_{x \to -\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \hat{\lambda}_1(p) \text{ uniformly for a.a. } z \in \Omega;$$

$$-\xi_1 \leq f(z, x)x - pF(z, x) \text{ for a.a. } z \in \Omega, \text{ all } x \leq 0;$$

(iv) $f_x'(z, 0) = \lim_{x \to 0} \frac{f(z, x)}{x}$ uniformly for a.a. $z \in \Omega$ and there exists an integer $m \geq 2$ s.t.

$$f_x'(z, 0) \in \left[\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)\right] \text{ a.e. in } \Omega,$$

$$f_x'(\cdot, 0) \not\equiv \hat{\lambda}_m(2), \quad f_x'(\cdot, 0) \not\equiv \hat{\lambda}_{m+1}(2).$$

**Remark 4.1.** Since $f(z, \cdot) \in C^1(\mathbb{R})$, from the mean value theorem and hypothesis $H_3(i)$, we see that for a.a. $z \in \Omega$, $f(z, \cdot)$ is locally Lipschitz. The above hypotheses imply that $\varphi \in C^2(W_0^{1,p}(\Omega))$.

**Proposition 4.1.** *If hypotheses $H_3$ hold, then $C_k(\varphi, \infty) = 0$ for all $k \geq 0$.*

*Proof.* Let $\partial B_1^+ = \{u \in W_0^{1,p}(\Omega) : \|u\| = 1$ and $u^+ \neq 0\}$. We consider the deformation $h : [0,1] \times \partial B_1^+ \to \partial B_1^+$ defined by

$$h(t,u) = \frac{(1-t)u + t\hat{u}_1(p)}{\|(1-t)u + t\hat{u}_1(p)\|} \text{ for all } (t,u) \in [0,1] \times \partial B_1^+.$$

Note that

$$h(0,\cdot) = id|_{\partial B_1^+}, \quad h(1,u) = \frac{\hat{u}_1(p)}{\|\hat{u}_1(p)\|}.$$

Therefore the set $\partial B_1^+$ is contractible in itself. Hypothesis $H_3(ii)$ implies that given any $\xi > 0$, we can find $M_{12} = M_{12}(\xi) > 0$ s.t.

$$(74) \qquad F(z,x) \geq \frac{\xi}{p}x^p \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_{12}.$$

Also hypothesis $H_3(iii)$ implies that we can find $c_{12} > 0$ and $\hat{M}_{12} > 0$ s.t.

$$(75) \qquad F(z,x) \geq -\frac{c_{12}}{p}|x|^p \text{ for a.a. } z \in \Omega, \text{ all } x \leq -\hat{M}_{12}.$$

Finally hypothesis $H_3(i)$ guarantees that

$$(76) \quad |F(z,x)| \leq c_{13} \text{ for a.a. } z \in \Omega, \text{ all } x \in \widehat{I} = [-\hat{M}_{12}, M_{12}] \text{ and some } c_{13} > 0.$$

Suppose that $u \in \partial B_1^+$ and $t > 0$. We have

$$\begin{aligned}
\varphi(tu) &= \frac{t^p}{p}\|Du\|_p^p + \frac{t^2}{2}\|Du\|_2^2 - \int_\Omega F(z,tu)dz \\
&= \frac{t^p}{p}\|Du\|_p^p + \frac{t^2}{2}\|Du\|_2^2 - \\
&\quad - \int_{\{tu \geq M_{12}\}} F(z,tu)dz - \int_{\{tu \leq -\hat{M}_{12}\}} F(z,tu)dz - \int_{\hat{I}} F(z,tu)dz \\
&\leq \frac{t^p}{p}\|Du\|_p^p + \frac{t^2}{2}\|Du\|_2^2 - \frac{t^p}{p}\xi \int_{\{tu \geq M_{12}\}} u^p dz + \frac{t^p}{p}c_{12} \int_{\{tu \leq -\hat{M}_{12}\}} |u|^p dz + c_{13}|\Omega|_N.
\end{aligned}$$

$$(77) \leq \frac{t^p}{p}\left[\left(1 + \frac{c_{12}}{\hat{\lambda}_1(p)}\right)\|Du\|_p^p - \xi \int_{\{tu \geq M_{12}\}} u^p dz\right] + \frac{t^2}{2}\|Du\|_2^2 + c_{13}|\Omega|_N.$$

Since $u \in \partial B_1^+$, we can find $t^* > 0$ and $\gamma > 0$ s.t.

$$\int_{\{tu \geq M_{12}\}} u^p dz \geq \gamma \text{ for all } t > t^*.$$

Recall that $\xi > 0$ is arbitrary. So, we choose $\xi > 0$ big s.t.

$$\xi\gamma - \left(1 + \frac{c_{12}}{\hat{\lambda}_1(p)}\right)\|Du\|_p^p > 0.$$

Then from (77) and since $p > 2$, we infer that

(78) $$\varphi(tu) \to -\infty \text{ as } t \to -\infty.$$

Hypothesis $H_3(ii)$ implies that there exist $c_{14} > 0$ and $M_{13} > 0$ s.t.

(79) $$f(z,x)x - pF(z,x) \geq c_{14}x^\tau \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_{13}.$$

On the other hand hypotheses $H_3(i)(iii)$ imply that

(80) $$f(z,x)x - pF(z,x) \geq -c_{15} \text{ for a.a. } z \in \Omega, \text{ all } x < M_{13} \text{ and some } c_{15} > 0.$$

Then for all $u \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega [f(z,u)u - pF(z,u)]dz =$$

$$= \int_{\{u \geq M_{13}\}} [pF(z,u) - f(z,u)u]dz + \int_{\{u < M_{13}\}} [pF(z,u) - f(z,u)u]dz$$

(81) $$\leq -c_{14} \int_{\{u \geq M_{13}\}} u^\tau dz + c_{15}|\Omega|_N, \text{ (see (79), (80))}.$$

Let $u \in \partial B_1^+$ and $t \geq 1$. We have

$$\frac{d}{dt}\varphi(tu) = \langle \varphi'(tu), u \rangle$$

$$= \frac{1}{t}\langle \varphi'(tu), tu \rangle$$

$$= t^{p-1}\|Du\|_p^p + \|Du\|_2^2 - \int_\Omega f(z,tu)u\,dz$$

(82) $$\leq \frac{1}{t}[p\varphi(tu) + c_{15}|\Omega|], \text{ (see (81) and recall } p > 2.)$$

From (78) and (82) it follows that for $t \geq 1$ big enough, we have

(83) $$\frac{d}{dt}\varphi(tu) < 0.$$

Let $W_+ = \{u \in W_0^{1,p}(\Omega) : u(z) \geq 0, \text{a.e. in } \Omega\}$. From Proposition 3.2, we have that $\varphi_{|-W_+}$ is coercive. Therefore, there exists $c_{16} > 0$ s.t. $\varphi_{|-W_+} \geq -c_{16}$. We choose

$$\lambda < \min\left\{-\frac{c_{15}|\Omega|}{p}, -c_{16}, \inf_{\partial B_1} \varphi\right\} < 0,$$

(here $\partial B_1 = \{u \in W_0^{1,p}(\Omega) : \|u\| = 1\}$). From (83) we see that there exists unique $\mu(u) > 1$ s.t.

(84) $$\begin{cases} \varphi(tu) > \lambda, & \text{if } 0 \leq t < \mu(u); \\ \varphi(tu) = \lambda, & \text{if } t = \mu(u); \\ \varphi(tu) < \lambda, & \text{if } t > \mu(u). \end{cases}$$

Moreover, the implicit function theorem implies that $\mu \in C\left(\partial B_1^+, (1, \infty)\right)$. From the choice of $\lambda < 0$, we have

$$\varphi^\lambda \subseteq \{tu : \ u \in \partial B_1^+, t \geq \mu(u)\}.$$

Let $D_+ = \{tu : \ u \in \partial B_1^+, t \geq 1\}$. Then $\varphi^\lambda \subseteq D_+$. We consider the deformation $\hat{h} : [0,1] \times D_+ \to D_+$ defined, for all $(s, u) \in [0,1] \times D_+$, by

$$\hat{h}(s, u) = \begin{cases} (1-s)tu + s\mu(u)u, & \text{if } 1 \leq t \leq \mu(u); \\ tu, & \text{if } t > \mu(u). \end{cases}$$

Note that $\hat{h}(1, D_+) \subseteq \varphi^\lambda$ and $\hat{h}(s, \cdot)|_{\varphi^\lambda} = id|_{\varphi^\lambda}$ for all $s \in [0,1]$ (see (84)). Hence $\varphi^\lambda$ is a strong deformation retract of $D_+$. Using the radial retraction, we see that $D_+$ and $\partial B_1^+$ are homotopy equivalent. Therefore it follows that

$$(85) \qquad H_k\left(W_0^{1,p}(\Omega), \varphi^\lambda\right) = H_k\left(W_0^{1,p}(\Omega), D_+\right) = H_k\left(W_0^{1,p}(\Omega), \partial B_1^+\right) \text{ for all } k \geq 1.$$

Recall that we have already established that $\partial B_1^+$ is contractible in itself. Hence from Granas-Dugundji [18, p.389], we have

$$H_k\left(W_0^{1,p}(\Omega), \partial B_1^+\right) = 0, \text{ for all, } k \geq 0$$

and owing to (85), we obtain $C_k(\varphi, \infty) = 0$ for all $k \geq 0$ (by choosing $\lambda < 0$ even smaller if necessary). $\qquad\square$

**Remark 4.2.** A similar computation of critical groups at infinity was first conducted by Wang [19] for semilinear equations driven by the Laplacian and with a $C^1$ reaction $x \to f(x)$ which exhibits symmetric asymptotic behavior at $\pm\infty$ and satisfies the AR-condition.

**Theorem 4.1.** *If hypotheses $H_3$ hold, then problem* (P) *has at least three nontrivial solutions*

$$u_0 \in -int C_+, \quad and, \ \hat{u}, \widetilde{u} \in C_0^1(\overline{\Omega}) \setminus \{0\}.$$

*Proof.* From Theorem 3.1 we already have two nontrivial solutions

$$u_0 \in -int C_+, \quad \hat{u} \in C_0^1(\overline{\Omega}).$$

We know that $u_0$ is a local minimizer of the energy functional $\varphi$. Hence

$$(86) \qquad\qquad C_k\left(\varphi, u_0\right) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0.$$

Also, from the proof of Proposition 3.4, we know that $\hat{u}$ is a critical point of $\varphi$ of mountain pass type. Therefore, $C_1(\varphi, \hat{u}) \neq 0$ (see Motreanu-Motreanu-Papageorgiou [12, p.176]). Since $\varphi \in C^2(W_0^{1,p}(\Omega))$, from Papageorgiou-Smyrlis [20], it follows that

$$(87) \qquad\qquad C_k\left(\varphi, \hat{u}\right) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0.$$

From the proof of Proposition 3.4, we have

$$(88) \qquad\qquad C_k\left(\varphi, 0\right) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0.$$

From Proposition 4.1, we have

$$(89) \qquad C_k(\varphi, \infty) = 0 \quad \text{for all } k \geq 0.$$

Suppose $K_\varphi = \{0, u_0, \hat{u}\}$. Then from $(86) - (89)$ and the Morse relation with $t = -1$ (see (7)), we have $(-1)^{d_m} = 0$, a contradiction. This means that there exists $\tilde{u} \in K_\varphi$ with $\tilde{u} \notin \{0, u_0, \hat{u}\}$. So, $\tilde{u}$ is the third nontrivial solution of problem (P) and the nonlinear regularity theory implies that $\tilde{u} \in C_0^1(\overline{\Omega})$. $\qquad\square$

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