ON THE EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR DIRICHLET'S PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS

Diego Averna [1], Stepan Tersian [2], Elisabetta Tornatore [3]

Abstract

In this paper, by using variational methods and critical point theorems, we prove the existence and multiplicity of solutions for boundary value problem for fractional order differential equations where Riemann-Liouville fractional derivatives and Caputo fractional derivatives are used. Our results extend the second order boundary value problem to the non integer case. Moreover, some conditions to determinate nonnegative solutions are presented and examples are given to illustrate our results.

MSC 2010: Primary 34A08; Secondary 58E30, 26A33, 35A15

Key Words and Phrases: fractional differential equations, critical points theorem, variational methods, multiple solutions

1. Introduction

The aim of this paper is to establish the existence results for the following boundary value problem for fractional order differential equations

$$\begin{cases} {}_xD_b^\alpha \left({}_a^CD_x^\alpha u(x)\right) + u(x) = \lambda f(x, u(x)) & \text{in }]a,b[\\ u(a) = u(b) = 0 \end{cases} \tag{1.1}$$

where ${}_xD_b^\alpha$, ${}_a^CD_x^\alpha$ are the right Riemann-Liouville fractional derivative and the left Caputo fractional derivative of order $\frac{1}{2} < \alpha \leq 1$ respectively, λ is a positive real parameter and $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is a given continuous function. Boundary value problems for fractional order differential equations appear in many fields of mathematics sciences and engineering such as rheology, visco-elasticity, electrical circuits. The need of such equations arises from the fact that many phenomena can not be modeled by differential equations

pp. xxx–xxx

with integer order derivatives, for details see the monographs [11], [12], [14] and among the papers, see for example [8], [9] and the references therein. If $\alpha = 1$, the problem (1.1) reduces to the second order boundary value problem

$$\begin{cases} -u''(x) + u(x) = \lambda f(x, u(x)) & \text{in }]a, b[\\ u(a) = u(b) = 0 \end{cases} \tag{1.2}$$

which is widely studied by using variational methods and critical point theorems, among the papers we cite [2], [3]. The variational approach has been also employed to study fractional differential problem when $\frac{1}{2} < \alpha < 1$. In [10] the authors studied a fractional boundary value problem by using the Mountain pass theorem, under the usual Ambrosetti-Rabinowitz condition [1], the same model has been studied in [7], in which the existence and multiplicity of solutions for boundary value problem for fractional equations is considered under three distinct types conditions on the potential function, called super quadratic, asymptotically quadratic and sub quadratic cases. The authors used embedding theorems of [10] and mountain-pass theorem and Clarks theorem (see [1]). In [6] and [13] the authors studied the fractional boundary value problem (1.1) with impulsive effects and proved the existence of at least three solutions. In this paper, by using recent results of Bonanno ([4], [5]), we present results on the existence of at least two non zero solutions of problem (1.1) in which the Ambrosetti-Rabinowitz condition ([1]) is used (Theorem **3.1**) and on the existence of one non-zero solution (Theorem **3.2**) for each λ in an appropriate interval.

As a special case of Theorem **3.1** and Theorem **3.2** to the autonomous case, here we point out the following results on the existence of nonnegative solutions. Consider the following problem

$$\begin{cases} {}_xD_b^\alpha \left({}_a^C D_x^\alpha u(x) \right) + u(x) = \lambda f(u(x)) & \text{in }]a, b[\\ u(a) = u(b) = 0 \end{cases} \tag{1.3}$$

and put $F(t) = \int_0^t f(\xi) d\xi$ for all $t \in \mathbb{R}$.

Theorem **1.1**. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty, \tag{1.4}$$

and assume that there are two positive constants $\mu > 2$ and $R > 0$ such that for all $\xi \geq R$ one has

$$0 < \mu F(\xi) \leq \xi f(\xi). \tag{1.5}$$

Then, for each $\lambda \in]0, \lambda^*[$, *where* $\lambda^* = \frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha}} \sup_{c>0}\{\frac{c^2}{F(c)}\}$, *problem* (1.3) *admits at least two nonnegative solutions.*

THEOREM **1.2**. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a nonnegative continuous and non zero function such that*

$$\lim_{t\to+\infty} \frac{f(t)}{t} = 0. \tag{1.6}$$

Then, the problem (1.3) *for each* $\lambda > \bar{\lambda}$, *with* $\bar{\lambda} = \frac{2k}{b-a} \inf_{d>0}\{\frac{d^2}{F(d)}\}$ *admits at least one non-zero and nonnegative solution.*

The paper is arranged as follows. In Section **2** we recall the used tools on variational methods and fractional derivatives. The main results are proved in Section **3**. Some examples are given.

2. Preliminaries and basic notations

In this section, we recall some definitions and theorems used in the paper.

Let $(X, \|\cdot\|)$ be a real Banach space and Φ, $\Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals and $r \in]-\infty, +\infty]$. We say that a functional $I = \Phi - \Psi$ satisfies the *Palais-Smale condition cut off upper at* r (in short $(PS)^{[r]}$-condition) if any sequence $\{u_n\}$ in X such that

(α_1) $\{I(u_n)\}$ is bounded,
(α_2) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
(α_3) $\Phi(u_n) < r \quad \forall n \in \mathbb{N}$,

has a convergent subsequence.

When $r = +\infty$ the previous definition coincides with the classical (PS)-condition, while if $r < \infty$ such condition is more general than the classical one. We refer to [4] for more details.

We recall a recent result obtained in [5] that insures the existence of two critical points in which the boundedness of the functional is not required.

THEOREM **2.1**. ([5, Theorem 1.3]). *Let* X *be a real Banach space and let* Φ, $\Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there exist* $r \in \mathbb{R}$ *and* $\bar{u} \in X$, *with* $0 < \Phi(\bar{u}) < r$, *such that*

(j) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$

(jj) *for each* $\lambda \in \Lambda := \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$ *the functional* $I_\lambda = \Phi - \lambda\Psi$ *is unbounded from below and satisfies (PS)-condition.*

Then, for each $\lambda \in \Lambda$, *the functional* $I_\lambda = \Phi - \lambda\Psi$ *has at least two distinct critical points in* X.

Finally, we also recall a recent result obtained in [4] which insures the existence of at least one critical point. We mention here its version as presented in [5].

THEOREM **2.2**. ([5, Theorem 2.3]). *Let* X *be a real Banach space, and let* Φ, $\Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there exist* $r \in \mathbb{R}$ *and* $\bar{u} \in X$, *with* $0 < \Phi(\bar{u}) < r$, *such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})} \tag{2.1}$$

and, for each $\lambda \in \Lambda := \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[$ *the functional* $I_\lambda = \Phi - \lambda\Psi$ *satisfied* $(PS)^{[r]}$*-condition.*

Then, for each $\lambda \in \Lambda := \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[$, *there is* $u_\lambda \in \Phi^{-1}(]0, r[)$ *such that* $I_\lambda(u_\lambda) \le I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]0, r[)$ *and* $I'_\lambda(u_\lambda) = 0$.

For readers' convenience let us introduce here some definitions and lemmas from the theory of fractional calculus that will be used in the paper.

DEFINITION **2.1**. The Left and Right Riemann-Liouville fractional integrals of order $\alpha > 0$ for a continuous function $u : [a, b] \to \mathbb{R}$ denoted by ${}_aD^{-\alpha}u(x)$, ${}_xD_b^{-\alpha}u(x)$ respectively, are defined by

$${}_aD_x^{-\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}u(s)ds \qquad x \in [a, b]$$

$${}_xD_b^{-\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}u(s)ds \qquad x \in [a, b]$$

where Γ is the gamma function.

If $\alpha = 0$ we put ${}_aD_x^0u(x) = u(x)$ and ${}_xD_b^0u(x) = u(x)$.

Definition **2.2**. The Left and Right Riemann-Liouville fractional derivatives of order $0 \leq \alpha < 1$ for a continuous function $u : [a,b] \to \mathbb{R}$ denoted by ${}_aD_x^\alpha u(x)$, ${}_xD_b^\alpha u(x)$ respectively, are defined by

$${}_aD_x^\alpha u(x) = \frac{d}{dx}{}_aD_x^{\alpha-1}u(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-s)^{-\alpha}u(s)ds \qquad x \in [a,b]$$

$${}_xD_b^\alpha u(x) = -\frac{d}{dx}{}_xD_b^{\alpha-1}u(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b (s-x)^{-\alpha}u(s)ds \quad x \in [a,b].$$

Definition **2.3**. The Left and Right Caputo fractional derivatives of order $0 < \alpha \leq 1$ for an absolutely continuous function $u : [a,b] \to \mathbb{R}$ denoted by ${}_a^CD_x^\alpha u(x)$, ${}_x^CD_b^\alpha u(x)$ respectively, are defined by

$${}_a^CD_x^\alpha u(x) = {}_aD_x^{\alpha-1}u'(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-s)^{-\alpha}u'(s)ds \qquad x \in [a,b]$$

$${}_x^CD_b^\alpha u(x) = -{}_xD_b^{\alpha-1}u'(x) = -\frac{1}{\Gamma(1-\alpha)}\int_x^b (s-x)^{-\alpha}u'(s)ds \qquad x \in [a,b].$$

Note that when $\alpha = 1$, then ${}_a^CD_x^1 u(x) = u'(x)$, ${}_x^CD_b^1 u(x) = -u'(x)$.

Let E_0^α $(0 < \alpha \leq 1)$ be the fractional derivative space defined by the closure of $C_0^\infty([a,b])$ with respect on the norm

$$\|u\| = \left(\|u\|_{L^2([a,b])}^2 + \|{}_a^CD_x^\alpha u\|_{L^2([a,b])}^2\right)^{\frac{1}{2}}.$$

E_0^α is a reflexive and separable Banach space (for more details see [10]), moreover E_0^α is a Hilbert space with the inner product

$$(u,v)_\alpha = (u,v)_{L^2([a,b])} + ({}_a^CD_x^\alpha u, {}_a^CD_x^\alpha v)_{L^2([a,b])} \qquad \forall u,v \in E_0^\alpha.$$

It is well known that if $\frac{1}{2} < \alpha \leq 1$, then $(E_0^\alpha, \|\cdot\|)$ is embedded in $(C^0([a,b]), \|\cdot\|_\infty)$ and one has

$$\|u\|_\infty \leq \frac{(b-a)^{\frac{2\alpha-1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\|u\| \qquad \forall u \in E_0^\alpha. \tag{2.2}$$

Lemma **2.1**. ([10]) *Let $\frac{1}{2} < \alpha \leq 1$ and be $\{u_n\}$ a sequence such that $u_n \rightharpoonup u$ in E_0^α, then $\{u_n\}$ converges strongly to u in $C^0([a,b])$.*

We will use the following integration by parts formulae.

Proposition **2.1**. ([11]) Let $0 < \alpha \leq 1$, $f, g \in L^2([a,b])$, then

$$\int_a^b {}_xD_b^{\alpha-1} f(x)g(x)dx = \int_a^b f(x)_a D_x^{\alpha-1} g(x)dx.$$

Proposition **2.2**. ([11]) Let $0 < \alpha \leq 1$, $f, g \in E_0^\alpha$, then

$$\int_a^b ({}_a^C D_x^\alpha f(x))({}_a^C D_x^\alpha g(x))dx = \int_a^b {}_xD_b^\alpha ({}_a^C D_x^\alpha f(x))g(x)dx.$$

A function $u \in AC([a,b])$ is said to be a solution to problem (1.1), if

- ${}_xD_b^\alpha({}_a^C D_x^\alpha u(x))$ exists a.e in $[a,b]$,
- u safisfies (1.1).

A function $u \in E_0^\alpha$ is said to be a weak solution to problem (1.1) if

$$\int_a^b \left(({}_a^C D_x^\alpha u(x))({}_a^C D_x^\alpha v(x)) + u(x)v(x)\right) dx = \lambda \int_a^b f(x,u(x))v(x)dx$$

for every $v \in E_0^\alpha$.

In order to study problem (1.1), we will use the functionals Φ, Ψ : $E_0^\alpha \to \mathbb{R}$ defined by putting

$$\Phi(u) := \frac{\|u\|^2}{2}, \quad \Psi(u) := \int_a^b F(x,u(x))dx \tag{2.3}$$

for every $u \in E_0^\alpha$, where $F(x,t) = \int_0^t f(x,\xi)d\xi$ for all $(x,t) \in [a,b] \times \mathbb{R}$.

Clearly, Φ is coercive continuous and convex, hence it is weakly sequentially lower semicontinuous. Moreover Φ is continuously Gâteaux differentiable and its Gâteaux derivative admits an continuous inverse. Gâteaux derivative of Φ at point $u \in E_0^\alpha$ is defined by

$$\Phi'(u)(v) = \int_a^b ({}_a^C D_x^\alpha u(x))({}_a^C D_x^\alpha v(x))dx + \int_a^b u(x)v(x)dx$$

for every $v \in E_0^\alpha$.

On the other hand, Ψ is continuously Gâteaux differentiable and its Gâteaux derivative at point $u \in E_0^\alpha$ is defined by

$$\Psi'(u)(v) = \int_a^b f(x,u(x))v(x)dx,$$

for every $v \in E_0^\alpha$.

Moreover, if we assume that $\frac{1}{2} < \alpha \leq 1$, then Ψ' is compact.

We observe that $\Phi(0) = \Psi(0) = 0$.

A critical point for the functional $I_\lambda := \Phi - \lambda\Psi$ is any $u \in E_0^\alpha$ such that

$$\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0 \qquad \forall v \in E_0^\alpha.$$

We can prove the following lemma.

LEMMA **2.2**. *If* $\frac{1}{2} < \alpha \leq 1$, *then* $u \in E_0^\alpha$ *is a weak solution of* (1.1) *if and only if it is a solution of* (1.1).

Hence, the critical points for functional $I_\lambda := \Phi - \lambda\Psi$ are exactly the solutions to problem (1.1).

Now, put

$$k = \frac{b-a}{3} + \frac{6\alpha^2 - 19\alpha + 16}{2(1-\alpha)^2(3-2\alpha)(2-\alpha)\Gamma^2(1-\alpha)} \left(\frac{b-a}{4}\right)^{1-2\alpha}. \quad (2.4)$$

3. Main results

Our main results are the following theorems.

The following result, in which the global Ambrosetti-Rabinowitz condition is also used, ensures the existence at least two solutions of problem (1.1).

THEOREM **3.1**. *We suppose that* $f(x,0) \neq 0$ *for every* $x \in [a,b]$. *We assume that there exist four positive constants* c, d, $\mu > 2$ *and* R *such that*

(i_1) $F(x,t) \geq 0$ *for every* $(x,t) \in [a,b] \times [0,d]$

(i_2)

$$0 < \mu F(x,t) \leq t f(x,t)$$

for all $x \in [a,b]$ *and* $|t| \geq R$;

(i_3)

$$d < \sqrt{\frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha-1}k}}c$$

$$\frac{\int_a^b \max_{|\xi|\leq c} F(x,\xi)dx}{c^2} < \frac{\Gamma^2(\alpha)(2\alpha-1)}{2(b-a)^{2\alpha-1}k} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)dx}{d^2}.$$

where k *is given by* (2.4).

Then, for each $\lambda \in \left] \frac{kd^2}{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)dx}, \frac{\Gamma^2(\alpha)(2\alpha-1)c^2}{2(b-a)^{2\alpha-1}\int_a^b \max_{|\xi|\leq c} F(x,\xi)dx} \right[$, *the problem (1.1) has at least two non trivial solutions.*

P r o o f. Our goal is to apply Theorem **2.1**. Consider the Banach space $X = E_0^\alpha$ and the operators defined in (2.3). We observe that the regularity assumptions on Φ and Ψ are satisfied. Set $r = \frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha-1}}c^2$ and consider the function $\bar{u} \in E_0^\alpha$ defined by putting

$$\bar{u}(t) := \begin{cases} \frac{4d}{b-a}(x-a) & \text{if } a \le x \le a + \frac{b-a}{4} \\ d & \text{if } a + \frac{b-a}{4} < x \le b - \frac{b-a}{4} \\ \frac{4d}{b-a}(b-x) & \text{if } b - \frac{b-a}{4} < x \le b \end{cases}. \tag{3.1}$$

Clearly, one has

$$\bar{u}'(t) := \begin{cases} \frac{4d}{b-a} & \text{if } a < x < a + \frac{b-a}{4} \\ 0 & \text{if } a + \frac{b-a}{4} < x < b - \frac{b-a}{4} \\ -\frac{4d}{b-a} & \text{if } b - \frac{b-a}{4} < x < b \end{cases}$$

and

$${}_a^C D_x^\alpha \bar{u}(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-\tau)^{-\alpha} \bar{u}'(\tau) d\tau$$

$$= \frac{1}{\Gamma(1-\alpha)} \begin{cases} \frac{4d}{b-a} \frac{(x-a)^{1-\alpha}}{1-\alpha} & \text{if } a \le x \le a + \frac{b-a}{4} \\ \frac{4d}{b-a} \frac{(\frac{b-a}{4})^{1-\alpha}}{1-\alpha} & \text{if } a + \frac{b-a}{4} \le x \le b - \frac{b-a}{4} \\ \frac{4d}{(1-\alpha)(b-a)}[(\frac{b-a}{4})^{1-\alpha} \\ \quad - \left(x - (b - \frac{b-a}{4})\right)^{1-\alpha}] & \text{if } b - \frac{b-a}{4} \le x \le b \end{cases}$$

so that

$$\Phi(\bar{u}) = kd^2 \tag{3.2}$$

where k is given by (2.4). And, taking into account that $d < \sqrt{\frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha-1}k}}c$, we have

$$0 < \Phi(\bar{u}) < r$$

In virtue of (i_1) we have

$$\Psi(\bar{u}) \ge \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)dx.$$

Therefore, one has

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \ge \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)dx}{kd^2}. \tag{3.3}$$

From (2.2) if $\Phi(u) \le r$, we have $\|u\| \le c$ therefore

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) \le \int_a^b \max_{|\xi| \le c} F(x,\xi)dx. \tag{3.4}$$

Hence, owing to (3.3), (3.4) and (i_3) condition (j) of Theorem **2.1** is verified.

From (i_2) by standard computations, one has that I_λ is unbounded from below and satisfies the Palais-Smale condition. Then the condition (jj) of Theorem **2.1** is verified.

Therefore, all the assumptions of Theorem **2.1** are satisfied.

So, for each

$$\lambda \in \left] \frac{kd^2}{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)dx}, \frac{\Gamma^2(\alpha)(2\alpha-1)c^2}{2(b-a)^{2\alpha-1}\int_a^b \max_{|\xi|\leq c} F(x,\xi)dx} \right[,$$

the functional I_λ has at least two non-zero critical point that are non trivial solutions of (1.1). □

Now, we point out the following result on the existence of at least one non zero solution of problem (1.1).

THEOREM **3.2**. *We assume that there exist two positive constants c, d with $d < \sqrt{\frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha-1}k}}c$ such that*

(h_1) *$F(x,t) \geq 0$ for every $(x,t) \in [a,b] \times [0,d]$;*

(h_2)

$$\frac{\int_a^b \max_{|\xi|\leq c} F(x,\xi)dx}{c^2} < \frac{\Gamma^2(\alpha)(2\alpha-1)}{2(b-a)^{2\alpha-1}k}\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)dx}{d^2}$$

where k is given by (2.4).

Then, for each $\lambda \in \left] \frac{kd^2}{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)dx}, \frac{\Gamma^2(\alpha)(2\alpha-1)c^2}{2(b-a)^{2\alpha-1}\int_a^b \max_{|\xi|<c} F(x,\xi)dx} \right[$, the problem (1.1) has at least one non zero solution.

P r o o f. Our goal is to apply Theorem **2.2**. Consider the Banach space $X = E_0^\alpha$ and the operators defined in (2.3). We observe that the regularity assumptions on Φ and Ψ are satisfied. Arguing as in the proof of Theorem **3.1**, put $\bar{u}$ as in (3.1) and $r = \frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha-1}}c^2$, bearing in mind that $d < \sqrt{\frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha-1}k}}c$, by using (h_1) and (3.2) we obtain

$$0 < \Phi(\bar{u}) < r,$$

$$\Psi(\bar{u}) \geq \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)dx$$

and

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) \leq \int_a^b \max_{|\xi|<c} F(x,\xi)dx.$$

We observe that, for all $r>0$, fixed a sequence $\{u_n\}\in X$ such that (α_3) holds, in virtue of the coercivity of Φ it is bounded in E_0^α so taking into account the regularity assumptions on Φ and Ψ we prove that the functional I_λ satisfies $(\mathrm{PS}^{[r]})$-condition.

Hence, owing to these relations and (j_2) condition (3) of Theorem **2.2** is verified.

Therefore, all the assumptions of Theorem **2.2** are satisfied.

So, for each

$$\lambda \in \left]\frac{kd^2}{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)dx}, \frac{\Gamma^2(\alpha)(2\alpha-1)c^2}{2(b-a)^{2\alpha-1}\int_a^b \max_{|\xi|\leq c} F(x,\xi)dx}\right[\subseteq \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[,$$

the functional I_λ has at least one non-zero critical point that is solution of problem (1.1). □

Now we suppose that $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a nonnegative continuous function, we point out the following weak maximum principle.

PROPOSITION **3.1**. *Suppose that $u\in E_0^\alpha$ is a weak solution of problem* (1.1)*, then u is nonnegative.*

P r o o f. Put $u^- = -\min\{u,0\}$, one has $u^-\in E_0^\alpha$, so taking into account that u is a weak solution and choosing $v=u^-$, one has

$$0 \leq \int_a^b \left(({}_a^C D_x^\alpha u(x))({}_a^C D_x^\alpha u^-(x)) + u(x)u^-(x)\right)dx$$
$$= -\left[\int_a^b \left(({}_a^C D_x^\alpha u^-(x))^2 + (u^-(x))^2\right)dx\right] = -\|u^-\|,$$

that is $u^-=0$ a.e. in $[a,b]$. Hence, our claim is proved. □

Now, we can give the proofs of the theorems in Introduction.

P r o o f. (**Proof of Theorem 1.1**)

Fix $\lambda\in]0,\lambda^*[$, then there is $c>0$ such that $\lambda < \frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha}}\frac{c^2}{F(c)}$. Since f is nonnegative one has that $\max_{|\xi|\leq c} F(\xi) = F(c)$ for every $c\in\mathbb{R}_+$. From

(1.4), there is $d < \sqrt{\frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha-1}k}}c$ such that

$$\frac{b-a}{2k}\frac{F(d)}{d^2} > \frac{1}{\lambda}$$

where k is given by (2.4).

Hence, Theorem **3.1** and Proposition **3.1** ensures the conclusion. □

P r o o f. (**Proof of Theorem 1.2**)

Fix $\lambda > \bar{\lambda}$, there exists $d > 0$ such that $\lambda > \frac{2k}{b-a}\frac{d^2}{F(d)}$. Since f is nonnegative one has that $\max_{|\xi|\le c} F(\xi) = F(c)$ for every $c \in \mathbb{R}_+$. From (1.6) there is $c > 0$ such that $c > \sqrt{\frac{2(b-a)^{2\alpha-1}k}{(2\alpha-1)\Gamma^2(\alpha)}}d$ and

$$\frac{2(b-a)^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\frac{F(c)}{c^2} < \frac{1}{\lambda},$$

where k is given by (2.4). Hence, the conclusion follows from Theorem **3.2** taking into account Proposition **3.1**. □

Now, we present some examples that illustrate our results.

Example **3.1**. Let us take $a = 0, b = c = 1, \alpha = 0.7$. Using *Mathematica*, we choose

$$d = 0.23 < \sqrt{\frac{(2\alpha-1)\Gamma^2(\alpha)}{4k}} \approx 0.238445$$

and $g = 1/d = 4.3478$. We define the function $F_0 = F_0(u)$ as follows:

$$F_0(u) = \begin{cases} u^4, & |u| \ge 1, \\ \frac{u}{d^2}, & -\frac{d}{2} \le u \le d, \\ 1 + \frac{1-u}{d}, & d \le u \le 1, \\ 1 - \frac{(1+u)(1+2d)}{d(2-d)}, & -1 \le u \le -\frac{d}{2}. \end{cases}$$

Note that

$$\max\{F_0(x) : |x| \le 1\} = F_0(d) = 1/d = 4.3478.$$

We have $F_0 \in C(\mathbb{R})$ but $F_0 \notin C^1(\mathbb{R})$. Now, we define a C^1 function F, modifying F_0 around the points $-1, -d/2, d$ and 1 by cubic functions (splines). Take $n : 0 < 1/n < d/2$. Define the function F as $F_0\,(u)$ if

$$u \in \mathbb{R}\backslash\big\{\big[-1, -1+\frac{1}{n}\big] \cup \big[-\frac{d}{2}-\frac{1}{n}, -\frac{d}{2}+\frac{1}{n}\big] \cup \big[d-\frac{1}{n}, d+\frac{1}{n}\big] \cup \big[1-\frac{1}{n}, 1\big]\big\}$$

and by cubic functions φ_1, φ_2 and φ_3 for $0 \le u \le 1$ as:

$$\begin{aligned} \varphi_1(d-1/n) &= F_0(d-1/n), \varphi_1'(d-1/n) = 1/d^2, \\ \varphi_1(d) &= 1/d, \varphi_1'(d) = 0, \\ \varphi_2(d) &= 1/d, \varphi_2'(d) = 0, \\ \varphi_2(d+1/n) &= F_0(d+1/n), \varphi_2'(d+1/n) = -1/d, \\ \varphi_3(1-1/n) &= F_0(1-1/n), \varphi_3'(1-1/n) = -1/d, \\ \varphi_3(1) &= 1, \varphi_3'(1) = 4. \end{aligned}$$

Similarly, we modify the function F_0 for $-1 \le u \le 0$.

Moreover, if we choose $R = 1$, $\mu = 3$ assumptions (i_1) and (i_2) of Theorem **3.1** are satisfied.

Note that the left bound for λ is

$$\frac{2kd^2}{F(d)} = 2kd^3 = 0.0721147$$

and the right bound for λ is

$$\frac{\Gamma^2(\alpha)(2\alpha-1)}{2\frac{1}{d}} = \frac{\Gamma^2(\alpha)(2\alpha-1)d}{2} = 0.0775076.$$

We can take $\lambda \in]0.0722, 0.0775[$, which implies the existence of at least two nontrivial solutions of the problem (1.1) with above data.

Example **3.2**. Consider the function $f : \mathbb{R} \to]0, +\infty[$ defined by $f(t) = 3t^2+1$. We have $\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty$ and (1.5) is satisfied. Moreover, one has $\lambda^* = \frac{(2\alpha-1)\Gamma^2(\alpha)}{2} \sup_{c>0}\{\frac{c^2}{F(c)}\} = \frac{(2\alpha-1)\Gamma^2(\alpha)}{4}$. Due to Theorem **1.1**, for each $\lambda \in]0, \frac{(2\alpha-1)\Gamma^2(\alpha)}{4}[$ and for each $\alpha \in]\frac{1}{2}, 1]$ the problem

$$\begin{cases} {}_xD_b^\alpha \left({}_a^CD_x^\alpha u(x)\right) + u(x) = \lambda(3u^2+1) & \text{in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

admits at least two nonnegative solutions.

Example **3.3**. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} te^{-t} & t \ge 0 \\ 0 & t < 0 \end{cases}.$$

Since f is nonnegative and $\lim_{t\to+\infty} \frac{f(t)}{t} = 0$ then, from Theorem **1.2** there is

$$\bar{\lambda} = 2k \inf_{d>0}\{\frac{d^2}{F(d)}\}$$

such that for all $\lambda > \bar{\lambda}$ and for each $\alpha \in]\frac{1}{2}, 1]$ the problem

$$\begin{cases} {}_xD_b^\alpha \left({}_a^C D_x^\alpha u(x) \right) + u(x) = \lambda u(x) e^{-u(x)} & \text{in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

admits at least one nonnegative solution.

References

[1] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, # 65 of CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, USA (1986).

[2] D. Averna, G. Bonanno, A three critical points theorem and its applications to the ordinary Dirichlet problem. *Topol. Math. Nonlinear Anal.* **2** (2003), 93–103.

[3] G. Bonanno, A. Sciammetta, An existence result of non trivial solution for two points boundary value problems. *Bull. Aust. Math. Soc.* **84** (2011), 288–299.

[4] G. Bonanno, A critical point theorem via Ekeland variational principle. *Nonlinear Anal.* **75** (2012), 2992–3007.

[5] G. Bonanno, Relations between the mountain pass theorem and local minima. *Adv. Nonlinear Anal.* **1**, No 3 (2012), 205–220.

[6] G. Bonanno, R. Rodriguez-Lopez, S. Tersian, Existence of solutions to boundary value problem for impulsive fractional differential equations. *Fract. Calc. Appl. Anal.* **17**, No 3 (2014), 717–744; DOI: 10.2478/s13540-014-0196-y.

[7] J. Chen, X. H. Tang, Existence and multiplicity of solutions for some fractional Boundary value problem via critical point theory. *Hindawi Publishing Corporation, Abstract Appl. Anal.* **2012** (2012), Article ID 648635, 21 pages, doi:10.1155/2012/648635

[8] G. Cottone, M. Di Paola, M. Zingales, Elastic waves propagation in 1D fractional non local contiuum. *Physica E* **42** (2009), 95–103.

[9] M. Di Paola, M. Zingales, Long-range cohesive interactions of non local continuum faced by fractional calculus. *Internat. J. of Solids and Structures* 45 (2008), 5642–5659.

[10] F. Jiao, Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory. *Comput. Math. Appl.* **62** (2011), 1181–1199.

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applicationa of Fractional Differential equations*, North- Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam (2006).

[12] I. Podlubny, *Fractional Differential Equations.* Mathematics in Science and Engineering # 198, Academic Press, Boston etc. (1999).

[13] R. Rodriguez-Lopez, S. Tersian, Existence of solutions to boundary value problem for impulsive fractional differential equations. *Fract. Calc. Appl. Anal.* **17**, No 4 (2014), 1016–1038; DOI: 10.2478/s13540-014-0212-2.

[14] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach, Longhorne, PA – USA (1993).

[1] *Department of Mathematics and Computer Science*
University of Palermo
90123 – Palermo, ITALY
e-mail: diego.averna@unipa.it

[2] *Department of Mathematics*
University of Ruse
7017 – Ruse, BULGARIA
e-mail: sterzian@uni-ruse.bg

Received: April 30, 2015

[3] *Department of Mathematics and Computer Science*
University of Palermo
90123 – Palermo, ITALY
e-mail: elisa.tornatore@unipa.it