The arithmetic decomposition of central Cantor sets

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A B S T R A C T

Every central Cantor set of positive Lebesgue measure is the arithmetic sum of two central Cantor sets of Lebesgue measure zero. Under some mild condition this result can be strengthened by stating that the summands can be chosen to be C^s regular if the initial set is of this class.

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1. Introduction

Cantor sets (by which name in this note we understand non-empty bounded nowhere dense perfect subsets of $\mathbb{R}$) and their arithmetic sums appear in many different settings. Related with the study of bifurcations of generic one-parameter families of surface diffeomorphisms having a generic homoclinic tangency at a parameter value, J. Palis [10] asked if the arithmetic sum (or difference) of two Cantor sets, both with Lebesgue measure zero is either of Lebesgue measure zero or it contains an interval. This is false in full generality (see [11], [2], [1]), but Moreira and Yoccoz in the ingenious paper [9] have shown that it is generically true for dynamically defined Cantor sets.

The main result of our note shows that every central Cantor set of positive Lebesgue's measure gives rise to a counterexample to the Palis hypothesis. In our second theorem we use the powerful characterization of degree of regularity of central Cantor sets – established in [2] – for strengthening our decomposition result in terms of regularity.

Given a convergent series $\sum a_n$ of positive and nonincreasing terms, we will denote the set of its subsums by $E(a_n)$, that is,

$$E(a_n) \ := \left\{ x \in \mathbb{R} : \ \exists A \subset \mathbb{N} \qquad x \ = \sum_{n \in A} a_n \right\}.$$

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The n-th remainder of the series $\sum a_n$ will be denoted by r_n, that is, $r_n := \sum_{k=n+1}^{\infty} a_k$. The classical result of S. Kakeya says that if $a_n > r_n$ for all n (we say in this case that $\sum a_n$ is fast convergent) then $E(a_n)$ is a Cantor set and its Lebesgue measure is $\mu E(a_n) = \lim_n 2^n r_n$ [6–8,5].

A central Cantor set in $\mathbb{R}$ is constructed from a sequence $(\lambda_i)_{i\in\mathbb{N}}$, with $\lambda_i < \frac{1}{2}$ for all $i \in \mathbb{N}$, in the following way: choose an arbitrary closed interval K_0 and delete the middle open interval of length $|K_0| - 2\lambda_1|K_0|$ leaving two intervals each of length $\lambda_1|K_0|$. Call this process "process λ_1 on K_0". Let K_1 be the union of the remaining two intervals. Now do the "process λ_2 on each of the two intervals of K_1" obtaining a compact set K_2 which is the union of 2^2 intervals each of length $\lambda_1\lambda_2|K_0|$. Proceeding inductively, doing the "process λ_{i+1} on each of the 2^i intervals of K_i", one constructs for each $n \in \mathbb{N}$ a set K_n that is the union of 2^n intervals each of length $\prod_{i=1}^{n} \lambda_i |K_0|$. The central Cantor set is given by the intersection of these sets. That is, $C_{(\lambda_i)} = \bigcap_{i=0}^{\infty} K_i$.

There is a natural duality between central Cantor sets and convergent series of positive terms. Namely a set C with $\min C = 0$ and $\max C = s > 0$ is a central Cantor set if and only if it is the set of subsums of a fast convergent series $\sum a_n$ of positive terms and of sum s. Indeed, if $C = C_{(\lambda_i)}$ with the initial interval $[0, s]$, then $C = E(a_n)$ where $r_0 = s$, $r_n := s\prod_{i=1}^{n} \lambda_i$ and $a_n = r_{n-1} - r_n = s(1-\lambda_n)\prod_{i=1}^{n-1} \lambda_i$ for $n \in \mathbb{N}$. Conversely, if $\sum a_n$ is a fast convergent series of positive terms and of sum s, then $E(a_n) = C_{(\lambda_i)}$ where $\lambda_i = \frac{r_i}{r_{i-1}}$ for $i \in \mathbb{N}$. Moreover,

$$\mu E(a_n) \;=\; \mu C_{(\lambda_i)} \;=\; \lim_{n\to\infty} 2^n r_n \;=\; \prod_{i=1}^{\infty}(2\lambda_i).$$

A Cantor set $C \subset [0,\, 1]$ is said to be C^r regular for $r \geq 0$ if there is a C^r map $\phi:\ P \to [0,\, 1]$ such that:

(i) P is the union of a finite family of at least two pairwise disjoint closed subintervals of $[0,\, 1]$ such that $0,\, 1 \in P$. (The component intervals of P will be enumerated in a natural way from the left to the right. Then $P = P_1 \cup \ldots \cup P_N$, $P_i < P_{i+1}$, $\min P_1 = 0$, $\max P_N = 1$.)
(ii) ϕ is continuous (or, equivalently, for each $i = 1, \ldots, N$ the restriction $\phi\big|_{P_i}$ is continuous).
(iii) Each $\phi\big|_{P_i}$ is expanding, that is,

$$\forall i = 1, \ldots, N\ \exists\ \alpha_i > 1\ \forall\ x, y \in P_i \quad |\phi(x) - \phi(y)| \;\geq\; \alpha_i |x - y|.$$

(iv) For every $i = 1, \ldots, N$ the set $\phi(P_i)$ is the convex hull of a nonempty subfamily of $\{P_k :\ k = 1, \ldots, N\}$.
(v) $\forall i = 1, \ldots, N\ \exists\ m_i \in \mathbb{N} \quad P \subset \phi^{m_i}(P_i)$.
(vi)

$$C \;=\; \bigcap_n \phi^{-n}([0,\, 1]).$$

Every central Cantor set is regular of class C^0. It is not easy to characterize C^s regular central Cantor sets for $s \geq 1$, but such an analytic characterization was found by Bamón, Plaza and Vera in [2] under some rather mild additional assumptions on the sequence (λ_i). We are now going to rephrase their Theorem 1 in terms of conditions on $\sum a_n$ instead of on the sequence (λ_i).

The Bamón–Plaza–Vera Theorem. *Let $\sum a_n$ be a fast convergent series of positive terms satisfying the conditions:*

(B_1) *there exists a limit $\gamma := \lim_{n\to\infty} \frac{a_n}{r_n} > 1$;*
(B_2) *either $\frac{a_n}{r_n} > \gamma$ for all n or $\frac{a_n}{r_n} < \gamma$ for all n,*

and let s be a positive integer. Then the set $E(a_n)$ is regular of class C^s if and only if

$$\lim_{n\to\infty} \frac{a_n - \gamma r_n}{(r_{n-1})^s} \;=\; 0.$$

Indeed, denoting $\gamma_i := \frac{a_i}{r_i}$, we have $\lambda_i = \frac{1}{1+\gamma_i}$, and the condition $\lambda := \lim \lambda_i \in (0, \frac{1}{2})$ is equivalent to $\gamma := \lim \gamma_i > 1$. Moreover, $\lambda_i < \lambda$ holds if and only if $\gamma_i > \gamma$. Hence the condition (A_1) from [2] is equivalent to our (B_1) and the weakened (A_2) is equivalent to (B_2).

The last auxiliary result needed for our note is an enhancement of a less known version of the Stolz–Cesaro Theorem (see [4]). To avoid ambiguity let us say explicitly that a sequence (x_n) is said to be convergent to a limit $g \in \overline{\mathbb{R}}$ from the right (left) if $x_n \to g$ and all terms of the sequence (x_n) are greater (smaller) than the limit.

The Stolz–Cesaro Theorem. *Let the sequence (b_n) decrease strictly to 0 and the sequence (c_n) tends to 0. If the sequence $\left(\frac{c_n - c_{n+1}}{b_n - b_{n+1}}\right)_{n\in\mathbb{N}}$ tends to a limit (not necessarily finite) from one side, then the sequence $\left(\frac{c_n}{b_n}\right)_{n\in\mathbb{N}}$ tends to the same limit from the same side.*

Proof. We will consider only the case $\frac{c_n - c_{n+1}}{b_n - b_{n+1}} \to g$ from the right, $g \in \mathbb{R}$, since the remaining three cases are very similar. Given an $\epsilon > 0$, there is an integer N such that $0 < \frac{c_i - c_{i+1}}{b_i - b_{i+1}} - g < \epsilon$ for all $i \geq N$. Thus, for all $m > n \geq N$,

$$g(b_n - b_m) \;<\; c_n - c_m \;<\; (g+\epsilon)(b_n - b_m). \tag{1}$$

Passing to limits as $m \to \infty$, we obtain

$$g \leq \frac{c_n}{b_n} \leq g + \epsilon \tag{2}$$

for $n \geq N$ and since ϵ was arbitrary it follows that $\frac{c_n}{b_n} \to g$.

Since the left inequality in (1) holds for all n, the left inequality in (2) is valid for all n as well. It remains to exclude the possibility that $\frac{c_n}{b_n} = g$ for some n. Indeed, if it were the case, then we would get from (1)

$$g \;=\; \frac{c_n}{b_n} \;<\; \frac{c_n - c_{n+1}}{b_n - b_{n+1}},$$

and thus $c_{n+1} b_n < c_n b_{n+1}$, yielding

$$\frac{c_{n+1}}{b_{n+1}} \;<\; \frac{c_n}{b_n} \;=\; g$$

which contradicts the left inequality in (2). $\square$

Let us finish the introduction with the following remark. Our claim that it makes no difference whether one considers the sum $A + B$ or the difference $A - B$ of two Cantor sets is true if they both are central Cantor sets, since $A - B$ is then a translation of $A + B$. Our claim fails for more general Cantor sets. A nice and brief historical note on that can be read in the introduction to the paper [3].

2. The arithmetic decomposition

Our main decomposition theorem is based on an idea used by Anisca and Ilie [1, Prop. 2].

Theorem 1. *Every central Cantor set is the arithmetic sum of two central Cantor sets of Lebesgue measure zero.*

Proof. Let C be a central Cantor set. Without loss of generality we may assume that $\min C = 0$. Then there is a unique convergent series $\sum a_n$ of positive, nonincreasing terms such that $C = E(a_n)$ and $a_n > r_n = \sum_{k=n+1}^{+\infty} a_k$ for all n. Moreover the Lebesgue measure of C is

$$\mu(C) = \lim_{n\to\infty} 2^n r_n \tag{3}$$

We define $a'_n := a_{2n-1}$ and $a''_n := a_{2n}$ for all $n \in \mathbb{N}$. Let r'_n and r''_n be the n-th remainders of the two series $\sum a'_n$ and $\sum a''_n$. We have clearly $r'_n < a'_n$ and $r''_n < a''_n$ for all n and hence the corresponding sets of subsums $C_1 := E(a'_n)$ and $C_2 := E(a''_n)$ are central Cantor sets such that $C = C_1 + C_2$.

Moreover $r'_n + r''_n = r_{2n}$ and $r'_n \geq r''_n$ and hence $2r''_n \leq r_{2n}$. Thus

$$0 < 2^n r''_n < \frac{2^{2n} r_{2n}}{2^{n+1}}$$

It implies by (3)

$$\mu(C_2) = \lim 2^n r''_n = 0$$

Starting with equality $r'_n + r'_{n-1} = r_{2n-1}$ we obtain $\mu(C_1) = 0$ in similar way which completes the proof of Theorem 1. $\square$

We will say that a series $\sum a_n$ satisfies the condition (B_3) if the sequence $\left(\frac{a_i}{r_i}\right)$ is monotone, but not eventually constant.

Using the powerful characterization from [2] we are able to prove a partially stronger version of our Theorem 1, a version that discusses the degree of regularity of both summands of the decomposition.

Theorem 2. *Let $C = E(a_n)$ be a C^s-regular central Cantor set satisfying the conditions (B_1) and (B_3). Then C is the arithmetic sum of two C^s-regular central Cantor sets of Lebesgue measure zero.*

Proof. We know from the proof of Theorem 1 that $C = C_1 + C_2$ where $C_1 = E(a'_n)$ and $C_2 = E(a''_n)$ are central Cantor set of Lebesgue measure zero where $a'_n = a_{2n-1}$ and $a''_n = a_{2n}$. We are going to prove that C_1 is regular of class C^s.

Since, by (B_1), $\lim_{n\to\infty} \frac{a_n}{r_n} = \gamma > 1$, we have

$$\lim_{k\to+\infty} \frac{a_k}{a_{k+1}} = \lim_{k\to\infty} \frac{a_k}{r_k}\left(1 + \frac{1}{\frac{a_{k+1}}{r_{k+1}}}\right) = \gamma + 1 \tag{4}$$

Moreover if $\lim_{k\to+\infty} \frac{a_k}{r_k} = \gamma$ from the right, then $\lim_{k\to+\infty} \frac{a_k}{a_{k+1}} = \gamma + 1$ from the right as well, and if $\lim_{k\to+\infty} \frac{a_k}{r_k} = \gamma$ from the left, then $\lim_{k\to+\infty} \frac{a_k}{a_{k+1}} = \gamma + 1$ from the left.

We observe also that, by Stolz–Cesaro Theorem,

$$\lim_{n\to\infty} \frac{r_{2n-1}}{r'_n} = \lim_{n\to\infty} \frac{\sum_{k=2n}^{+\infty} a_k}{\sum_{k=n+1}^{+\infty} a_{2k-1}} = \lim_{n\to+\infty} \frac{a_{2n} + a_{2n+1}}{a_{2n+1}} = \gamma + 2 \tag{5}$$

Moreover, the sequence $\left(\frac{r_{2n-1}}{r'_n}\right)$ tends to $\gamma + 2$ from the same side as $\left(\frac{a_n}{r_n}\right)$ tends to γ.

Similarly, by Stolz–Cesaro Theorem we obtain also

$$\lim_{n\to\infty} \frac{r'_n}{r_{2n}} = \frac{\gamma+1}{\gamma+2} \tag{6}$$

Further we have

$$\lim_{n\to\infty} \frac{a'_n}{r'_n} = \lim_{n\to\infty} \frac{a_{2n-1}}{r_{2n-1}} \frac{r_{2n-1}}{r'_n} = \gamma(\gamma+2) > 1$$

This means that the set C_1 satisfies the condition (B_1).

Denote the last limit by γ'. Since C satisfies (B_3), the sequence $(\frac{a'_n}{r'_n})$ tends to γ' from the same side as $(\frac{a_n}{r_n})$ tends to γ. In particular the series $\sum a'_n$, and so the set C_1, satisfies the condition (B_2) of the Bamón–Plaza–Vera Theorem. Therefore, by the Bamón–Plaza–Vera Theorem, in order to show that C_1 is regular of class C^s, it suffices to show that

$$\lim_{n\to\infty} \frac{a'_n - \gamma' r'_n}{(r'_{n-1})^s} \quad = \quad 0.$$

We will use the equality

$$\frac{a'_n - \gamma' r'_n}{(r'_{n-1})^s} \cdot \left(\frac{r'_{n-1}}{r_{2n-2}}\right)^s \quad = \quad \frac{a_{2n-1} - \gamma(\gamma+2) r'_n}{(r_{2n-2})^s}\,.$$

Since the second factor on the left has a finite non-zero limit by (6), it suffices to show that the right-hand side tends to zero. Writing the right-hand side in the form of a sum

$$\frac{a_{2n-1} - \gamma r_{2n-1}}{(r_{2n-2})^s} \;+\; \gamma\, \frac{r_{2n-1} - (\gamma+2) r'_n}{(r_{2n-2})^s}, \tag{7}$$

we see that the first summand in (7) tends to 0 by the Bamón–Plaza–Vera Theorem, because C is regular of class C^s.

In order to deal with the second summand in (7) we are now going to observe that if $\left(\frac{a_n}{r_n}\right)$ tends to γ from the right, then

$$0 \;<\; r_{2n-1} \;-\; (\gamma+2) r'_n \;<\; a_{2n-1} \;-\; \gamma r_{2n-1} \tag{8}$$

for all n. The first inequality follows from the remark we made after (5). Similarly, the remark after (4) says that $\frac{a_{2n-1}}{a_{2n}} > \gamma + 1$ for all n. Thus,

$$(1+\gamma) \sum_{k=n}^{\infty} a_{2k} \;<\; a_{2n-1} \;+\; \sum_{k=n+1}^{\infty} a_{2k-1}$$

and hence

$$(1+\gamma) \sum_{k=2n}^{\infty} a_k \;<\; a_{2n-1} \;+\; \sum_{k=n+1}^{\infty} a_{2k-1} \;+\; (1+\gamma) \sum_{k=n+1}^{\infty} a_{2k-1}$$

which means that

$$(1+\gamma) r_{2n-1} \;<\; a_{2n-1} \;+\; (2+\gamma) r'_n$$

and the second inequality in (8) follows.

It can be proven in an analogous way that if $\left(\frac{a_n}{r_n}\right)$ tends to γ from the left, then

$$a_{2n-1} \; - \; \gamma r_{2n-1} \; < \; r_{2n-1} \; - \; (\gamma+2)r'_n \; < \; 0$$

for all n. Thus, if

$$\lim_{n\to\infty} \frac{a_n - \gamma r_n}{(r_{n-1})^s} \; = \; 0,$$

then

$$\lim_{n\to\infty} \frac{r_{2n-1} - (\gamma+2)r'_n}{(r_{2n-2})^s} \; = \; 0$$

as well. Hence, if C is regular or class C^s then so is C_1 by the Bamón–Plaza–Vera Theorem.

The proof that C_2 is then regular of class C^s is very similar and we leave it out. $\square$

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