Existence results for periodic boundary value problem with a convenction term

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Abstract: By using an abstract coincidence point theorem for sequentially weakly continuous maps the existence of at least one positive solution is obtained for a periodic second order boundary value problem with a reaction term involving the derivative u' of the solution u; the so called convention term. As consequence of the main result also the existence of at least one positive solution is obtained for a parameter-depending problem.
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1 Introduction

The aim of this paper is to obtain new existence results for the following periodic boundary value problem

$$\begin{cases} -u'' + M(t)u = f(t, u, u') & \text{in } (0, T) \\ u(T) - u(0) = u'(T) - u'(0) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $M : [0, T] \to \mathbb{R}$ is a continuous and positive function and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(t, 0, 0) \neq 0$, for every $t \in [0, T]$.

As usual, here we say that problem (1.1) has a convention term because the nonlinearity f depends both on the function u and its derivative u'.

Concerning boundary value problems there is a well consolidated literature where many pioneering results are obtained by several scholars using different tools, as for instance, a priori bounds and topological degree [8, 10, 22]; upper and lower methods [7, 14, 24] and fixed point theory [1] and [11].

In particular, as pointed out in [25], the application of the fixed point theorem in studying problem (1.1) is strictly connected to the sign properties of the Green's function associated to the linear homogeneous problem, that is $f \equiv 0$.

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Recently, many authors paid attention to this topic and very interesting results are pointed out in [2, 5, 12, 13, 15, 18, 20, 26, 27].

Here, for obtaining our main results, we apply a coincidence point theorem for sequentially weakly continuous maps [3], see Theorem 2.1 below, in the variational setting used in [23]. Such approach in spirit is based on an useful version of K. Fan's fixed point theorem [9] contained in [4]. However, we do not use the Green's function to get the solutions of problem (1.1). Moreover, we do not require any asymptotic growth condition on the nonlinearity f at zero and/or at infinity. We just assume condition (3.2) below, together $f(t,0,0) \neq 0$, for every $t \in [0,T]$ to guarantee the existence of a nontrivial solution which become positive provided that $f(t,0,0) > 0$ for every $t \in [0,T]$.

However, as far as we know, there are few papers dealing with problem (1.1). For example, in [19], applying a coincidence degree theorem and when the nonlinear term is of the form $f(t,x,y) = h(t)g(x,y)$, the existence of at least one positive solution is ensured in terms of the relative behaviors of $\frac{g(x,y)}{|x|+|y|}$ for $|x|+|y|$ near 0 and $+\infty$, where

(H) $h : [0,T] \to [0,+\infty)$ and $g : [0,+\infty) \times \mathbb{R} \to [0,+\infty)$ are continuous, $h(t) \not\equiv 0$.

Furthermore, for the readers interested to the applications of periodic BVP in physics and engineering, we again mention [19] and the references therein.

On the other hand, it seems that much more attention is paid to problems without convention terms and depending from a positive parameter λ. An example is the following

$$\begin{cases} -u'' + M(t)u = \lambda g(t,u) & \text{in } (0,T) \\ u(T) - u(0) = u'(T) - u'(0) = 0, \end{cases} \tag{1.2}$$

where $T > 0$, $M : [0,T] \to \mathbb{R}$ is a continuous and positive function and $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In this case, many existence, non-existence and multiplicity results have been obtained, for instance, in [12, 13, 16, 17, 20, 21, 27], requiring suitable asymptotic behaviors of the "slope" $f(t,u)/u$ of f at zero and at infinity.

Finally, for the sake of completeness, we wish to stress that in [3] and [6] a similar approach to those proposed in the present note has been adopted for the study of a Dirichlet and a Neumann boundary value problem respectively.

2 Preliminaries

We recall that the weak derivative of a function $u \in L^1([0,T])$ is a function $u' \in L^1([0,T])$ such that

$$\int_0^T u(t)\varphi'(t)\,dt = -\int_0^T u'(t)\varphi(t)\,dt$$

for every $\varphi \in C_T^\infty$, where C_T^∞ is the space of indefinitely differentiable T-periodic functions (see [23]).

Let us denote by H_T the Sobolev space of functions $u \in L^2([0,T])$ having a weak derivative $u' \in L^2([0,T])$, while

$$H_T^2 = \{u \in H_T : \ u' \in H_T\}.$$

According to ([23, pp. 6-7]), for every $u \in H_T^2$ one has that

$$\int_0^T u'(t)\ dt = \int_0^T u''(t)\ dt = 0,$$

hence the periodic conditions $u(T) - u(0) = u'(T) - u'(0) = 0$ hold. Moreover, if we endow H_T^2 with the norm

$$\|u\| = \|u\|_2 + \|u'\|_2 + \|u''\|_2$$

for every $u \in H_T^2$ and on $C^1([0,T])$ we consider the norm

$$\|u\|_{C^1} = \max\{\|u\|_\infty, \|u'\|_\infty\},$$

H_T^2 is compactly embedded in $C^1([0,T])$, see [23, Proposition 1.2]. In particular, if $u \in H_T^2$ observe that

$$\begin{aligned} |u(t)| &= \frac{1}{T} \left| \int_0^T u(s) + \int_0^T \left(\int_s^t u'(x)\ dx \right) ds \right| \\ &\leq \frac{1}{T}\|u\|_1 + \|u'\|_1 \leq T^{-1/2}\|u\|_2 + T^{1/2}\|u'\|_2 \\ &\leq \max\{T^{-1/2},\ T^{1/2}\}\|u\| \end{aligned}$$

for every $t \in [0,T]$. Thus, if we put

$$c_T = \max\{T^{-1/2},\ T^{1/2}\}, \tag{2.1}$$

one can conclude that

$$\|u\|_\infty \leq c_T\|u\|. \tag{2.2}$$

Similarly one can obtains

$$\|u'\|_\infty \leq c_T\|u\|, \tag{2.3}$$

namely

$$\|u\|_{C^1} \leq c_T\|u\|. \tag{2.4}$$

Incidentally, observe that if $0 < T \leq 1$ then $c_T = T^{-1/2}$ and one can realize the equality in (2.4) choosing u constant. Namely, if $0 < T < 1$ the constant introduced in (2.1) is the best one of the embedding. Some sharp estimates for the norms of functions in H_T can be found in [23, Proposition 1.3].

A direct computation based on (2.4) shows that for every $r > 0$

$$B_r = \{u \in H_T^2 : \|u\| \leq r\} \subseteq \{u \in C^1([0,1]) : \ \|u\|_{C^1} \leq c_T r\}. \tag{2.5}$$

The following coincidence point theorem represents the key tool for the proof of our main results.

Theorem 2.1. *Let X, Y be real Banach spaces, let K be a weakly compact, convex subset of X, and let F, G be sequentially weakly continuous functions from K into Y, that is, if $x_n \rightharpoonup x$ in K then $F(x_n) \rightharpoonup F(x)$ and $G(x_n) \rightharpoonup G(x)$ in Y. Assume that $F^{-1}(y)$ is a nonempty convex set for all $y \in G(K)$. Then there exists $x_0 \in K$ such that $F(x_0) = G(x_0)$.*

3 Main results

Here is the first existence result for the considered periodic problem.

Theorem 3.1. *Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Put*

$$\tau = \frac{\mu}{c_T\sqrt{T}\,[1 + (T+1)(\|M\|_\infty + \mu)]}, \tag{3.1}$$

with $\mu = \min_{t\in[0,T]} M(t)$, and assume that there exists $r > 0$ such that

$$\max_{(t,x,y)\in[0,T]\times[-r,r]\times[-r,r]} |f(t,x,y)| \le \tau \cdot r. \tag{3.2}$$

Then, problem (1.1) admits at least one classical solution $\tilde{u}$ such that

$$(\tilde{u}(t), \tilde{u}'(t), \tilde{u}''(t)) \in \ [-r,r] \times [-r,r] \times [-(\|M\|_\infty + \tau)r, (\|M\|_\infty + \tau)r].$$

Proof. We will apply Theorem 2.1 with $X = H_T^2$, $Y = X^*$, $K = B_\rho$, being $\rho = \frac{r}{c_T}$, and $F, G : X \to X^*$ the functions defined as follows

$$F(u)(v) = \int_0^T (u'(t)v'(t) + M(t)u(t)v(t))\; dt,$$

$$G(u)(v) = \int_0^T f(t, u(t), u'(t))\; dt$$

for every $u,\ v \in X$. Indeed, K is weakly compact in view of the reflexivity of X, while the compactness of the embedding of X into $C^1([0,T])$ assures that both F and G are sequentially weakly continuous functions from X to X^*.
We claim that

$$G(K) \subseteq F(K). \tag{3.3}$$

Fix $w^* \in G(K)$ and let $w \in K$ be such that $G(w) = w^*$. Put

$$g(t) = f(t, w(t), w'(t))$$

for all $t \in [0,T]$ and observe that $g \in C^0([0,T])$. Hence, applying the Minty-Browder theorem (or the Lax-Milgram theorem) in the space H_T, the following problem

$$\begin{cases} -u'' + M(t)u = g(t) & \text{in } (0,T) \\ \\ u(T) - u(0) = u(T) - u(0) = 0 \end{cases} \tag{3.4}$$

admits a unique weak solution $u_w \in H_T$ and, in particular, thanks to the classical regularity theory, one has that $u_w \in C^2([0,T])$ and it is a classical solution.
If we localize $u_w \in H^2_T$ and prove that

$$u_w \in B_\rho, \tag{3.5}$$

we can conclude that (3.3) holds, since $F(u_w) = G(w) = w^*$.
To this end, we first point out that

$$\|u_w\|_\infty \leq \frac{\|g\|_\infty}{\mu}, \tag{3.6}$$

$$\|u'_w\|_\infty \leq T\left(\frac{\|M\|_\infty}{\mu} + 1\right)\|g\|_\infty, \tag{3.7}$$

and

$$\|u''_w\|_\infty \leq \left(\frac{\|M\|_\infty}{\mu} + 1\right)\|g\|_\infty. \tag{3.8}$$

Indeed, fix $k = \frac{\|g\|_\infty}{\mu}$ and put $\varphi(t) = (u_w - k)^+$. Obviously $\varphi \in H_T$ and $\varphi' = u'_w \cdot \chi_{\{u_w \geq k\}}$. Hence, from (3.4) one has

$$\int_0^T (u'_w\varphi' + M(t)u_w\varphi)\, dt = \int_0^T g\varphi\, dt$$

that is

$$\begin{aligned} 0 &\leq \int_0^T M(t)(u_w - k)(u_w - k)^+\, dt \\ &\leq \int_0^T ((u'_w)^2\chi_{\{u_w \geq k\}} + M(t)(u_w - k)(u_w - k)^+)\, dt \\ &= \int_0^T (g - M(t)k)(u_w - k)^+\, dt \leq 0, \end{aligned}$$

and this implies that $(u_w - k)(u_w - k)^+ \equiv 0$, namely

$$u_w(t) \leq k \tag{3.9}$$

for every $t \in [0,T]$. Arguing in a similar way, one has that

$$-k \leq u_w(t) \tag{3.10}$$

for every $t \in [0,T]$. Clearly (3.9) and (3.10) lead to (3.6).
Moreover, since $u_w(0) = u_w(T)$, there exists $t_0 \in (0,T)$ such that $u'_w(t_0) = 0$

and, in view of (3.6), for every $t \in [0,T]$ one has

$$\begin{aligned} |u'_w(t)| &= \left| \int_{t_0}^t u''_w(s)\ ds \right| \\ &= \left| \int_{t_0}^t (M(s)u_w(s) - g(s))\ ds \right| \\ &\le T(\|M\|_\infty \|u_w\|_\infty + \|g\|_\infty) \\ &\le T\left(\frac{\|M\|_\infty}{\mu} + 1\right)\|g\|_\infty, \end{aligned}$$

namely (3.7) holds.
Exploiting again that u_v is a classical solution of problem (3.4), from (3.6) one derives

$$\|u''_w\|_\infty \le \left(\frac{\|M\|_\infty}{\mu} + 1\right)\|g\|_\infty$$

and (3.8) is verified.
Now observe that from (2.5) it follows that $\|w\|_{C^1} \le r$, hence, in view of assumption (3.2), $\|g\|_\infty \le \tau \cdot r$. Putting together (3.6)-(3.8) and this last estimate, one has

$$\|u_w\|_2 + \|u'_w\|_2 + \|u''_w\|_2 \le \tau \frac{\sqrt{T}}{\mu}\left[1 + (T+1)(\|M\|_\infty + \mu)\right] r = \frac{r}{c_T} = \rho,$$

namely (3.5) holds and (3.3) is verified.
It is simple to verify that F is injective, hence $F^{-1}(w^*) = \{u_w\}$ for every $w^* \in G(K)$ and all the assumptions of Theorem 2.1 are satisfied. Thus, there exists $\tilde{u} \in K$ such that

$$F(\tilde{u})(v) = G(\tilde{u})(v)$$

for every $v \in H^2_T$. But $C^\infty_T \subset H^2_T$ implies that $\tilde{u}' \in H_T$, being $M(t)\tilde{u} - f(t,\tilde{u},\tilde{u}')$ its weak derivative. The regularity theory assures that $\tilde{u} \in C^2([0,T])$ and it is a classical solution of (1.1). The proof is complete since $\|\tilde{u}\|_\infty$, and $\|\tilde{u}'\|_\infty$ can be estimated recalling (2.5), while $\|\tilde{u}''\|_\infty$ can be estimated exploiting the fact that $\tilde{u}$ solves (1.1). □

As a consequence of the previous result, we can state the main constant sign periodic solution theorem.

Theorem 3.2. *Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(t,0,0) > 0$ for every $t \in [0,T]$. Let $\tau > 0$ as defined in (3.1) and assume that*

$$\max_{(t,x,y)\in[0,T]\times[0,r]\times[-r,r]} |f(t,x,y)| \le \tau \cdot r. \tag{3.11}$$

Then, problem (1.1) admits at least one positive classical solution $\tilde{u}$ such that such that

$$(\tilde{u}(t), \tilde{u}'(t), \tilde{u}''(t)) \in\ (0,r] \times (0,r] \times [-(\|M\|_\infty + \tau)r, (\|M\|_\infty + \tau)r].$$

Proof. We make use of some truncation arguments. Let $\hat{f} : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\hat{f}(t, x, y) = \begin{cases} f(t, x, y) & \text{if } x \geq 0 \\ f(t, 0, y) & \text{if } x < 0. \end{cases} \tag{3.12}$$

If we consider the following auxiliary periodic problem

$$\begin{cases} -u'' + M(t)u = \hat{f}(t, u, u') & \text{in } [0, T] \\ u(T) - u(0) = u'(T) - u(0) = 0, \end{cases} \tag{3.13}$$

it is evident that the non negative solutions of (3.13) are also constant sign solutions of problem (1.1). At this point, we can observe that, thanks to (3.11) and (3.12), $\hat{f}$ satisfies all the assumptions of Theorem 3.1. Hence, problem (3.13) admits at least one classical solution $\tilde{u} \in C^2([0, T])$. Finally, the proof is complete if we verify that

$$\min_{t \in [0,T]} \tilde{u}(t) > 0. \tag{3.14}$$

Suppose (3.14) false, namely, there exists $t^* \in [0, T]$ such that

$$\tilde{u}(t^*) = \min_{t \in [0,T]} \tilde{u}(t) \leq 0.$$

Thus, we have that

$$\tilde{u}'(t^*) = 0, \quad \tilde{u}''(t^*) \geq 0. \tag{3.15}$$

Indeed, if $t^* \in (0, T)$ then (3.15) is obvious. Otherwise, suppose that $t^* = 0$ (the other case $t^* = T$ is analogous). Since 0 is a minimizer of $\tilde{u}$ one has that $\tilde{u}'(0) \geq 0$, but the periodic boundary conditions lead to $\tilde{u}'(0) = 0$. Otherwise, if $\tilde{u}'(0) > 0$ one has $\tilde{u}'(T) > 0$ and for t close to T one achieves the contradiction $\tilde{u}(t) < \tilde{u}(T) = \tilde{u}(0) = \min_{[0,T]} \tilde{u}$.
Moreover, if it was $\tilde{u}''(0) < 0$, since $\tilde{u} \in C^2([0, T])$, one could find a suitable $\delta > 0$ such that $\tilde{u}'(t) < 0$ for all $t \in (0, \delta)$, in contradiction with the fact that $t^* = 0$ is a minimizer.
At this point, exploiting (3.15) one is lead to the evident contradiction

$$0 \geq -\tilde{u}''(t^*) + M(t^*)\tilde{u}(t^*) = \hat{f}(t^*, \tilde{u}(t^*), \tilde{u}'(t^*)) = f(t^*, 0, 0) > 0.$$

In conclusion, (3.14) holds and the proof is completed. □

Remark 3.3. The existence of a negative classical solution can be similarly proved if one assumes that $f(t, 0, 0) < 0$ for every $t \in [0, T]$, in place of $f(t, 0, 0) > 0$.

Corollary 3.4. *Let $T > 0$, $M : [0, T] \to \mathbb{R}$ a continuous and positive function and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ a continuous function. Then, there exists $\lambda^* > 0$ such that, for each $\lambda \in]-\lambda^*, \lambda^*[$, problem (1.2) admits at least one classical solution.*

Proof. Let τ as given in (3.1) and put

$$\lambda^* = \tau \sup_{r>0} \frac{r}{\max\limits_{[0,T]\times[-r,r]} |g(t,x)|}.$$

Therefore, fixed λ such that $|\lambda| < \lambda^*$, it is clear that there exists $r > 0$ such that

$$\max_{(t,x)\in[0,T]\times[-r,r]} |\lambda g(t,x)| < \tau r.$$

In few words, the function λg fulfils condition (3.2) of Theorem 3.1 and our conclusion follows. □

Example 3.5. The following problem

$$\begin{cases} -u'' + \frac{u}{2} = \frac{2+\sin(t)}{40\pi^2}(1-u^3)(1-u'^4) & \text{in } [0,2\pi] \\ u(2\pi) - u(0) = u'(2\pi) - u'(0) = 0, \end{cases} \tag{3.16}$$

admits at least one positive and non constant solution.

Indeed, we can apply Theorem 3.2 if we consider $r = 1$, $M(t) \equiv 1/2$ and put

$$f(t,x,y) = \frac{2+\sin(t)}{40\pi^2}(1-x^3)(1-y^4)$$

for every $(t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$, simple computation shows that

$$\begin{aligned} \max_{[0,1]\times[0,1]\times[-1,1]} |f(t,x,y)| &= \max_{[0,1]\times[0,1]\times[-1,1]} \frac{2+\sin(t)}{40\pi^2}(1-x^3)(1-y^4) \\ &= \frac{3}{40\pi^2}, \end{aligned}$$

namely (3.11) is satisfied, being $\tau = \frac{1}{8\pi(1+\pi)}$. Hence, (3.16) has at least one positive classical solution u_0 such that $(u_0(t), u_0'(t), u_0''(t)) \in (0,1] \times (0,1] \times \left[-\frac{1}{2} - \frac{1}{8\pi(1+\pi)}, \frac{1}{2} + \frac{1}{8\pi(1+\pi)}\right]$ for every $t \in [0,1]$. Finally, it is easy to verify that (3.16) does not admits constant solutions..

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