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Long time behavior for a dissipative shallow water model

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Abstract

We consider the two-dimensional shallow water model derived in [29], describing the motion of an incompressible fluid, confined in a shallow basin, with varying bottom topography. We construct the approximate inertial manifolds for the associated dynamical system and estimate its order. Finally, working in the whole space $\mathbb{R}^2$, under suitable conditions on the time dependent forcing term, we prove the L^2 asymptotic decay of the weak solutions.

Résumé

Nous considérons le modèle d'eau peu profonde à deux dimensions dérivé dans [29], décrivant le mouvement d'un fluide incompressible, confinèe dans un bassin peu profond, avec topographie du fond variable. Nous construisons des variétés inertielles approximatives pour le système dynamique associé et nous estimons son ordre. Finalement, pour le espace $\mathbb{R}^2$ avec des conditions appropriées pour la force, nous prouvons la L^2 décroissance asymptotique des solutions faibles.

Keywords: Inertial manifolds; Attractors; Time decay; Fourier splitting method; Incompressible viscous fluids; Navier–Stokes equations

1. Introduction

In [29], the authors derived the following shallow water model:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p + \eta \boldsymbol{u} = \tag{1.1a}$$
$$= b^{-1} \nabla \cdot [b\nu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T - \mathbf{I} \nabla \cdot \boldsymbol{u})] + \mathbf{f},$$

$$\nabla \cdot (b\boldsymbol{u}) = 0, \tag{1.1b}$$

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$$\boldsymbol{u}(\boldsymbol{x}, t=0) = \boldsymbol{u}_0, \tag{1.1c}$$

$$\boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \qquad \boldsymbol{x} \in \partial\Omega, \tag{1.1d}$$

$$\boldsymbol{\tau} \cdot (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T) \cdot \boldsymbol{\nu} = -\beta \boldsymbol{u} \cdot \boldsymbol{\tau} \quad \boldsymbol{x} \in \partial\Omega. \tag{1.1e}$$

In the above system $\Omega \subset \mathbb{R}^2$ is a bounded domain with sufficiently regular boundary $\partial\Omega$ and $\boldsymbol{u}(\boldsymbol{x}, t)$ denotes the velocity of the fluid at $\boldsymbol{x} \in \Omega$ and at time t. The smooth function $b(\boldsymbol{x})$ describes the bottom topography and satisfies $0 < b_i \le b(\boldsymbol{x}) \le b_s$, $\nu(\boldsymbol{x})$ is the viscosity, $\eta(\boldsymbol{x})$ is a positive smooth bounded function defined in Ω representing the combined actions of the friction at the bottom and the wind pressure, $\mathbf{I}$ is the identity, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are respectively the unity tangent and normal vector to the boundary $\partial\Omega$, $\beta(z)$ is a regular function defined in $\partial\Omega$ giving the friction coefficient at the boundary, and $\boldsymbol{f}(\boldsymbol{x})$ is the force term which describes the wind stress.

System (1.1a)–(1.1e) was derived in [29] from a three-dimensional anisotropic eddy viscosity model of an incompressible fluid confined to a shallow basin with varying bottom topography. To obtain the shallow water model (1.1a)–(1.1e), the authors assumed that the depth of the basin is much smaller than the typical horizontal length, and the typical velocity of the fluid is much smaller than the velocity of the gravity waves. This last assumption is equivalent to consider the fluid motion on time scales much longer than the period of the gravity waves so that averaging on time suppresses gravity waves. The same assumptions had been used in [5] starting from the Euler equations to derive the so called lake equations. The system (1.1a)–(1.1e) is therefore a generalization of the lake equations [28,27] as the effects of the viscous stresses are taken into account. In [29] the well posedness of the model was also established. Given that in large scale flows the Reynolds number can reach values like 10^9 or higher, the problem of the vanishing viscosity limit for models of geophysical interest is considered to be relevant, see e.g. [24] and references therein; the zero viscosity of the system (1.1a)–(1.1e) was in fact addressed in [16], while the case of degenerate topography was considered in [3,17].

In this paper we construct approximate inertial manifolds whose order decreases exponentially with respect to the dimension of the manifold. We give the dependence of all the constants with respect to the corresponding physical parameters and in particular we give explicitly the order of the approximate inertial manifolds.

When $\Omega = \mathbb{R}^2$ we address the problem of the asymptotic decay of the solutions. Under suitable conditions on the forcing term and of the initial datum, we show that the energy norm of weak solution has non-uniform decay. A weak solution which satisfies a generalized energy inequality is constructed following [32,35,21]. Then using the Fourier splitting method [42,43,49] non-uniform L^2 decay is obtained.

Similar decay questions were originally proposed by Leray in [25,26] for the Navier–Stokes equations. The first proof for decay without a rate was given by Masuda in [32] and by Kato in [22] in the case of null force and strong solutions with small data. Schonbek [42,43], using the Fourier Splitting Method, obtained the algebraic rate of decay for weak solution with large data. See also [2,15,20,23,30,48].

The main technical difficulties in the application of the above mentioned theories to system (1.1a)–(1.1e) originate: first from the fact that the incompressibility condition (1.1b) is weighted with the bottom topography; second from the presence, in (1.1a), of a non-standard dissipative operator. Therefore, besides several technical difficulties with respect to the classical 2D Navier–Stokes system, here we had to derive the appropriate exponential dichotomy as well adapted Agmon and Brezis–Gallouet inequalities. The technical details are postponed to an Appendix. Concerning the time decay in $\mathbb{R}^2$, a modified energy inequality allows us to use a modified Fourier splitting method but, the presence of bottom topography, gives rise to more complicated terms that require ad hoc estimates.

The plan of the paper is the following. In the next section, after introducing the appropriate mathematical settings for the model equations, we prove the existence of the Approximate Inertial Manifolds (AIM) and, then give the thickness of the thin neighborhood in terms of the data.

In section 3.1 we give the preliminary results to establish the decay of the solutions. In section 3.2 we prove the non-uniform asymptotic decay of the L^2 norm of the weak solution.

2. Bounded domain: approximate inertial manifolds

The concept of inertial manifold was introduced in [12], as part of the theory of dissipative differential equations. An inertial manifold for a semigroup associated to a dissipative dynamical system, is a finite dimensional Lipschitz manifold which is positively invariant, and attracts all the orbits exponentially [38,44,46]. To prove the existence of the inertial manifold it is necessary that the so called *spectral gap* condition [46] is verified. Unfortunately, this

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spectral gap condition is not verified for Navier–Stokes equations. For this reason the notion of approximate inertial manifolds (AIM) was introduced [6,8,11,10,38,39,41,46,47]. The existence of these manifolds does not require the spectral gap condition and therefore can be obtained for a broader class of dissipative dynamic systems. The AIM can be defined as a Lipschitz manifold surrounded by a thin neighborhood and each orbit of the system must enter in a finite time. The order of the manifold is the width of the thin neighborhood and is exponentially small compared to the size of the AIM, hence the AIM gives an approximation of the attractor of exponential order. The AIM theory plays an important role in the development of new numerical algorithms suitable to the approximation of dissipative systems for long times [7,11,18,19,13,34,31].

In this section we construct a sequence of approximate inertial manifolds $\mathcal{M}_N$ for system (1.1a)–(1.1e). Moreover, we show that the AIM $\mathcal{M}_N$ approximate the global attractor exponentially. For the proof of the existence of $\mathcal{M}_N$ and to estimate the semidistance of the attractor to $\mathcal{M}_N$, we shall follow the ideas of [6,8,39,46].

2.1. The mathematical setting

In this section we shall briefly introduce the mathematical setting appropriate for system (1.1a)–(1.1e). More details can be found in [29]. One introduces the following Hilbert spaces:

$$H = \{\boldsymbol{u} : \boldsymbol{u} \in L^2_b,\ \nabla\cdot(b\boldsymbol{u}) = 0,\ \boldsymbol{\nu}\cdot\boldsymbol{u} = 0\,\boldsymbol{x} \in \partial\Omega\} \tag{2.1}$$

$$V = \{\boldsymbol{u} : \boldsymbol{u} \in H^1_b,\ \nabla\cdot(b\boldsymbol{u}) = 0,\ \boldsymbol{\nu}\cdot\boldsymbol{u} = 0\,\boldsymbol{x} \in \partial\Omega\} \tag{2.2}$$

where L^2_b and H^1_b are Sobolev spaces with scalar products and weighted norms defined as:

$$(\boldsymbol{u},\boldsymbol{v})_b = \int_\Omega b\boldsymbol{u}\cdot\boldsymbol{v}d\boldsymbol{x}\,, \quad |\boldsymbol{u}|_b^2 = \int_\Omega b|\boldsymbol{u}|^2 d\boldsymbol{x}\,,$$

$$((\boldsymbol{u},\boldsymbol{v}))_b = \int_\Omega b\nabla\boldsymbol{u} : \nabla\boldsymbol{v}d\boldsymbol{x}\,, \quad \|\boldsymbol{u}\|_b^2 = \int_\Omega b|\nabla\boldsymbol{u}|^2 d\boldsymbol{x}\,.$$

The following Poincaré inequality holds:

$$|\boldsymbol{u}|_b \le \Pi\|\boldsymbol{u}\|_b, \tag{2.3}$$

where $\Pi = \Pi(\Omega)$.

We take the L^2_b scalar product of equation (1.1a) with a generic function $\boldsymbol{v} \in V$ and write (1.1a) in the following weak form (see [45]):

$$\frac{d}{dt}(\boldsymbol{u},\boldsymbol{v})_b + [\boldsymbol{u},\boldsymbol{v}]_{b\nu} + (\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})_b + (\eta\boldsymbol{u},\boldsymbol{v})_b = (\boldsymbol{f},\boldsymbol{v})_b, \tag{2.4}$$

where $[\cdot,\cdot]_{b\nu} : V \times V \to \mathbb{R}$, is a bilinear form defined as

$$[\boldsymbol{u},\boldsymbol{v}]_{b\nu} = \int_\Omega b\nu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T - \mathbf{I}\nabla\cdot\boldsymbol{u}\right) : \left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T - \mathbf{I}\nabla\cdot\boldsymbol{v}\right)d\boldsymbol{x} + \\ + \int_{\partial\Omega} b\nu\beta\boldsymbol{u}\cdot\boldsymbol{v}ds, \tag{2.5}$$

and, $(\cdot,\cdot,\cdot)_b : V \times V \times V \to \mathbb{R}$, is a trilinear form defined by

$$(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})_b = \int_\Omega b\,(\boldsymbol{u}\cdot\nabla\boldsymbol{w})\,\boldsymbol{v}d\boldsymbol{x}. \tag{2.6}$$

The trilinear form defines a continuous bilinear operator $B(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{u}\cdot\nabla\boldsymbol{v}$ from $V \times V$ into V' such that

$$(B(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w})_b = (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})_b. \tag{2.7}$$

With $A_{b\nu}$ we denote the operator from $V \to V'$ defined by

$$\langle A_{b\nu}\boldsymbol{u},\boldsymbol{v}\rangle = [\boldsymbol{u},\boldsymbol{v}]_{b\nu}\,. \tag{2.8}$$

We note that $A_{b\nu}$ is a linear unbounded operator on H with domain

$$D(A_{b\nu}) = \{\boldsymbol{u}\in H_b^2(\Omega), \nabla\cdot b\boldsymbol{u}=0 \quad \text{in} \quad \Omega, \quad \boldsymbol{u}\cdot\boldsymbol{\nu}=0,$$
$$\boldsymbol{\tau}\cdot(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T)\cdot\boldsymbol{\nu} = -\beta\boldsymbol{u}\cdot\boldsymbol{\tau} \quad \text{on} \quad \partial\Omega\},$$

and $D(A_{b\nu})\subset V\subset H\subset V'$, where the inclusions are continuous and dense. Moreover V is compactly embedded in H.

We observe that $B(\boldsymbol{u},\boldsymbol{v}): D(A_{b\nu})\times D(A_{b\nu})\to H$ (see again [45]).
Using (2.7) and (2.8), we can write (2.4), the weak form of equation (1.1a), as:

$$\frac{d}{dt}\boldsymbol{u} + A_{b\nu}\boldsymbol{u} + B(\boldsymbol{u},\boldsymbol{u}) + \eta\boldsymbol{u} = \boldsymbol{f}. \tag{2.9}$$

Note also that the bilinear form $[\cdot,\cdot]_{b\nu}$ is coercive, if $\beta(\boldsymbol{x})\geq\kappa(\boldsymbol{x})$, where κ is the curvature of $\partial\Omega$: Supposing this hypothesis on β we have

$$(A_{b\nu}\boldsymbol{u},\boldsymbol{u})_b \geq \bar{b}\nu_i\|\boldsymbol{u}\|_b^2, \tag{2.10}$$

where

$$\bar{b} = \frac{b_i}{b_s}, \quad \nu_i = \inf_\Omega \nu(x),$$

and

$$b_i = \inf_\Omega b(x)\,, \quad b_s = \sup_\Omega b(x).$$

For a proof of the coercivity inequality (2.10) see [29].

In [29] the authors established the well-posedness of (2.9). For completeness we state their main result:

Theorem 1. *(Theorem 4.1 of [29]) Let Ω be smooth. Suppose that $b(\boldsymbol{x})$, $\nu(\boldsymbol{x})$ and $\eta(\boldsymbol{x})$ are non-negative function over $\bar{\Omega}$. Suppose, moreover that $b\nu\geq C>0$ and that $\beta(\boldsymbol{x})\geq\kappa(\boldsymbol{x})$ on $\partial\Omega$, where $\kappa(\boldsymbol{x})$ is the curvature of $\partial\Omega$ at $\boldsymbol{x}$. Let $\boldsymbol{u}_{in}\in H_b^2\cap V$ and $\boldsymbol{f}\in L_b^2$.*
Then the system (1.1a)–(1.1e) has a unique solution $\boldsymbol{u}\in L^\infty\left([0,T],H_b^2\right)\cap C([0,T],V)$. Moreover, $\partial_t\boldsymbol{u}\in L^\infty([0,T],H)\cup L^2([0,T],V)$.

The spectral problem associated to the compact self-adjoint operator $A_{b\nu}$ admits solution in H [9], and from the coercivity (2.10) derives the existence of a non-decreasing sequence of positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ with (see [33])

$$\lambda_n\sim n\,, \qquad \text{for} \quad n\to\infty\,, \tag{2.11}$$

and a sequence of eigenfunctions forming an orthonormal basis in H. We denote by P_n the projection onto the finite dimensional space generated by the first n eigenfunctions and $Q_n = I - P_n$:

$$P_n\boldsymbol{u} = \boldsymbol{y}, \qquad Q_n\boldsymbol{u} = \boldsymbol{z} \qquad \text{and} \quad \boldsymbol{u} = \boldsymbol{y}+\boldsymbol{z}. \tag{2.12}$$

In Lemma 9, whose proof is postponed to the Appendix, we state that [14]:

$$|e^{-A_{b\nu}t}Q_n|_{\mathcal{L}(H,V)} \leq \bar{b}^{-\frac{1}{2}}\left((\nu_i t)^{-\frac{1}{2}} + \lambda_{n+1}^{\frac{1}{2}}\right)e^{-\lambda_{n+1}t}, \quad t>0, \tag{2.13}$$

$$|(I+\tau A_{b\nu})P_n|_{\mathcal{L}(V)} \leq (1+\tau\lambda_n) \leq e^{\tau\lambda_n}, \tag{2.14}$$

$$|I|_{\mathcal{L}(P_nH,P_nV)} \leq \left(\frac{\bar{b}\nu_i}{\lambda_n}\right)^{-\frac{1}{2}}. \tag{2.15}$$

If we consider an initial datum $\boldsymbol{u}_{in}$ in a ball of H with center at the origin and radius R, then there exists a time $t_0(R)$, depending on R and on ν, $\boldsymbol{f}$, Λ, b, such that for $t\geq t_0$:

$$|\boldsymbol{u}(t)|_b \leq \rho_0, \qquad \|\boldsymbol{u}(t)\|_b \leq \rho_1, \tag{2.16}$$

where ρ_0 and ρ_1 are the radii of the absorbing balls in H and V, respectively, whose explicit expressions is given in [36,40].

Moreover, following [11,10], it is possible to prove that:

$$\left|\frac{d^k\boldsymbol{u}}{dt^k}\right|_b \leq \frac{2^k k!}{\alpha^k}\rho_0, \qquad \left\|\frac{d^k\boldsymbol{u}}{dt^k}\right\|_b \leq \frac{2^k k!}{\alpha^k}\rho_1, \tag{2.17}$$

for $t \geq 2\alpha$, where $\alpha = \alpha(\Omega, |\mathbf{f}|_b, \|\boldsymbol{u}_{in}\|_b, \nu_i)$ defines the domain of time analyticity

$$\Delta = \{\xi \in \mathbb{C} : \Re\xi \leq \alpha \text{ and } |\Im\xi| \leq \Re\xi \quad \text{or} \quad \Re\xi \geq \alpha \text{ and } |\Im\xi| \leq \alpha\}. \tag{2.18}$$

From (2.16), following [46], one can derive the existence of a compact global attractor $\mathcal{A}$, connected and maximal in H, and its Hausdorff dimension $\tilde{m}$ satisfies the following estimate (see [36,40])

$$\tilde{m} - 1 \leq \frac{\bar{b}}{b_s^{1/2}}\widetilde{c}_\Omega \frac{|\boldsymbol{f}|_b \Pi^{1/2}}{\nu_i^2} < \tilde{m}.$$

For completeness, we recall that a global attractor $\mathcal{A}$ for a semigroup $S(t)$ defined in H, is a subset of H which satisfies the following properties:

- $\mathcal{A}$ is an invariant set, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for every $t \geq 0$,
- for every $\boldsymbol{u}_0 \in H$, it holds that

$$\operatorname{dist}(S(t)\boldsymbol{u}_0, \mathcal{A}) := \inf_{\boldsymbol{v}\in\mathcal{A}} |S(t)\boldsymbol{u}_0 - \boldsymbol{v}|_b \to 0, \quad \text{as} \quad t \to +\infty.$$

As it is usual in the theory of inertial manifold, we consider the associated equation derived from (1.1a), setting the non-linear term $B(\boldsymbol{u}, \boldsymbol{u})$ identically zero when $\boldsymbol{u}$ is outside the absorbing ball in V. Specifically, let $\theta \in \mathcal{C}^1$ be defined on $\mathbb{R}_+$ which is 1 in $[0, 1]$ and 0 in $[2, +\infty[$. Denote by

$$B_\theta \boldsymbol{u} = B_\theta(\boldsymbol{u}, \boldsymbol{u}) = \theta\left(\frac{\|\boldsymbol{u}\|_b}{\rho_1}\right) B(\boldsymbol{u}, \boldsymbol{u}). \tag{2.19}$$

In the sequel consider the system:

$$\frac{d\boldsymbol{u}}{dt} + A_{b\nu}\boldsymbol{u} + \eta\boldsymbol{u} = B_\theta\boldsymbol{u} + \boldsymbol{f}. \tag{2.20}$$

The problem (2.20) is well posed and has the same attractor as (2.9). Moreover there exist two constants M_0 and M_1, see Lemma 12 in the Appendix, such that for every $\boldsymbol{u}, \boldsymbol{v} \in V$,

$$|B_\theta\boldsymbol{u}|_b \leq M_0, \qquad |B_\theta\boldsymbol{u} - B_\theta\boldsymbol{v}|_b \leq M_1\|\boldsymbol{u} - \boldsymbol{v}\|_b. \tag{2.21}$$

2.2. *Existence of approximate inertial manifolds*

An Inertial Manifold (IM) $\mathcal{M} = \{\boldsymbol{y}, \Phi(\boldsymbol{y})\}$ is a positively invariant manifold defined as the graph of a Lipschitz function Φ, defined from P_nH to Q_nH, which attracts all trajectories of (2.20) exponentially.
We briefly outline the Lyapunov Perron Method ([6,8,11,10,38,39,41,46]) which will be used in our proof to construct an IM.

We decompose equation (2.20) using the projections P_n and Q_n to obtain a solution in $\mathcal{M}$

$$\frac{d\boldsymbol{y}}{dt} + A_{b\nu}\boldsymbol{y} + \eta\boldsymbol{y} = P_n B_\theta(\boldsymbol{y} + \Phi(\boldsymbol{y})) + P_n\boldsymbol{f} \tag{2.22}$$

$$\frac{d\Phi(\boldsymbol{y})}{dt} + A_{b\nu}\Phi(\boldsymbol{y}) + \eta\Phi(\boldsymbol{y}) = Q_n B_\theta(\boldsymbol{y} + \Phi(\boldsymbol{y})) + Q_n\boldsymbol{f}. \tag{2.23}$$

The finite dimensional system of ordinary differential equations (2.22) is called the inertial system associated to $\mathcal{M}$. Given the initial condition $\boldsymbol{u}_{in} = \boldsymbol{y}_{in} + \Phi(\boldsymbol{y}_{in})$, since Φ is a Lipschitz function, then for every $t \in \mathbb{R}$, the equation (2.22) determines a unique $\boldsymbol{y}(t) = \boldsymbol{y}(t; \boldsymbol{y}_{in}, \Phi)$.

Assuming that Φ is bounded, to determine the function Φ, integrate system (2.23) in time to obtain

$$\Phi(\boldsymbol{y}_{in}) = \int_{-\infty}^{0} e^{A_{bv}s}[Q_n(B_\theta(\boldsymbol{y}(s)+\Phi(\boldsymbol{y}(s)))+\boldsymbol{f})-\eta\Phi(\boldsymbol{y}(s))]ds. \tag{2.24}$$

The function Φ is the fixed point of the map $\phi \to \mathcal{F}\phi$ defined by

$$\mathcal{F}\phi(\boldsymbol{y}_{in}) = \int_{-\infty}^{0} e^{A_{bv}s}[Q_n(B_\theta(\boldsymbol{y}(s)+\phi(\boldsymbol{y}(s)))+\boldsymbol{f})-\eta\phi(\boldsymbol{y}(s))]ds, \tag{2.25}$$

where $\phi : P_nH \to Q_nH$ is a bounded Lipschitz function. The existence of an inertial manifold is achieved by showing that the map $\mathcal{F}$ is a contractive map in the complete metric space

$$\mathcal{F}_{l,L} = \{\phi : P_nV \to Q_nV : \mathrm{Lip}(\phi) \le l,\ |\phi|_\infty = \sup_{\boldsymbol{y}\in P_nV} \|\phi(\boldsymbol{y})\|_b \le L\}, \tag{2.26}$$

and $\boldsymbol{y}$ is a solution of the system (2.22) with $\boldsymbol{y}(t=0) = \boldsymbol{y}_{in}$. We recall that the proof of the existence of the Inertial Manifold is based on the spectral gap condition.

If the spectral gap condition is not verified, there is no standard proof for the existence of the IM. However, it is possible to construct a sequence of Approximate Inertial Manifolds [6,8,11,10,38,39,41,46]. This is what we will do for equation (2.9).

To obtain the AIM we construct an approximating sequence of solutions to the system (2.22) as follows: let $\boldsymbol{y}_0 = \boldsymbol{y}_{in} \in P_nV$, and $\tau > 0$ be the discrete time step and define $\boldsymbol{y}_k$, $k \ge 0$, by the following Euler explicit discretization of (2.22):

$$\frac{\boldsymbol{y}_{k+1}-\boldsymbol{y}_k}{-\tau} + A_{bv}\boldsymbol{y}_k = P_nB_\theta(\boldsymbol{y}_k+\phi(\boldsymbol{y}_k)) - \eta\boldsymbol{y}_k + P_n\boldsymbol{f}. \tag{2.27}$$

Fix the positive integers n and N, to construct the approximation function $\boldsymbol{y}_\tau$ to $\boldsymbol{y}$:

$$\begin{aligned} &\boldsymbol{y}_\tau(s) = \boldsymbol{y}_k \quad \text{for} \quad -(k+1)\tau < s \le -k\tau, \quad k = 0,\dots,N-1,\\ &\boldsymbol{y}_\tau(s) = \boldsymbol{y}_N \quad \text{for} \quad s \le -N\tau. \end{aligned} \tag{2.28}$$

The approximation $\mathcal{F}_\tau^N$ of $\mathcal{F}$ is defined substituting $\boldsymbol{y}$ by $\boldsymbol{y}_\tau$ in (2.25). Explicitly

$$\begin{aligned} &\mathcal{F}_\tau^N\phi(\boldsymbol{y}_0) = \\ &-(A_{bv})^{-1}(I-e^{-A_{bv}})\sum_{k=0}^{N-1} e^{-kA_{bv}\tau}[Q_n\left(B_\theta(\boldsymbol{y}_k+\phi(\boldsymbol{y}_k))+\boldsymbol{f}\right)-\eta\phi(\boldsymbol{y}_k)]\\ &\quad -(A_{bv})^{-1}e^{-NA_{bv}\tau}[Q_n\left(B_\theta(\boldsymbol{y}_N+\phi(\boldsymbol{y}_N))+\boldsymbol{f}\right)-\eta\phi(\boldsymbol{y}_N)]. \end{aligned} \tag{2.29}$$

To obtain the family of AIM, consider a sequence of positive numbers $(\tau_N)_{N\in\mathbb{N}}$ and define the manifolds $\mathcal{M}_N$ as the graph of the functions Φ_N constructed recursively, for $N \ge 0$, by

$$\Phi_0 = 0, \quad \Phi_{N+1} = \mathcal{F}_{\tau_N}^N(\Phi_N). \tag{2.30}$$

The main result of this section is to prove, for every $N \ge 0$, the existence of Φ_N in $\mathcal{F}_{l,L}$. Before proceeding with the formulation of the main theorem of this section and its proof, we recall some preliminary properties, which guaranties the consistence of the approximation scheme described above. To ease the notation in the sequel, we denote τ_N by τ. Write (2.27) as

$$\begin{aligned} \boldsymbol{y}(-(k+1)\tau) &= (I+\tau A_{bv})\boldsymbol{y}(-k\tau)\\ &\quad -\tau P_n(B_\theta(\boldsymbol{y}(-k\tau)+z(-k\tau))+\boldsymbol{f})+\tau\eta\boldsymbol{y}(-k\tau), \end{aligned} \tag{2.31}$$

and the approximation error

$$\epsilon_k = \boldsymbol{y}(-(k+1)\tau) - \boldsymbol{y}(-k\tau) - \tau\frac{d\boldsymbol{y}}{dt}(-k\tau). \tag{2.32}$$

The following Lemmas hold:

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Lemma 1. *Suppose that $\boldsymbol{u}(t)$ is a complete trajectory inside the global attractor $\mathcal{A}$, then:*

$$\|\epsilon_k\|_b \le \tau^2\beta_1, \qquad k = 0, \dots, N-1, \tag{2.33}$$

$$\left\|\frac{d\boldsymbol{u}}{dt}\right\|_b \le \beta_2, \qquad t < 0, \tag{2.34}$$

with $\beta_1 \le \frac{8}{\alpha^2}\rho_1$ and $\beta_2 \le \frac{2\rho_1}{\alpha}$, where α defines the domain of analyticity in (2.17) and ρ_1 is the radius of the absorbing balls in V in (2.16).

Proof. If the trajectory $\boldsymbol{u}(t)$ is a complete trajectory inside the global attractor $\mathcal{A}$, one can easy obtain (2.33) and (2.34) with

$$\beta_1 = \sup_{[\boldsymbol{u}(t)]_{t\in\mathbb{R}}\in\mathcal{A}} \sup_{t\in\mathbb{R}} \left\|\frac{d^2\boldsymbol{u}}{dt^2}\right\|_b \qquad \beta_2 = \sup_{[\boldsymbol{u}(t)]_{t\in\mathbb{R}}\in\mathcal{A}} \sup_{t\in\mathbb{R}} \left\|\frac{d\boldsymbol{u}}{dt}\right\|_b$$

Using (2.17) one derives the desired bounds on β_1 and β_2. □

Lemma 2. *Let be $i = 1, 2$, and let be $\boldsymbol{y}_0^i \in P_nV$. Define $\boldsymbol{y}_k^i$, $k = 0, \dots, N$ by (2.27) and (2.31) with $\boldsymbol{y}_0 = \boldsymbol{y}_0^i$ and construct $\boldsymbol{y}_\tau^i(s)$ using (2.28). Then, for every $s \le 0$,*

$$\|\boldsymbol{y}_\tau^1(s) - \boldsymbol{y}_\tau^2(s)\|_b \le e^{-s\left[\lambda_n + \left(\frac{\bar{b}\nu_i}{\lambda_n}\right)^{-\frac{1}{2}}(M_1+\Pi\bar{\eta})(1+l)\right]} \|\boldsymbol{y}_0^1 - \boldsymbol{y}_0^2\|_b, \tag{2.35}$$

where $\bar{\eta} = \sup_\Omega \eta$, and M_1 is given in (2.21).

Proof. Denoting by $\boldsymbol{y}_k = \boldsymbol{y}_k^1 - \boldsymbol{y}_k^2$ and subtracting (2.27) or (2.31) for $i = 1, 2$, and using (2.21), (2.14) and the Lipschitz constant l of ϕ, we obtain

$$\begin{aligned}
\|\boldsymbol{y}_{k+1}\|_b &\le (1+\tau\lambda_n)\|\boldsymbol{y}_k\|_b + \tau\left(\frac{\bar{b}\nu_i}{\lambda_n}\right)^{-\frac{1}{2}} \cdot \\
&\quad \cdot \left[|B_\theta(\boldsymbol{y}_k^1 + \phi(\boldsymbol{y}_k^1)) - B_\theta(\boldsymbol{y}_k^2 + \phi(\boldsymbol{y}_k^2))|_b + \bar{\eta}|\boldsymbol{y}_k + \phi(\boldsymbol{y}_k)|_b\right] \\
&\le (1+\tau\lambda_n)\|\boldsymbol{y}_k\|_b + \tau\left(\frac{\bar{b}\nu_i}{\lambda_n}\right)^{-\frac{1}{2}}(M_1+\Pi\bar{\eta})(1+l)\|\boldsymbol{y}_k\|_b \\
&\le \exp\{k\tau[\lambda_n + \left(\frac{\bar{b}\nu_i}{\lambda_n}\right)^{-\frac{1}{2}}(M_1+\Pi\bar{\eta})(1+l)]\}\|\boldsymbol{y}_0\|_b
\end{aligned}$$

for $k = 0, \dots, N$. From the definition of $\boldsymbol{y}_\tau^i(s)$ by (2.28), we obtain (2.35). □

In the sequel we use the notation $\gamma = \int_{-\infty}^0 |s|^{-1/2} e^s\, ds$. We are now ready to establish the main theorem.

Theorem 2. *Suppose that the constants δ_1 and δ_2 satisfy*

$$(N+1)\tau \le \frac{\delta_1}{(M_1+\Pi\bar{\eta})}\left(\frac{\bar{b}\nu_i}{\lambda_n}\right)^{\frac{1}{2}}, \tag{2.36}$$

and

$$\lambda_n \ge \delta_2, \tag{2.37}$$

then there exist l and L_0 such that $\mathcal{F}_\tau^N : \mathcal{F}_{l,L} \to \mathcal{F}_{l,L}$, for all $L \ge L_0$.

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Proof. We show that the following constant L_0 and l are appropriate

$$L_0 = \bar{b}^{-1/2}(|\boldsymbol{f}|_b + M_0 + \bar{\eta}\rho_0)(\gamma\nu_i^{-1/2} + 1)\lambda_{n+1}^{-1/2}. \tag{2.38}$$

$$l = 6\left(\frac{1}{2} + \sup_n\left(\frac{\nu_i\lambda_{n+1}}{\lambda_n}\right)^{\frac{1}{2}}\right) \quad \text{and} \quad \delta_1 = \min\left(\delta_0, \frac{\log(3/2)}{l}\right). \tag{2.39}$$

Let $\phi \in \mathcal{F}_{l,L}$, suppose $\boldsymbol{y}_0 \in P_n V$ and $(\boldsymbol{y}_k)_{k=0,\dots,N}$ and $\boldsymbol{y}_\tau(s)$ be given by (2.27), (2.28).

Using (2.21), (2.13) and recalling that by definition $\phi(y) \in Q_n V \subseteq V$, we have:

$$\begin{aligned}
&\left\|\mathcal{F}_\tau^N \phi(\boldsymbol{y}_0)\right\|_b \le \qquad (2.40)\\
&\le \int_{-\infty}^{0} \left|e^{A_{b\nu}s} Q_n\right|_{\mathcal{L}(H,V)} |\boldsymbol{f} + B_\theta(\boldsymbol{y}(s) + \phi(\boldsymbol{y}(s))) - \eta(\boldsymbol{y}(s) + \phi(\boldsymbol{y}(s)))|_b ds\\
&\le \bar{b}^{-1/2}(|\boldsymbol{f}|_b + M_0 + \bar{\eta}\rho_0)\int_{-\infty}^{0}\left(|\nu_i s|^{-1/2} + \lambda_{n+1}^{1/2}\right)e^{\lambda_{n+1}s} ds\\
&\le \bar{b}^{-1/2}(|\boldsymbol{f}|_b + M_0 + \bar{\eta}\rho_0)(\gamma\nu_i^{-1/2} + 1)\lambda_{n+1}^{-1/2}.
\end{aligned}$$

From the previous inequality we deduce that $\left\|\mathcal{F}_\tau^N \phi(\boldsymbol{y}_0)\right\|_b \le L$, for every $L \ge L_0$, where L_0 was defined by (2.38). Now, we show that l is our Lipschitz constant. For this scope, let $\boldsymbol{y}_0^i \in P_n V$ and $(\boldsymbol{y}_k^i)_{k=0,\dots,N}$ and $\boldsymbol{y}_\tau^i(s)$ constructed by (2.27) and (2.28), for $i = 1, 2$. Therefore, write

$$\begin{aligned}
\mathcal{F}_\tau^N \phi(\boldsymbol{y}_0^1) - \mathcal{F}_\tau^N \phi(\boldsymbol{y}_0^2) = &\int_{-(N+1)\tau}^{0} e^{A_{b\nu}s}[Q_n(B_\theta(\boldsymbol{y}_\tau^1(s) + \phi(\boldsymbol{y}_\tau^1(s)))\\
&- B_\theta(\boldsymbol{y}_\tau^2(s) + \phi(\boldsymbol{y}_\tau^2(s)))) - \eta(\phi(\boldsymbol{y}_\tau^1(s)) - \phi(\boldsymbol{y}_\tau^2(s)))]ds\\
&+ (A_{b\nu})^{-1}e^{-(N+1)A_{b\nu}\tau}[Q_n(B_\theta(\boldsymbol{y}_N^1 + \phi(\boldsymbol{y}_N^1))\\
&- B_\theta(\boldsymbol{y}_N^2 + \phi(\boldsymbol{y}_N^2))) - \eta(\phi(\boldsymbol{y}_N^1) - \phi(\boldsymbol{y}_N^2))].
\end{aligned} \tag{2.41}$$

Using again (2.21), (2.21) and (2.13), we have:

$$\begin{aligned}
&\left\|\mathcal{F}_\tau^N \phi(\boldsymbol{y}_0^1) - \mathcal{F}_\tau^N \phi(\boldsymbol{y}_0^2)\right\|_b \le \bar{b}^{-\frac{1}{2}}(M_1 + \Pi\bar{\eta})(l+1)\cdot\\
&\quad\cdot\int_{-(N+1)\tau}^{0}\left(|\nu_i s|^{-1/2} + \lambda_{n+1}^{1/2}\right)e^{\lambda_{n+1}s}\|\boldsymbol{y}_\tau^1(s) - \boldsymbol{y}_\tau^2(s)\|_b ds\\
&+ \bar{b}^{-\frac{1}{2}}(M_1 + \Pi\bar{\eta})(l+1)\nu_i^{-1/2}\lambda_{n+1}^{-1/2}e^{-\lambda_{n+1}(N+1)\tau}\|\boldsymbol{y}_N^1 - \boldsymbol{y}_N^2\|_b.
\end{aligned}$$

Using (2.35), since $\lambda_{n+1} - \lambda_n \ge 0$, using (2.36), we obtain

$$\left\|\mathcal{F}_\tau^N \phi(\boldsymbol{y}_0^1) - \mathcal{F}_\tau^N \phi(\boldsymbol{y}_0^2)\right\|_b \le \Xi\|\boldsymbol{y}_0^1 - \boldsymbol{y}_0^2\|_b, \tag{2.42}$$

with

$$\begin{aligned}
\Xi = \bar{b}^{-\frac{1}{2}}(l+1)e^{\delta_1(l+1)}&\left[2(M_1 + \Pi\bar{\eta})^{\frac{3}{2}}\left(\frac{\bar{b}}{\nu_i\lambda_n}\right)^{\frac{1}{4}} +\right.\\
&\left. + (M_1 + \Pi\bar{\eta})^2(\nu_i\lambda_{n+1})^{-\frac{1}{2}}\right] + e^{\delta_1(l+1)}\left(\frac{\nu_i\lambda_{n+1}}{\lambda_n}\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.43}$$

We now choose δ_1 and δ_2 to ensure that $\Xi \le l$ then the proof of the theorem will be complete. First choose $\delta_0 > 0$, with $\delta_1 \le \delta_0$, and choose δ_2 in (2.37) sufficiently large so that

$$\bar{b}^{-\frac{1}{2}}\left[2\,(M_1+\Pi\bar{\eta})^{\frac{3}{2}}\left(\frac{\bar{b}}{\nu_i\lambda_n}\right)^{\frac{1}{4}}+(M_1+\Pi\bar{\eta})^2(\nu_i\lambda_{n+1})^{-\frac{1}{2}}\right]e^{\delta_1}\le$$
$$\le\bar{b}^{-\frac{1}{2}}\left[2\,(M_1+\Pi\bar{\eta})^{\frac{3}{2}}\left(\frac{\bar{b}}{\nu_i\delta_2}\right)^{\frac{1}{4}}+(M_1+\Pi\bar{\eta})^2(\nu_i\delta_2)^{-\frac{1}{2}}\right]e^{\delta_0}\le$$
$$\le\frac{1}{2}. \tag{2.44}$$

Therefore, with this choice of δ_0, δ_2 and $\delta_1\le\delta_0$ we have

$$\Xi\le\left(\frac{l}{2}+\frac{1}{2}+\sup_n\left(\frac{\nu_i\lambda_{n+1}}{\lambda_n}\right)^{\frac{1}{2}}\right)e^{l\delta_1}\le l, \tag{2.45}$$

by choosing l as defined in (2.39) at the beginning of the Theorem. This completes the proof of the Theorem. □

2.3. Approximation of the attractor

In this section we prove that the approximate inertial manifolds $\mathcal{M}_N$ built in the previous section as a graph of the Φ_N, approximates the global attractor $\mathcal{A}$.

We first try to estimate the semi-distance in V of $\mathcal{A}$ to $\mathcal{M}_N$

$$\varrho_N=d_V(\mathcal{A},\mathcal{M}_N)=\sup_{\boldsymbol{v}\in\mathcal{A}}\inf_{\boldsymbol{w}\in\mathcal{M}_N}\|\boldsymbol{v}-\boldsymbol{w}\|_b. \tag{2.46}$$

We continue to use the notations of the previous sections.

Lemma 3. *Let be $\boldsymbol{u}_0\in\mathcal{A}$ and let be $\boldsymbol{y}_0=P_n\boldsymbol{u}_0$ and $z_0=Q_n\boldsymbol{u}_0$, with P_n and $Q_n=I-P_n$ the projection operators defined in (2.12). Suppose that (2.33), (2.34), (2.36) and (2.37) are satisfied. Then for every $\phi\in\mathcal{F}_{l,L}$ results that*

$$\|\mathcal{F}_N^\tau\phi(\boldsymbol{y}_0)-z_0\|_b\le$$
$$\le\bar{b}^{-\frac{1}{2}}(M_1+\Pi\bar{\eta})\left[l(\lambda_n\nu_i)^{-\frac{1}{2}}+\frac{(\gamma\nu_i^{-\frac{1}{2}}+1)}{\lambda_{n+1}^{\frac{1}{2}}}\right]\sup_{\boldsymbol{y}+z\in\mathcal{A}}\|\phi(\boldsymbol{y})-z\|_b$$
$$+\left[\beta_1 l\lambda_n^{-1}+\beta_2\bar{b}^{-\frac{1}{2}}(M_1+\Pi\bar{\eta})(1+l)\frac{(\gamma\nu_i^{-\frac{1}{2}}+1)}{\lambda_{n+1}^{\frac{1}{2}}}\right]\tau$$
$$+2\bar{b}^{-\frac{1}{2}}(M_0+\bar{\eta}\rho_0)\frac{[\nu_i(N+1)\tau]^{-\frac{1}{2}}+\lambda_{n+1}^{\frac{1}{2}}}{\lambda_{n+1}}e^{-\lambda_{n+1}(N+1)\tau}. \tag{2.47}$$

Proof. Take $\phi\in\mathcal{F}_{l,L}$ and $\boldsymbol{u}_0=\boldsymbol{y}_0+z_0\in\mathcal{A}$ a point in the global attractor. Denote with $(\boldsymbol{u}(t))_{t\in\mathbb{R}}$ the trajectory in $\mathcal{A}$ which pass through $\boldsymbol{u}_0$ at $t=0$. Consider $\boldsymbol{y}(t)=P_n\boldsymbol{u}(t)$, $z(t)=Q_n\boldsymbol{u}(t)$. Define $\widetilde{\boldsymbol{y}}_k=\boldsymbol{y}(-k\tau)$ and $(\boldsymbol{y}_k)_{k=0,\dots,N}$ with (2.27); and consider $\boldsymbol{y}_\tau$ constructed by (2.28). Using (2.21) and (2.13), the Lipschitz property of ϕ, the Poincaré inequality (2.3) and (2.16), we have:

$$\|\mathcal{F}_N^\tau\phi(\boldsymbol{y}_0)-z_0\|_b\le\bar{b}^{-\frac{1}{2}}(M_1+\Pi\bar{\eta})(1+l)\cdot \tag{2.48}$$
$$\cdot\int_{-(N+1)\tau}^{0}(|\nu_i s|^{-\frac{1}{2}}+\lambda_{n+1}^{\frac{1}{2}})e^{\lambda_{n+1}s}\|\boldsymbol{y}_\tau(s)-\boldsymbol{y}(s)\|_b ds$$
$$+(M_1+\Pi\bar{\eta})\bar{b}^{-\frac{1}{2}}(\gamma\nu_i^{-\frac{1}{2}}+1)\lambda_{n+1}^{-1}\sup_{\boldsymbol{y}+z\in\mathcal{A}}\|\phi(\boldsymbol{y})-z\|_b$$

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$$+ 2\bar{b}^{-\frac{1}{2}}(M_0 + \bar{\eta}\rho_0)\frac{[\nu_i(N+1)\tau]^{-\frac{1}{2}} + \lambda_{n+1}^{\frac{1}{2}}}{\lambda_{n+1}} e^{-\lambda_{n+1}(N+1)\tau}.$$

To estimate the integral on (2.48), from (2.34), for every s in $(-(k+1)\tau, -k\tau]$, we have

$$\|\boldsymbol{y}_\tau(s) - \boldsymbol{y}(s)\|_b \le \|\boldsymbol{e}_k\|_b + |k\tau + s| \sup_{\zeta \le 0} \|\frac{d\boldsymbol{y}}{dt}(\zeta)\|_b \le \|\boldsymbol{e}_k\|_b + \tau\beta_2,$$

with $\boldsymbol{e}_k = \boldsymbol{y}_k - \widetilde{\boldsymbol{y}}_k$. Using (2.31) and (2.32), we have

$$\|\boldsymbol{e}_{k+1}\|_b \le (1 + \tau\lambda_n)\|\boldsymbol{e}_k\|_b + \tau\left(\frac{\bar{b}\nu_i}{\lambda_n}\right)^{-1/2}(M_1 + \Pi\bar{\eta})(1+l)\|\boldsymbol{e}_k\|_b$$
$$+ \tau\left(\frac{\bar{b}\nu_i}{\lambda_n}\right)^{-1/2}(M_1 + \Pi\bar{\eta})\|\phi(\widetilde{\boldsymbol{y}}_k) - \boldsymbol{z}(-k\tau)\|_b + \|\epsilon_k\|_b,$$

and from (2.33), we have

$$\|\boldsymbol{e}_k\|_b \le [(M_1 + \Pi\bar{\eta})\left(\bar{b}\bar{\nu}\lambda_n\right)^{-\frac{1}{2}} \sup_{\boldsymbol{y}+\boldsymbol{z}\in\mathcal{A}} \|\phi(\boldsymbol{y}) - \boldsymbol{z}\|_b + \tau\beta_1\lambda_n^{-1}]$$
$$\cdot \exp\{k\tau[\lambda_n + \left(\frac{\bar{b}\nu_i}{\lambda_n}\right)^{-\frac{1}{2}}(M_1 + \Pi\bar{\eta})(1+l)]\}.$$

Now we are ready to estimate the first integral on (2.48):

$$\bar{b}^{-1/2}(M_1 + \Pi\bar{\eta})(1+l)\cdot$$
$$\cdot \int_{-(N+1)\tau}^{0} (|\nu_i s|^{-1/2} + \lambda_{n+1}^{1/2})e^{\lambda_{n+1}s}\|\boldsymbol{y}_\tau(s) - \boldsymbol{y}(s)\|_b ds$$
$$\le l[(M_1 + \Pi\bar{\eta})\left(\bar{b}\lambda_n\nu_i\right)^{-\frac{1}{2}} \sup_{\boldsymbol{y}+\boldsymbol{z}\in\mathcal{A}} \|\phi(\boldsymbol{y}) - \boldsymbol{z}\|_b + \tau\beta_1\lambda_n^{-1}]$$
$$+ \tau\beta_2\bar{b}^{-\frac{1}{2}}(M_1 + \Pi\bar{\eta})(1+l)(\gamma\nu_i^{-\frac{1}{2}} + 1)\lambda_{n+1}^{-\frac{1}{2}}.$$

Combining the previous estimate with (2.48), we have

$$\|\mathcal{F}_N^\tau\phi(\boldsymbol{y}_0) - \boldsymbol{z}_0\|_b \le$$
$$\le l[(M_1 + \Pi\bar{\eta})\left(\bar{b}\nu_i\lambda_n\right)^{-\frac{1}{2}} \sup_{\boldsymbol{y}+\boldsymbol{z}\in\mathcal{A}} \|\phi(\boldsymbol{y}) - \boldsymbol{z}\|_b + \tau\beta_1\lambda_n^{-1}]$$
$$+ \tau\beta_2\bar{b}^{-\frac{1}{2}}(M_1 + \Pi\bar{\eta})(1+l)(\gamma\nu_i^{-\frac{1}{2}} + 1)\lambda_{n+1}^{-1/2}$$
$$+ (M_1 + \Pi\bar{\eta})\bar{b}^{-\frac{1}{2}}(\gamma\nu_i^{-\frac{1}{2}} + 1)\lambda_{n+1}^{-\frac{1}{2}} \sup_{\boldsymbol{y}+\boldsymbol{z}\in\mathcal{A}} \|\phi(\boldsymbol{y}) - \boldsymbol{z}\|_b$$
$$+ 2\bar{b}^{-\frac{1}{2}}(M_0 + \bar{\eta}\rho_0)\frac{[\nu_i(N+1)\tau]^{-\frac{1}{2}} + \lambda_{n+1}^{1/2}}{\lambda_{n+1}} e^{-\lambda_{n+1}(N+1)\tau}, \tag{2.49}$$

which is the (2.47). □

In the next theorem we give an estimate on the number n of modes to yield an exponential approximation of $\mathcal{M}_N$ of the attractor, for N large.

Theorem 3. *Suppose that the hypothesis (2.36) and (2.37) of Theorem 1 hold and that (2.33) and (2.34) of Lemma 1 hold. Assume, moreover, that the sequence τ_N satisfies*

$$\chi \leq \tau_N(N+1) \leq \frac{\delta_1}{(M_1 + \Pi\bar{\eta})}\left(\frac{\bar{b}\nu_i}{\lambda_n}\right)^{1/2}, \tag{2.50}$$

for all $N \in \mathbb{N}$; where χ is any fixed constant less then δ_1. There exists a constant δ_3 such that if n is fixed by

$$\lambda_n \geq \max(\delta_2, \delta_3), \tag{2.51}$$

then the approximate inertial manifolds $\mathcal{M}_N$, constructed in Theorem 1, satisfy

$$d_V(\mathcal{A}, \mathcal{M}_N) \leq 4\bar{b}^{-1/2}(M_0 + \bar{\eta}\rho_0)\frac{1}{\lambda_{n+1}^{1/2}}e^{-\lambda_{n+1}\chi}, \tag{2.52}$$

for N sufficiently large.

Proof. Using the expression (2.47) in the previous Lemma, we have $\varrho_{N+1} \leq \mu\varrho_N + \sigma_N$, where

$$\mu = \bar{b}^{\frac{1}{2}}(M_1 + \Pi\bar{\eta})\left[l(\lambda_n\nu_i)^{-\frac{1}{2}} + (\gamma\nu_i^{-\frac{1}{2}} + 1)\lambda_{n+1}^{-\frac{1}{2}}\right], \tag{2.53}$$

and

$$\begin{aligned}\sigma_N = \tau_N &\left[\beta_1 l\lambda_n^{-1} + \beta_2\bar{b}^{-\frac{1}{2}}(M_1 + \Pi\bar{\eta})(1+l)\frac{(\gamma\nu_i^{-\frac{1}{2}} + 1)}{\lambda_{n+1}^{\frac{1}{2}}}\right] \\ &+ 2\bar{b}^{-\frac{1}{2}}(M_0 + \bar{\eta}\rho_0)\frac{[\nu_i(N+1)\tau_N]^{-\frac{1}{2}} + \lambda_{n+1}^{\frac{1}{2}}}{\lambda_{n+1}}e^{-\lambda_{n+1}(N+1)\tau_N}.\end{aligned} \tag{2.54}$$

Iterating, we obtain $\varrho_N \leq \mu^N\varrho_0 + \sum_0^{N-1}\sigma_{N-j-1}\mu^j$, with $\varrho_0 = \sup_{\boldsymbol{u}_0\in\mathcal{A}}\|z_0\|_b$. By (2.16), (2.21), (2.13):

$$\|z\|_b \leq \bar{b}^{-\frac{1}{2}}(M_0 + |\boldsymbol{f}|_b + \bar{\eta}\rho_0)(\gamma\nu_i^{-1/2} + 1)\lambda_{n+1}^{-1/2}. \tag{2.55}$$

Using (2.50) we obtain:

$$\begin{aligned}\sum_0^{N-1}\sigma_{N-j-1}\xi^j &= 2\bar{b}^{-\frac{1}{2}}(M_0 + \bar{\eta}\rho_0)\frac{1}{\lambda_{n+1}^{1/2}}e^{-\lambda_{n+1}\chi}\left(\sum_0^{N-1}\xi^j\right) \\ &+ \left[\beta_1 l\lambda_n^{-1} + \beta_2\bar{b}^{-\frac{1}{2}}(M_1 + \Pi\bar{\eta})(1+l)\frac{(\gamma\nu_i^{-\frac{1}{2}} + 1)}{\lambda_{n+1}^{\frac{1}{2}}}\right]\left(\sum_0^{N-1}\tau_{N-j-1}\xi^j\right),\end{aligned}$$

for $N \geq (\tau_N\nu_i)^{-1}$ and supposing that $\mu \leq \frac{1}{2}$, we have

$$\begin{aligned}\sum_0^{N-1}\sigma_{N-j-1}\mu^j &= 4\bar{b}^{-\frac{1}{2}}(M_0 + \bar{\eta}\rho_0)\frac{1}{\lambda_{n+1}^{1/2}}e^{-\lambda_{n+1}\chi} \\ &+ 2\left[\beta_1 l\lambda_n^{-1} + \beta_2\bar{b}^{-\frac{1}{2}}(M_1 + \Pi\bar{\eta})(1+l)\frac{(\gamma\nu_i^{-\frac{1}{2}} + 1)}{\lambda_{n+1}^{\frac{1}{2}}}\right]\sup_{0\leq j\leq N-1}\tau_{N-j-1}.\end{aligned} \tag{2.56}$$

Combining (2.55) and (2.56) we obtain that

$$\begin{aligned}d_V(\mathcal{A}, \mathcal{M}_N) &\leq 2^{-N}\bar{b}^{-\frac{1}{2}}(M_0 + \bar{\eta}\rho_0 + |\boldsymbol{f}|_b)(\gamma\nu_i^{-1/2} + 1)\lambda_{n+1}^{-1/2} \\ &+ 4\bar{b}^{-\frac{1}{2}}(M_0 + \bar{\eta}\rho_0)\frac{1}{\lambda_{n+1}^{1/2}}e^{-\lambda_{n+1}\chi} \\ &+ 2\left[\beta_1 l\lambda_n^{-1} + \beta_2\bar{b}^{-\frac{1}{2}}(M_1 + \Pi\bar{\eta})(1+l)\frac{(\gamma\nu_i^{-\frac{1}{2}} + 1)}{\lambda_{n+1}^{\frac{1}{2}}}\right]\sup_{0\leq j\leq N-1}\tau_{N-j-1}.\end{aligned} \tag{2.57}$$

Moreover, from (2.50) yields that $\tau_N \to 0$ as $N \to \infty$, hence from (2.57) we obtain (2.52) for $N \to \infty$. To complete the proof we determine λ_n in such a way that the previous estimates are satisfied. Choosing $\lambda_n \geq \delta_2$, we can write $l = 6\left(\frac{1}{2} + \sup_n \left(\frac{\nu_i \lambda_{n+1}}{\lambda_n}\right)^{\frac{1}{2}}\right)$, as given in (2.39). In this way, if $\lambda_n \geq \max(\delta_2, \delta_3)$ with

$$\delta_3 \geq 4\frac{(M_1 + \Pi\bar{\eta})^2}{\bar{b}}\left[\left(3 + 6\sup_n\left(\frac{\nu_i\lambda_{n+1}}{\lambda_n}\right)^{\frac{1}{2}}\right)\nu_i^{-\frac{1}{2}} + \left(\gamma\nu_i^{-\frac{1}{2}} + 1\right)\right]^2, \tag{2.58}$$

then condition $\mu \leq \frac{1}{2}$ is satisfied and the proof is complete. □

3. Unbounded domain: asymptotic L^2 decay

In this section we consider $\Omega = \mathbb{R}^2$, we suppose that the force term $\boldsymbol{f}$ is time dependent and $\boldsymbol{f} \in L^1\big([0, +\infty), L^2_b(\mathbb{R}^2)\big)$. Now the equations under consideration are as follows

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p + \eta\boldsymbol{u} = b^{-1}\nabla\cdot[\nu b(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T - \mathbf{I}\nabla\cdot\boldsymbol{u})] + \mathbf{f}, \tag{3.1a}$$

$$\nabla\cdot(b\boldsymbol{u}) = 0, \tag{3.1b}$$

$$\boldsymbol{u}(\boldsymbol{x}, t = 0) = \boldsymbol{u}_0, \tag{3.1c}$$

where $\boldsymbol{x} \in \mathbb{R}^2$, ν is the viscosity and we denote by $0 < \nu_i = \inf_{\mathbb{R}^2} \nu$, η is a smooth strictly positive function and $b(\boldsymbol{x})$ represents the bottom topography of the basin satisfying

$$0 < b_i \leq b(\mathbf{x}) \leq b_s.$$

The Fourier splitting method, will be used to establish the asymptotic L^2-decay of the weak solutions to the shallow water model with varying bottom topography.

3.1. Mathematical settings

We denote by $L^2_b(\mathbb{R}^2)$ the weighted $L^2(\mathbb{R}^2)$ space with scalar product and norm defined by

$$(\boldsymbol{u}, \boldsymbol{v})_b = \int_{\mathbb{R}^2} b\boldsymbol{u}\cdot\boldsymbol{v}\,d\boldsymbol{x}, \qquad \|\boldsymbol{u}\|^2_{b,2} = \int_{\mathbb{R}^2} b\,|\boldsymbol{u}|^2\,d\boldsymbol{x},$$

and $\|\cdot\|_p$ will denote the usual norm in $L^p(\mathbb{R}^2)$. We also use the following notation for our spaces

$$H = \left\{\boldsymbol{u} : \boldsymbol{u} \in L^2_b\left(\mathbb{R}^2\right), \quad \nabla\cdot(b\boldsymbol{u}) = 0\right\}, \tag{3.2}$$

$$V = \left\{\boldsymbol{u} : \boldsymbol{u} \in H^1_b\left(\mathbb{R}^2\right), \quad \nabla\cdot(b\boldsymbol{u}) = 0\right\}, \tag{3.3}$$

and

$$V_o = \left\{\boldsymbol{u} : \quad \boldsymbol{u} \in H^1_b\left(\mathbb{R}^2\right) \cap \mathbb{S}'\left(\mathbb{R}^2\right), \quad \nabla\cdot(b\boldsymbol{u}) = 0\right\}, \tag{3.4}$$

where $\mathbb{S}\left(\mathbb{R}^2\right)$ is the Schwartz class of smooth, rapidly decreasing functions.

A function $\boldsymbol{u}(\boldsymbol{x}, t) \in C_w([0, \infty), H)$ if $\boldsymbol{u} \in L^\infty([0, \infty), H)$ and $(\boldsymbol{u}, \boldsymbol{\phi})_b$ is continuous with respect to time $t \geq 0$, for all $\boldsymbol{\phi} \in H'$.

As usual, the Fourier transform of an integral function $\boldsymbol{v}(\boldsymbol{x}) \in L^2\left(\mathbb{R}^2\right)$ is $\hat{\boldsymbol{v}}(\xi) = \int_{\mathbb{R}^2} \boldsymbol{v}(\boldsymbol{x})e^{-i\boldsymbol{x}\cdot\xi}\,d\boldsymbol{x}$.

A weak solution $\boldsymbol{u}$ of problem (3.1a)–(3.1c) is a function belonging to $C_w([0, T], H) \cap L^2([0, T], V_o)$ for each $T > 0$, satisfying the integral relation

$$(\boldsymbol{u}(t), \boldsymbol{\phi}(t))_b + \int_0^t \Big\{ -\left(\boldsymbol{u}, \frac{\partial \boldsymbol{\phi}}{\partial t}\right)_b +$$

$$+\nu\left(\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T - \mathbf{I}\nabla\cdot\boldsymbol{u}\right) : \left(\nabla\boldsymbol{\phi} + (\nabla\boldsymbol{\phi})^T - \mathbf{I}\nabla\cdot\boldsymbol{\phi}\right)\right)_b +$$

$$+ (\eta\boldsymbol{u}, \boldsymbol{\phi})_b + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{\phi})_b \Big\}\, d\tau = \int_0^t (\boldsymbol{\phi}, \boldsymbol{f})_b\, d\tau + (u_0, \boldsymbol{\phi}(0))_b\,,$$

for all $t \geq s \geq 0$ and for every smooth vector fields

$$\boldsymbol{\phi} \in C\left([0\,,+\infty)\,, V\right) \cap C^1\left([0\,,+\infty)\,, H\right).$$

It is easy to prove the following two Propositions where, respectively, the strong energy inequality and the generalized energy inequality are given for a weak solution of (3.1a)–(3.1c).

Proposition 1. *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^2)$ and $\boldsymbol{f} \in L^1\left([0,+\infty)\,, L^2_b(\mathbb{R}^2)\right)$. Then, for every $T > 0$, there exists a unique weak solution $\boldsymbol{u}(\boldsymbol{x},t) \in C_w\left([0,T]\,, H\right) \cap L^2\left([0,T]\,, V_o\right)$ of system (3.1a)–(3.1c), which satisfies the following strong energy inequality:*

$$b_i\, \|\boldsymbol{u}(t)\|_2^2 + 2\nu_i b_i \int_s^t \|\nabla\boldsymbol{u}(\tau)\|_2^2\, d\tau \leq b_s\, \|\boldsymbol{u}(s)\|_2^2 + 2\int_s^t (\boldsymbol{u}, \boldsymbol{f})_b\, d\tau, \tag{3.5}$$

for almost all $s \geq 0$ including $s = 0$ and all $t \geq s \geq 0$.

Proof. The existence and uniqueness of a weak solution to problem (3.1a)–(3.1c) satisfying the strong energy inequality (3.5) follows by an application of the standard Galerkin technique (see [45]). □

Let $\boldsymbol{u}(\boldsymbol{x},t) = (u_1(\boldsymbol{x},t), u_2(\boldsymbol{x},t))$ be a vector function and $\psi(\boldsymbol{x},t)$ be a scalar function. In the sequel we use the notation

$$\psi' = \partial_t\psi, \quad \psi * \boldsymbol{u} = (\psi * u_1, \psi * u_2)\,, \tag{3.6}$$

where the convolution is calculated with respect to the $\boldsymbol{x}$ variable.

Proposition 2. *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^2)$ and $\boldsymbol{f} \in L^1\left([0,+\infty)\,, L^2_b(\mathbb{R}^2)\right)$. Let $Z \in C^1[0,\infty)$ with $Z(t) \geq 0$, and $\psi(t) \in C^1\left([0,\infty)\,; \mathcal{S}\left(\mathbb{R}^2\right)\right)$ be arbitrary functions. Let $\boldsymbol{u}$ be a weak solution of system (3.1a)–(3.1c), then the following generalized energy inequality holds:*

$$\begin{aligned} Z(t) b_i\, \|\psi(t) * \boldsymbol{u}(t)\|_2^2 \leq\; & b_s Z(s)\, \|\psi(s) * \boldsymbol{u}(s)\|_2^2 \\ & + b_s \int_s^t Z'(\tau)\, \|\psi(\tau) * \boldsymbol{u}(\tau)\|_2^2\, d\tau \\ & + 2\int_s^t Z(\tau)\left(\psi'(\tau) * \boldsymbol{u}(\tau), \psi(\tau) * \boldsymbol{u}(\tau)\right)_b d\tau \\ & - 2\nu_i b_i \int_s^t Z(\tau)\, \|\psi(\tau) * \nabla\boldsymbol{u}(\tau)\|_2^2\, d\tau \\ & + 2\int_s^t Z(\tau)\left[(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \psi * \psi * \boldsymbol{u})_b\,(\tau)\right] d\tau + 2\int_s^t Z(\tau)\,(\psi * \boldsymbol{u}, \boldsymbol{f})_b\, d\tau. \end{aligned} \tag{3.7}$$

for almost all $s \geq 0$ including $s = 0$ and all $t \geq s \geq 0$.

Proof. To prove the generalized energy inequality (3.7) one can follow [21,35]. □

We give two preliminary Lemmas which are consequence of the generalized energy inequality (3.7).

Lemma 4. *Let $\boldsymbol{u}$ be a weak solution of (3.1a)–(3.1c) satisfying the generalized energy inequality (3.7) of Lemma 1. Then for every $\varphi \in \mathcal{S}(\mathbb{R}^2)$, we have:*

$$b_i \|\varphi * \boldsymbol{u}(t)\|_2^2 \le b_s \left\| e^{\frac{\nu_i b_i}{b}(t-s)\Delta} \varphi * \boldsymbol{u}(s) \right\|_2^2 + 2\int_s^t \left[\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, e^{2\frac{\nu_i b_i}{b}(t-\tau)\Delta}(\varphi*\varphi)*\boldsymbol{u} \right)_b (\tau) \right] d\tau + 2\int_s^t \left(e^{\frac{\nu_i b_i}{b}(t-\tau)\Delta}\varphi * \boldsymbol{u}(s), \boldsymbol{f} \right)_b d\tau, \tag{3.8}$$

for almost all $s \ge 0$ including $s = 0$ and all $t \ge s \ge 0$.

Proof. Apply (3.7) with $Z(t) = 1$ and $\psi = e^{\frac{\nu_i b_i}{b}(t+\delta-\tau)\Delta}\varphi$ and let $\delta \to 0$ (see [21,35]). □

Lemma 5. *Let $Z(t) \in C^1[0,+\infty)$ with $Z(t) \ge 0$. Let $\boldsymbol{u}$ be a weak solution of (3.1a)–(3.1c) satisfying the generalized energy inequality (3.7) of Lemma 2. Then for every $\varphi \in \mathcal{S}(\mathbb{R}^2)$, we have:*

$$Z(t) b_i \|\boldsymbol{u}(t) - \varphi * \boldsymbol{u}(t)\|_2^2 \le b_s Z(t) \|\boldsymbol{u}(s) - \varphi * \boldsymbol{u}(s)\|_2^2 \tag{3.9}$$

$$+ b_s \int_s^t Z'(\tau) \|\boldsymbol{u}(\tau) - \varphi * \boldsymbol{u}(\tau)\|_2^2 d\tau - 2 b_i \nu_i \int_s^t Z(\tau) \|\nabla\boldsymbol{u}(\tau) - \varphi * \nabla\boldsymbol{u}(\tau)\|_2^2 d\tau$$

$$+ 2\int_s^t Z(\tau) \left[(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \varphi*\varphi*\boldsymbol{u} - 2\varphi*\boldsymbol{u})_b(\tau) \right] d\tau + 2\int_s^t (\boldsymbol{u} - \varphi*\boldsymbol{u}, \boldsymbol{f})_b,$$

for almost $s \ge 0$ including $s = 0$ and all $t \ge s \ge 0$.

Proof. Apply (3.7) with $\psi = \zeta_n - \varphi$, with $\zeta_n(\boldsymbol{x}) = n^{-1}\zeta(\boldsymbol{x}/n)$ is a smooth and compactly supported approximation of the Dirac measure, and let $n \to \infty$ (see [21,35]). □

3.2. Non-uniform decay

We now state the main theorem of the section:

Theorem 4. *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^2)$ and $\boldsymbol{f} \in L^1\left([0,+\infty), L_b^2(\mathbb{R}^2)\right)$. Let $\boldsymbol{u}$ be a weak solution of problem (3.1a)–(3.1c), then*

$$\lim_{t\to+\infty} \|\boldsymbol{u}\|_2 = 0. \tag{3.10}$$

Proof. The proof is based on ideas of [21,35]:
We decompose the L^2-norm of the Fourier transform of the weak solution $\boldsymbol{u}$ as follows

$$\|\boldsymbol{u}(t)\|_2 = \left\|\hat{\boldsymbol{u}}(t)\right\|_2 \le \left\|\check{\varphi}\hat{\boldsymbol{u}}(t)\right\|_2 + \left\|\left(1-\check{\varphi}\right)\hat{\boldsymbol{u}}(t)\right\|_2, \tag{3.11}$$

where $\check{\varphi}(\boldsymbol{\xi}) = e^{-|\boldsymbol{\xi}|^2}$ is the inverse Fourier Transform of $\varphi(\boldsymbol{x}) = \dfrac{1}{4\pi}e^{-\frac{|x|^2}{4}}$, the fundamental solution of the heat equation at $t = 1$. We estimate separately the low frequencies and the high energy frequencies terms in (3.11).

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Low frequencies term estimate: Using Plancherel identity and (3.8), we have

$$b_i \left\|\check{\varphi}\hat{\boldsymbol{u}}(t)\right\|_2^2 = b_i \left\|\varphi * \boldsymbol{u}(t)\right\|_2^2 \le b_s \left\| e^{\frac{\nu_i b_i}{b}(t-s)\Delta} \varphi * \boldsymbol{u}(s)\right\|_2^2 + 2\int_s^t \left| \left(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, e^{2\frac{\nu_i b_i}{b}(t-\tau)\Delta}\varphi*\varphi*\boldsymbol{u}\right)_b(\tau)\right| d\tau + 2\int_s^t \left(e^{\frac{\nu_i b_i}{b}(t-\tau)\Delta}\varphi*\boldsymbol{u}(s), \boldsymbol{f}\right)_b d\tau.$$

Using (see [45]) the Schwarz, Hölder and Young inequalities and the Gagliardo–Nirenberg interpolation inequality, we have

$$\begin{aligned}\left| \left(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, e^{2\frac{\nu_i b_i}{b}(t-\tau)\Delta}\varphi*\varphi*\boldsymbol{u}\right)_b(\tau)\right| &\le C\left\|\boldsymbol{u}\right\|_4 \left\|\nabla\boldsymbol{u}\right\|_2 \left\|e^{2\frac{\nu_i b_i}{b}(t-s)\Delta}\varphi*\varphi*\boldsymbol{u}\right\|_4 \\ &\le C\left\|e^{2\frac{\nu_i b_i}{b}(t-s)\Delta}\varphi*\varphi\right\|_1 \left\|\nabla\boldsymbol{u}\right\|_2 \left\|\boldsymbol{u}\right\|_4^2 \\ &\le C\left\|e^{2\frac{\nu_i b_i}{b}(t-s)\Delta}\varphi*\varphi\right\|_1 \left\|\nabla\boldsymbol{u}\right\|_2^2 \left\|\boldsymbol{u}\right\|_2 .\end{aligned}$$

It is easy to prove (see [49]) that there exists a constant $\kappa = \kappa(\boldsymbol{u}_0, \boldsymbol{f})$ such that

$$\left|(\boldsymbol{u}, \boldsymbol{f})_b\right| \le \kappa \left\|\boldsymbol{f}\right\|_{b,2}, \quad \text{and} \quad \left|\left(e^{\frac{\nu_i b_i}{b}(t-\tau)\Delta}\varphi*\boldsymbol{u}, \boldsymbol{f}\right)_b\right| \le \kappa\left\|\boldsymbol{f}\right\|_{b,2}. \tag{3.12}$$

From the strong energy inequality (3.5) we have that

$$\left\|\boldsymbol{u}(t)\right\|_2^2 \le \left\|\boldsymbol{u}_0\right\|_2^2 + 2\kappa\int_0^t \left\|\boldsymbol{f}\right\|_{b,2} d\tau. \tag{3.13}$$

Hence

$$\left\|\check{\varphi}\hat{\boldsymbol{u}}(t)\right\|_2^2 \le \frac{b_s}{b_i}\left\|e^{\frac{\nu_i b_i}{b}(t-s)\Delta}\varphi*\boldsymbol{u}(s)\right\|_2^2 + \frac{C}{b_i}\left(\left\|\boldsymbol{u}_0\right\|_2^2 + 2\kappa\int_0^{+\infty}\left\|\boldsymbol{f}\right\|_{b,2} d\tau\right)^{\frac{1}{2}} \int_s^{+\infty}\left\|\nabla\boldsymbol{u}\right\|_2^2 d\tau + 2\frac{\kappa}{b_i}\int_s^{+\infty}\left\|\boldsymbol{f}\right\|_{b,2} d\tau.$$

By the Lebesgue dominated convergence theorem, it follows that, as $t \to +\infty$,

$$\begin{aligned}\left\|e^{\frac{\nu_i b_i}{b}(t-s)\Delta}\varphi*\boldsymbol{u}(s)\right\|_2^2 \le \left\|e^{\nu_i(t-s)\Delta}\varphi*\boldsymbol{u}(s)\right\|_2^2 = \\ = \left\|e^{-\nu_i(t-s)\boldsymbol{\xi}^2}\check{\varphi}\hat{\boldsymbol{u}}(s)\right\|_2^2 \to 0,\end{aligned} \tag{3.14}$$

for each $s \ge 0$, since $\check{\varphi}\hat{\boldsymbol{u}}(s) \in L^2(\mathbb{R}^2)$.

Since $\int_0^{+\infty} \left\|\nabla\boldsymbol{u}\right\|_2^2 d\tau < \infty$ by the strong energy inequality (3.5) and $\int_0^{+\infty}\left\|\boldsymbol{f}\right\|_{b,2} d\tau < \infty$ by hypothesis, the quantities $\int_s^{+\infty}\left\|\nabla\boldsymbol{u}\right\|_2^2 d\tau$ and $\int_s^{+\infty}\left\|\boldsymbol{f}\right\|_{b,2} d\tau$ are small for s suitable large, then $\left\|\check{\varphi}\hat{\boldsymbol{u}}(t)\right\|_2 \to 0$ as $t \to 0$.

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High frequencies term estimate: Use Corollary (3.9) with $\check{\varphi}(\boldsymbol{\xi}) = e^{-|\boldsymbol{\xi}|^2}$, and $Z(t)$ determined below. Consider a function $G(t) \geq 0$, to be determined below, and apply the Fourier splitting method to the first two terms in (3.9):

$$\int_s^t Z'(\tau)\,\|\boldsymbol{u}(\tau) - \varphi * \boldsymbol{u}(\tau)\|_2^2\,d\tau - 2b_s\int_s^t Z(\tau)\,\|\nabla\boldsymbol{u}(\tau) - \varphi * \nabla\boldsymbol{u}(\tau)\|_2^2\,d\tau$$

$$= \int_s^t Z'(\tau) \int_{|\boldsymbol{\xi}|>G} \left|\left(1-\check{\varphi}(\boldsymbol{\xi})\right)\hat{\boldsymbol{u}}(\boldsymbol{\xi},\tau)\right|^2 d\boldsymbol{\xi}\,d\tau$$

$$-2\int_s^t Z(\tau) \int_{|\boldsymbol{\xi}|>G} b_s\left||\boldsymbol{\xi}|\left(1-\check{\varphi}(\boldsymbol{\xi})\right)\hat{\boldsymbol{u}}(\boldsymbol{\xi},\tau)\right|^2 d\boldsymbol{\xi}\,d\tau$$

$$+\int_s^t Z'(\tau) \int_{|\boldsymbol{\xi}|\leq G} \left|\left(1-\check{\varphi}(\boldsymbol{\xi})\right)\hat{\boldsymbol{u}}(\boldsymbol{\xi},\tau)\right|^2 d\boldsymbol{\xi}\,d\tau$$

$$-2\int_s^t Z(\tau) \int_{|\boldsymbol{\xi}|\leq G} b_s\left||\boldsymbol{\xi}|\left(1-\check{\varphi}(\boldsymbol{\xi})\right)\hat{\boldsymbol{u}}(\boldsymbol{\xi},\tau)\right|^2 d\boldsymbol{\xi}\,d\tau.$$

Choose

$$Z(t) = (1+t)^{\alpha} \qquad \text{and} \qquad G^2 = \frac{\alpha}{2b_s(t+1)}, \tag{3.15}$$

with $\alpha > 0$ fixed, then $Z(t)$ and $G(t)$ satisfies the following equation:

$$Z'(t) - 2b_s Z(t) G^2(t) = 0.$$

Hence the last equation is reduced to

$$\int_s^t Z'(\tau) \int_{|\boldsymbol{\xi}|>G} \left|\left(1-\check{\varphi}(\boldsymbol{\xi})\right)\hat{\boldsymbol{u}}(\boldsymbol{\xi},\tau)\right|^2 d\boldsymbol{\xi}\,d\tau$$

$$-2\int_s^t Z(\tau) \int_{|\boldsymbol{\xi}|>G} b_s\left||\boldsymbol{\xi}|\left(1-\check{\varphi}(\boldsymbol{\xi})\right)\hat{\boldsymbol{u}}(\boldsymbol{\xi},\tau)\right|^2 d\boldsymbol{\xi}\,d\tau$$

$$\leq \int_s^t \left(Z' - 2b_s Z G^2\right) \int_{|\boldsymbol{\xi}|>\mathbf{0}} \left|\left(1-\check{\varphi}(\boldsymbol{\xi})\right)\hat{\boldsymbol{u}}(\boldsymbol{\xi},\tau)\right|^2 d\boldsymbol{\xi}\,d\tau = 0.$$

As $\left|1-\check{\varphi}(\boldsymbol{\xi})\right| \leq |\boldsymbol{\xi}|^2$, then for small $|\boldsymbol{\xi}|$ we have

$$\int_s^t Z'(\tau) \int_{|\boldsymbol{\xi}|\leq G} \left|\left(1-\check{\varphi}(\boldsymbol{\xi})\right)\hat{\boldsymbol{u}}(\boldsymbol{\xi},\tau)\right|^2 d\boldsymbol{\xi}\,d\tau \leq$$

$$\leq C\,\|\boldsymbol{u}_0\| \int_s^t Z'(\tau)G^4(\tau)d\tau \leq C\int_s^t (1+\tau)^{\alpha-3}\,d\tau.$$

The last two terms in (3.9) can be simplified denoting by $\chi = \varphi * \varphi - 2\varphi$, and combining (see [45]) the Schwarz, the Hölder and the Young inequalities, the Gagliardo–Nirenberg interpolation inequality, and the strong energy inequality (3.13),

$$\int_s^t Z(\tau)\left|(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\varphi*\varphi*\boldsymbol{u}-2\varphi*\boldsymbol{u})_b(\tau)\right|d\tau=$$

$$=\int_s^t Z(\tau)\left|(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\chi*\boldsymbol{u})_b(\tau)\right|d\tau\le\int_s^t Z(\tau)\,\|\boldsymbol{u}\|_4\,\|\nabla\boldsymbol{u}\|_2\,\|\chi*\boldsymbol{u}\|_4\,d\tau$$

$$\le C\,\|\chi\|_1\int_s^t Z(\tau)\,\|\boldsymbol{u}\|_4^2\,\|\nabla\boldsymbol{u}\|_2\,d\tau\le C\,\|\chi\|_1\int_s^t Z(\tau)\,\|\boldsymbol{u}\|_2\,\|\nabla\boldsymbol{u}\|_2^2\,d\tau$$

$$\le C\,\|\chi\|_1\left(\|\boldsymbol{u}_0\|_2^2+2\kappa\int_0^{+\infty}\|\boldsymbol{f}\|_{b,2}\,d\tau\right)^{\frac{1}{2}}\int_s^t Z(\tau)\,\|\nabla\boldsymbol{u}\|_2^2\,d\tau,$$

and

$$\int_s^t Z(\tau)\,(\varphi*\boldsymbol{u},\boldsymbol{f})_b\,d\tau\le\kappa\int_s^t Z(\tau)\,\|\boldsymbol{f}\|_{b,2}\,d\tau.$$

Combining the previous estimates yields

$$\left\|\left(1-\check{\varphi}\right)\hat{\boldsymbol{u}}(t)\right\|_2^2\le\frac{b_s Z(s)}{b_i Z(t)}\left\|\left(1-\check{\varphi}\right)\hat{\boldsymbol{u}}(s)\right\|_2^2+\frac{C}{Z(t)}\int_s^t(1+\tau)^{\alpha-3}d\tau$$
$$+\frac{1}{Z(t)}\left(C\int_s^t Z(\tau)\,\|\nabla\boldsymbol{u}\|_2^2\,d\tau+\kappa\int_s^t Z(\tau)\,\|\boldsymbol{f}\|_{b,2}\,d\tau\right).$$

We compute the $\limsup$ as $t\to+\infty$ for fixed $s>0$.
Since $Z(t)=(1+t)^\alpha$ for some $\alpha>0$, it follows that $\dfrac{Z(s)}{Z(t)}\to 0$ when $t\to+\infty$. Moreover we have that

$$\limsup_{t\to+\infty}\frac{1}{(t+1)^\alpha}\int_s^t(1+\tau)^{\alpha-3}d\tau=0.$$

As $\dfrac{Z(\tau)}{Z(t)}\le 1$ for $\tau\in[0,t]$, then

$$\limsup_{t\to+\infty}\left\|\left(1-\check{\varphi}\right)\hat{\boldsymbol{u}}(t)\right\|_2^2\le C\int_s^{+\infty}\|\nabla\boldsymbol{u}(\tau)\|_2^2\,d\tau+2\kappa\int_s^{+\infty}\|\boldsymbol{f}\|_{b,2}\,d\tau, \tag{3.16}$$

hence $\limsup_{t\to+\infty}\left\|\left(1-\check{\varphi}\right)\hat{\boldsymbol{u}}(t)\right\|_2^2 d=0$, for s sufficiently large. □

3.3. Uniform decay

In this section we want to prove the uniform rate of decay for the solutions of the viscous shallow water equations (3.1a)–(3.1c).

We suppose for simplicity, that ν is a constant, then (3.1a)–(3.1c) can be written as:

$$\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\eta\boldsymbol{u}+\nabla p=\frac{\nu}{b}\nabla\cdot[b(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T-\mathbf{I}\nabla\cdot\boldsymbol{u})]+\boldsymbol{f}, \tag{3.17}$$
$$\nabla\cdot(b\boldsymbol{u})=0, \tag{3.18}$$
$$\boldsymbol{u}(\boldsymbol{x},t=0)=\boldsymbol{u}_0. \tag{3.19}$$

Suppose that the force term $\boldsymbol{f}$ satisfies the following properties:

$$\boldsymbol{f} = D\mathbf{g}, \text{ where } D \text{ is any first order derivative} \quad (3.20)$$
$$\text{and } \mathbf{g} \in L^\infty\Big([0\,,\,+\infty)\,,\, L^1(\mathbb{R}^2)\Big),$$

$$\|\boldsymbol{f}\|_2 \le \kappa(e+t)^{-2}. \quad (3.21)$$

In particular we prove the following theorem:

Theorem 5. *Suppose that $\boldsymbol{u}_0 \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and let $\boldsymbol{u}$ be the weak solution of the viscous shallow water equations (3.17)–(3.19). Suppose that $\boldsymbol{f}$ satisfies (3.20) and (3.21), then*

$$\|\boldsymbol{u}\|_2 \le C\,(\log(e+t))^{-1/2}, \quad (3.22)$$

with C a constant which depends on $\boldsymbol{f}$, b, η and $\boldsymbol{u}_0$.

Before establishing the proof of the theorem, we give three preliminary lemmas:

Lemma 6. *$(L^p - L^q)$-type estimate: Let us consider $\boldsymbol{u}_0 \in L^q \cap L^2$, with $1 \le q < 2$, then*

$$\|e^{-[A_{b\nu}-\eta I]t}\boldsymbol{u}_0\|_2 \le Ct^{-(1/q-1/2)}(\|\boldsymbol{u}_0\|_2 + \|\boldsymbol{u}_0\|_q). \quad (3.23)$$

Proof. The proof follows from the well-known $(L^p - L^q)$ type estimate for the linear heat equation and observing that

$$(A_{b\nu}\boldsymbol{u}, \boldsymbol{u})_b \equiv -\|\nabla \boldsymbol{u}\|_2^2 \equiv (\Delta \boldsymbol{u}, \boldsymbol{u}), \quad (3.24)$$

then, denoting with $\boldsymbol{u}(t) = e^{-[A_{b\nu}-\eta I]t}\boldsymbol{u}_0$ we have:

$$\begin{aligned}\|e^{-[A_{b\nu}-\eta I]t}\boldsymbol{u}_0\|_2^2 &\le \|\boldsymbol{u}_0\|_2^2 + C\int_0^t (A_{b\nu}\boldsymbol{u}, \boldsymbol{u})_b d\tau - C\inf|\eta| \int_0^t \|\boldsymbol{u}\|_2^2 d\tau \\ &\le \|\boldsymbol{u}_0\|_2^2 + C\int_0^t \|\nabla \boldsymbol{u}\|_2^2 d\tau \\ &\le C\|e^{\Delta t}\boldsymbol{u}_0\|_2^2. \qquad \square\end{aligned}$$

Lemma 7. *Suppose that $\boldsymbol{u}_0 \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and that $\boldsymbol{f}$ satisfies (3.20) and (3.21). Then the weak solution $\boldsymbol{u}$ of the viscous shallow water equations (3.17)–(3.19) satisfies the following a priori estimate:*

$$\int_0^t \|\boldsymbol{u}(\tau)\|_2^4 d\tau \le C(e+t)^{-1}, \quad (3.25)$$

where C is a constant which depends on $\boldsymbol{u}_0$, η, $\boldsymbol{f}$ and b.

Proof. From Lemma 6, we have that

$$\begin{aligned}\|\boldsymbol{u}\|_2 \le C\|e^{-[A_{b\nu}-\eta I]t}\boldsymbol{u}_0\|_2 &+ C\int_0^t \|e^{-[A_{b\nu}-\eta I](t-\tau)} P\,(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\,\|_2 d\tau \\ &+ C\int_0^t \|e^{-[A_{b\nu}-\eta I](t-\tau)} P\,(\boldsymbol{f}, \boldsymbol{u})\,\|_2 d\tau\end{aligned}$$

$$\leq Ct^{-1/2}\|\boldsymbol{u}_0\|_1 + C\int_0^t (t-\tau)^{-1/2}\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_1 d\tau + C\int_0^t (t-\tau)^{-1/2}\|\boldsymbol{f}\boldsymbol{u}\|_1 d\tau$$

$$\leq C(e+t)^{-1/2}\|\boldsymbol{u}_0\|_1 + C\int_0^t (t-\tau)^{-1/2}\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2 d\tau +$$

$$+ C\int_0^t (t-\tau)^{-1/2}\|\boldsymbol{u}\|_2\|\boldsymbol{f}\|_2 d\tau \tag{3.26}$$

Consider the generalized Young's inequality [37,20] for convolution:
if $f \in L^p$ and $g \in L^{q,w}$, with $1 < p, q, r < \infty$ and $p^{-1} + r^{-1} = 1 + q^{-1}$ then

$$\|f \star g\|_q \leq C_{p,r}\|f\|_p\|g\|_{r,w}, \tag{3.27}$$

where the $L^{r,w}$ is the weak L^r space with norm $\|g\|_{r,w} + \sup_t (t^r \mu\{x : g(x) > t\})^{1/r}$.

Now, let $q = 4$ and $1 + \frac{1}{q} = \frac{1}{2} + \frac{q+2}{2q}$, and applying (3.27) to (3.26):

$$\left[\int_0^t \|\boldsymbol{u}(\tau)\|_2^q d\tau\right]^{1/q} \leq C\|\boldsymbol{u}_0\|_1(e+t)^{1/q-1/2}$$

$$+ C\left[\int_0^t (\|\boldsymbol{u}(\tau)\|_2\|\nabla\boldsymbol{u}(\tau)\|_2)^{\frac{2q}{q+2}} d\tau\right]^{\frac{q+2}{2q}}$$

$$+ C\left[\int_0^t (\|\boldsymbol{u}(\tau)\|_2\|\boldsymbol{f}(\tau)\|_2)^{\frac{2q}{q+2}} d\tau\right]^{\frac{q+2}{2q}}$$

$$\leq C\|\boldsymbol{u}_0\|_1(e+t)^{1/q-1/2} + C\left[\int_0^t \|\boldsymbol{u}(\tau)\|_2^q d\tau\right]^{\frac{1}{q}}\left[\int_0^t \|\nabla\boldsymbol{u}(\tau)\|_2^2 d\tau\right]^{\frac{1}{2}}$$

$$+ C\left[\int_0^t \|\boldsymbol{u}(\tau)\|_2^q d\tau\right]^{\frac{1}{q}}\left[\int_0^t \|\boldsymbol{f}\|_2^2 d\tau\right]^{\frac{1}{2}}$$

$$\leq C\|\boldsymbol{u}_0\|_1(e+t)^{1/q-1/2} + C(1+\|\boldsymbol{u}_0\|_2)\left[\int_0^t \|\boldsymbol{u}(\tau)\|_2^q d\tau\right]^{\frac{1}{q}},$$

and assuming that $C(1+\|\boldsymbol{u}_0\|_2) \leq 1/2$, we have (3.25). □

Lemma 8. *Suppose that $\boldsymbol{u}_0 \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and that $\boldsymbol{f}$ satisfies (3.20) and (3.21). Then the weak solution $\boldsymbol{u}$ of the viscous shallow water equations (3.17)–(3.19) satisfies the following a priori estimate:*

$$|\hat{\boldsymbol{u}}(\xi,t)| \leq \|\boldsymbol{u}_0\|_1 + C|\xi|t + C(1+|\xi|)\int_0^t \|\boldsymbol{u}(\tau)\|_2 d\tau$$

$$+ C(1+|\xi|)\int_0^t \|\boldsymbol{u}(\tau)\|_2^2 d\tau. \tag{3.28}$$

Proof. Write the viscous shallow water equations in the following way.

$$\frac{\partial \boldsymbol{u}}{\partial t} = \nu \Delta \boldsymbol{u} + G(\boldsymbol{u}) + \boldsymbol{f},$$
$$\nabla \cdot (b\boldsymbol{u}) = 0,$$

where

$$G(\boldsymbol{u}) = -\boldsymbol{u} \cdot \nabla \boldsymbol{u} - \eta \boldsymbol{u} - \nabla p + \nu \frac{\nabla b}{b} \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T - \nabla \cdot \boldsymbol{u} \mathbb{I} \right).$$

Hence

$$\hat{\boldsymbol{u}} = e^{-|\xi|^2 t} \hat{\boldsymbol{u}}_0 + \int_0^t e^{-|\xi|^2 (t-s)} \left(\widehat{P(G)} + \widehat{P\boldsymbol{f}} \right) ds, \tag{3.29}$$

where P is the projection form L_b^2 in H.

By assumption $\boldsymbol{f} = D\boldsymbol{g}$ where D is any first order derivative and $\boldsymbol{g} \in L^\infty \left([0, +\infty), L^1(\mathbb{R}^2) \right)$, hence

$$|\widehat{P\boldsymbol{f}}| \le C|\xi|.$$

We prove later that

$$|\widehat{P(G)}| \le C(1 + |\xi|)(\|\boldsymbol{u}(t)\|_2^2 + \|\boldsymbol{u}(t)\|_2), \tag{3.30}$$

where C is a constant which depends on b. Using (3.30) in (3.29) and integrate in time (3.29), we have

$$|\hat{\boldsymbol{u}}(\xi, t)| \le |\hat{\boldsymbol{u}}(\xi, 0)| + C|\xi|t + C(1 + |\xi|) \int_0^t \|\boldsymbol{u}(\tau)\|_2 d\tau + C(1 + |\xi|) \int_0^t \|\boldsymbol{u}(\tau)\|_2^2 d\tau,$$

which is (3.28). To complete the proof, we finally show that (3.30) holds,

$$|\widehat{P(G)}| \le C(1 + |\xi|)(\|\boldsymbol{u}(t)\|_2^2 + \|\boldsymbol{u}(t)\|_2).$$

Using (3.18)

$$\begin{aligned}
|P\widehat{(\boldsymbol{u} \cdot \nabla \boldsymbol{u})}| &\le \left| \int \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) e^{i\xi \cdot \boldsymbol{x}} d\boldsymbol{x} \right| + \left| \int \boldsymbol{u} \frac{\nabla b}{b} \cdot \boldsymbol{u} e^{i\xi \cdot \boldsymbol{x}} d\boldsymbol{x} \right| \\
&\le |\xi| \|\boldsymbol{u} \otimes \boldsymbol{u}\|_1 + \|\boldsymbol{u} \frac{\nabla b}{b} \cdot \boldsymbol{u}\|_1 \\
&\le |\xi| \|\boldsymbol{u}\|_2^2 + C\|\boldsymbol{u}\|_2^2 \le C(1 + |\xi|)\|\boldsymbol{u}(t)\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
&\left| P\left[\frac{\nabla b}{b} \widehat{(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T} - \nabla \cdot \boldsymbol{u} I) \right] \right| \le \left| \int \nabla \left(\frac{\nabla b}{b} \cdot \boldsymbol{u} \right) e^{i\xi \cdot \boldsymbol{x}} d\boldsymbol{x} \right| + \\
&\qquad + 2 \left| \int \nabla \left(\frac{\nabla b}{b} \right) \boldsymbol{u} e^{i\xi \cdot \boldsymbol{x}} d\boldsymbol{x} \right| + \left| \int \nabla \cdot \left(\frac{\nabla b}{b} \otimes \boldsymbol{u} \right) e^{i\xi \cdot \boldsymbol{x}} d\boldsymbol{x} \right| + \\
&\quad + \left| \int \left(\nabla \cdot \frac{\nabla b}{b} \right) \boldsymbol{u} e^{i\xi \cdot \boldsymbol{x}} d\boldsymbol{x} \right| + \left| \int \nabla \cdot \left(\boldsymbol{u} \otimes \frac{\nabla b}{b} \right) e^{i\xi \cdot \boldsymbol{x}} d\boldsymbol{x} \right| \\
&\le |\xi| \left(\|\frac{\nabla b}{b} \cdot \boldsymbol{u}\|_1 + \|\frac{\nabla b}{b} \otimes \boldsymbol{u}\|_1 + \|\boldsymbol{u} \otimes \frac{\nabla b}{b}\|_1 \right) + \\
&\quad + \| \left(\nabla \cdot \frac{\nabla b}{b} \right) \boldsymbol{u}\|_1 + \|\nabla \left(\frac{\nabla b}{b} \right) \boldsymbol{u}\|_1 \\
&\le C(1 + |\xi|)\|\boldsymbol{u}(t)\|_2.
\end{aligned}$$

Finally

$$|\widehat{P(\eta \boldsymbol{u})}| = \left| \int \eta \boldsymbol{u} e^{i\xi\cdot \boldsymbol{x}} d\boldsymbol{x} \right| \le \|\eta\|_2 \|\boldsymbol{u}(t)\|_2 \le C\|\boldsymbol{u}(t)\|_2. \qquad \square$$

We are now in the position to give the proof of the main theorem of this section:

Proof. Taking the scalar product of (3.17) with $\boldsymbol{u}$ and using Plancherel's theorem, we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi + \int_{\mathbb{R}^2} |\xi|^2 |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi \le |(\boldsymbol{f},\boldsymbol{u})_b|.$$

For the second term

$$\begin{aligned}\int_{\mathbb{R}^2} |\xi|^2|\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi &\ge \int_{G(t)^c} |\xi|^2 |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi \\ &\ge g^2(t)\int_{G(t)^c} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi \\ &= g^2(t)\int_{\mathbb{R}^2} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi - g^2(t)\int_{G(t)} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi\end{aligned}$$

where $G(t) = \left\{\xi \in \mathbb{R}^2 : |\xi| < g(t)\right\}$ and $g \in C\left([0,\infty]\,;\mathbb{R}_+\right)$ which can be determinate later.

Then

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi + g^2(t)\int_{\mathbb{R}^2} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi \le g^2(t)\int_{G(t)} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi + (\boldsymbol{f},\boldsymbol{u})_b,$$

and by Lemma 7 we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi + g(t)^2\int_{\mathbb{R}^2} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi \le \\ &\le 2\pi g^2(t)\int_0^{g(t)} \Bigg[\|\boldsymbol{u}_0\|_1 + Crt + C(1+r)\int_0^t \|\boldsymbol{u}(\tau)\|_2 d\tau \\ &\qquad + C(1+r)\int_0^t \|\boldsymbol{u}(\tau)\|_2^2 d\tau \Bigg]^2 r dr + |(\boldsymbol{f},\boldsymbol{u})_b|,\end{aligned}$$

and by Hölder inequality it is possible to write as:

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi + g(t)^2\int_{\mathbb{R}^2} |\hat{\boldsymbol{u}}(\xi,t)|^2 d\xi \le \\ &\le 2\pi g^2(t)\int_0^{g(t)} \Bigg[\|\boldsymbol{u}_0\|_1^2 + Cr^2t^2 + C(1+r^2)t^{\frac{3}{2}} \left[\int_0^t \|\boldsymbol{u}(\tau)\|_2^4 d\tau\right]^{\frac{1}{2}} + \\ &\qquad + C(1+r^2)t\int_0^t \|\boldsymbol{u}(\tau)\|_2^4 d\tau \Bigg] r dr + |(\boldsymbol{f},\boldsymbol{u})_b|.\end{aligned}$$

Integrating in time, we have:

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$$
\begin{aligned}
& e^{2\int_0^t g^2(s)ds}\|\boldsymbol{u}(t)\|_2^2 \le \|\boldsymbol{u}_0\|_2^2 + \\
& + 2\pi\|\boldsymbol{u}_0\|_1^2 \int_0^t e^{2\int_0^t g^2(s)ds} g^4(s)ds \\
& + 2\pi C \int_0^t e^{2\int_0^t g^2(s)ds} g^6(s)s^2 ds \\
& + C \int_0^t e^{2\int_0^t g^2(s)ds} (g^4(s)+g^6(s))s^{\frac{3}{2}} \left[\int_0^s \|\boldsymbol{u}(\tau)\|_2^4 d\tau\right]^{\frac{1}{2}} ds \\
& + C \int_0^t e^{2\int_0^t g^2(s)ds} (g^4(s)+g^6(s))s \left[\int_0^s \|\boldsymbol{u}(\tau)\|_2^4 d\tau\right] ds \\
& + \int_0^t e^{2\int_0^t g^2(s)ds} |(\boldsymbol{f},\boldsymbol{u})_b| ds.
\end{aligned}
$$

To obtain a basic estimate, we take

$$
\begin{aligned}
& g^2(t) = \frac{1}{(e+t)\log(e+t)}, \\
& e^{2\int_0^t g^2(s)ds} = \left[\log(e+t)\right]^2,
\end{aligned}
$$

then

$$
\begin{aligned}
& \int_0^t e^{2\int_0^t g^2(s)ds} g^4(s)ds \le C\int_0^t \frac{1}{(e+s)^2} ds \le C, \\
& \int_0^t e^{2\int_0^t g^2(s)ds} g^6(s)s^2 ds \le C\int_0^t \frac{s^2}{(e+s)^3\log(e+s)} ds \le C\log\left(\log(e+t)\right), \\
& \int_0^t e^{2\int_0^t g^2(s)ds} g^6(s)s^{\frac{3}{2}} \left[\int_0^s \|\boldsymbol{u}(\tau)\|_2^4 d\tau\right]^{\frac{1}{2}} ds \le \\
& \qquad \le C\int_0^t \frac{s^2\|\boldsymbol{u}_0\|_2^2}{(e+s)^3\log(e+s)} ds \le C\log\left(\log(e+t)\right), \\
& \int_0^t e^{2\int_0^t g^2(s)ds} g^6(s)s \left[\int_0^s \|\boldsymbol{u}(\tau)\|_2^4 d\tau\right] ds \le \\
& \qquad \le C\int_0^t \frac{s^2\|\boldsymbol{u}_0\|_2^4}{(e+s)^3\log(e+s)} ds \le C\log\left(\log(e+t)\right),
\end{aligned}
$$

and using Lemma 8:

$$
\int_0^t e^{2\int_0^t g^2(s)ds} g^4(s)s^{\frac{3}{2}} \left[\int_0^s \|\boldsymbol{u}(\tau)\|_2^4 d\tau\right]^{\frac{1}{2}} ds \le
$$

$$\leq C\int_0^t \frac{s^{\frac{3}{2}}}{(e+s)^{\frac{5}{2}}}ds \leq C\log(e+t),$$

$$\int_0^t e^{2\int_0^t g^2(s)ds} g^4(s)s\left[\int_0^s \|\boldsymbol{u}(\tau)\|_2^4 d\tau\right] ds \leq C\int_0^t \frac{s}{(e+s)^3}ds \leq C.$$

Finally, using the hypothesis (3.21) on $\boldsymbol{f}$ the last term is

$$\int_0^t e^{2\int_0^t g^2(s)ds} |(\boldsymbol{f},\boldsymbol{u})_b| ds \leq C\int_0^t \left[\log(e+s)\right]^2 \|\boldsymbol{f}\|_2 \|\boldsymbol{u}\|_2 ds$$

$$\leq C\|\boldsymbol{u}_0\|_2 \int_0^t \frac{\log^2(e+s)}{(e+s)^2}ds \leq C$$

Hence

$$\left[\log(e+t)\right]^2 \|\boldsymbol{u}(t)\|_2^2 \leq C\left[1+\log\left(\log(e+t)\right)+\log(e+t)\right],$$

and the theorem is proved. □

Conflict of interest statement

There is no conflict of interest.

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Appendix A. Technical lemmas

In this section we present some Lemmas that we have used in section 2 to prove the existence of the AIMs. We begin proving the estimates (2.13)–(2.15) for the continuous linear semigroup $\{e^{-A_{b\nu}t}\}_{t\geq 0}$ associated to the equation

$$\frac{d\boldsymbol{u}}{dt} + A_{b\nu}\boldsymbol{u} = 0. \tag{A.1}$$

Lemma 9. *Let P_n and Q_n be the projection operator defined in (2.12), and λ_n the n-th eigenvalue of $A_{b\nu}$, then*

$$|e^{-A_{b\nu}t}Q_n|_{\mathcal{L}(H,V)} \leq \overline{b}^{-\frac{1}{2}}\left((\nu_i t)^{-\frac{1}{2}} + \lambda_{n+1}^{\frac{1}{2}}\right)e^{-\lambda_{n+1}t}, \quad t>0,$$

$$|(I+\tau A_{b\nu})P_n|_{\mathcal{L}(V)} \leq (1+\tau\lambda_n) \leq e^{\tau\lambda_n},$$

$$|I|_{\mathcal{L}(P_nH,P_nV)} \leq \left(\frac{\overline{b}\nu_i}{\lambda_n}\right)^{-1/2}.$$

Proof. In the proof we shall use the notation introduced in (2.12). The second inequality derives from the obvious estimate

$$\tau \frac{\|A_{b\nu}\boldsymbol{y}\|_b^2}{\|\boldsymbol{y}\|_b^2} \le \tau\lambda_n^2, \tag{A.2}$$

while the third inequality is a simple consequence of the estimate

$$\frac{\|\boldsymbol{y}\|_b^2}{|\boldsymbol{y}|_b^2} \le \frac{1}{\overline{b}\nu_i} \frac{(A_{b\nu}\boldsymbol{y}, \boldsymbol{y})_b}{|\boldsymbol{y}|_b^2} \le \frac{1}{\overline{b}\nu_i}\lambda_n, \tag{A.3}$$

that can be easily proved using coercivity (2.10). To prove the first inequality, we compute the L_b^2-scalar product between the Q_n-projection of equation (A.1) and $A_{b\nu}z$; therefore, using coercivity (2.10), one can obtain

$$\frac{1}{2}\frac{d}{dt}\|z\|_b^2 + \lambda_{n+1}\|z\|_b^2 \le 0, \tag{A.4}$$

which gives

$$|e^{-tA_{b\nu}} Q_n|_{\mathcal{L}(V)} \le \exp(-\lambda_{n+1}t). \tag{A.5}$$

Analogously one can derive

$$|z|_b^2 \le |z_0|_b^2 \exp(-2\lambda_{n+1}t). \tag{A.6}$$

Now consider the Q_n-projected energy estimate, and multiply times $\exp(2\lambda_{n+1}t)$. One readily derives:

$$\frac{d}{dt}\left(\exp(2\lambda_{n+1}t)|z|_b^2\right) + 2\nu_i\overline{b}\exp(2\lambda_{n+1}t)\|z\|_b^2 \le 2\lambda_{n+1}\exp(2\lambda_{n+1}t)|z|_b^2$$

that, integrated in time, gives

$$2\nu_i\overline{b}\int_0^t \exp(2\lambda_{n+1}s)\|z\|_b^2 ds \le |z_0|_b^2 + 2\lambda_{n+1}\int_0^t \exp(2\lambda_{n+1}s)|z|_b^2 ds. \tag{A.7}$$

From (A.6), we obtain

$$\int_0^t \exp(2\lambda_{n+1}s)|z|_b^2 ds \le \int_0^t |z_0|_b^2 ds \le t|z_0|_b^2,$$

that, together with (A.7), gives

$$\int_0^t \exp(2\lambda_{n+1}s)\|z\|_b^2 ds \le \frac{1}{\overline{b}}|z_0|_b^2\left(\frac{1}{\nu_i} + \lambda_{n+1}t\right). \tag{A.8}$$

From (A.4), after multiplication times t, using the Gronwall lemma, and with the help of (A.8), one gets

$$t\|z\|_b^2 \le \frac{1}{\overline{b}}|z_0|_b^2\left(\frac{1}{\nu_i} + \lambda_{n+1}t\right)\exp(-2\lambda_{n+1}t).$$

The above estimate can be written as

$$|e^{-tA_{b\nu}} Q_n|_{\mathcal{L}(H,V)} \le \frac{1}{\overline{b}^{1/2}}\left(\left(\frac{1}{t\nu_i}\right)^{1/2} + \lambda_{n+1}^{1/2}\right)e^{-\lambda_{n+1}t}$$

which concludes the proof. □

We now want to prove that the operator $B_\theta : V \to H$, defined in (2.19), is bounded and Lipschitz. We first give two preliminary Lemmas.

Lemma 10. *Let $\boldsymbol{u} \in D(A_{b\nu})$, then there exist three constants χ_1, χ_2 and χ_3 such that the following inequalities hold:*

$$\|\boldsymbol{u}\|^2_{H^2_b(\Omega)} \le \chi_1 |A_{b\nu}\boldsymbol{u}|^2_b, \tag{A.9}$$

$$|\boldsymbol{u}|_{L^\infty(\Omega)} \le \chi_2 |\boldsymbol{u}|_b^{1/2} |A_{b\nu}\boldsymbol{u}|_b^{1/2}, \tag{A.10}$$

$$|\boldsymbol{u}|_{L^\infty(\Omega)} \le \chi_3 b_i^{-1/2} \|\boldsymbol{u}\|_b \left(1 + \log \frac{|A_{b\nu}\boldsymbol{u}|^2_b}{\lambda_1 \|\boldsymbol{u}\|^2_b}\right)^{1/2}. \tag{A.11}$$

Proof. The (A.9) is the regularity of the Stokes problem related to the elliptic operator $A_{b\nu}$ and was proved in [29].

The second inequality is the analogous of the classical Agmon inequality [1]. It comes from the interpolation inequality

$$|\boldsymbol{u}|_{L^\infty(\Omega)} \le \chi |\boldsymbol{u}|^{1/2} \|\boldsymbol{u}\|^{1/2}_{H^2},$$

and from (A.9). The last inequality is the Brezis–Gallouet inequality [4] adapted to our case. Following [4], one can show that there exists $\kappa > 0$ such that, for any $R > 0$

$$|\boldsymbol{u}|_{L^\infty(\Omega)} \le \kappa^2 b_i^{-1/2} \left(\|\boldsymbol{u}\|_b [\log(1+R)]^{1/2} + \|\boldsymbol{u}\|_{H^2_b(\Omega)} (1+R)^{-1} \right). \tag{A.12}$$

The proof is completed choosing $1 + R = \frac{|A_{b\nu}\boldsymbol{u}|^2_b}{\lambda_1 \|\boldsymbol{u}\|^2_b}$, and using (2.16) and (A.9). The constant χ_3 is given by:

$$\chi_3 = \max\{\kappa^2, \kappa^2 \chi_1 \lambda_1 \rho_1^2\}. \qquad \square \tag{A.13}$$

Next Lemma shows, for the solutions of (2.9), the existence in $L^\infty(\Omega)$ of an absorbing set:

Lemma 11. *Let $r > 0$, and suppose $\boldsymbol{u}_{in}$ satisfies $|\boldsymbol{u}_{in}|_b \le r$. Let $\boldsymbol{u}$ be a solution of (2.9) with initial datum $\boldsymbol{u}_{in} \in V \bigcap H^2_b$. Then there exist a time $\bar{t}(r)$ and $\rho_2 = \rho_2(\Omega, |\mathbf{f}|_b, \bar{\nu}, \|\boldsymbol{u}_{in}\|_b, r)$, such that*

$$|\boldsymbol{u}|_{L^\infty(\Omega)} \le \rho_2, \quad \text{for} \quad t \ge \bar{t}(r). \tag{A.14}$$

Proof. Taking the L^2_b norm of (2.9) we get

$$\begin{aligned} |A_{b\nu}\boldsymbol{u}|_b &\le \left|\frac{d\boldsymbol{u}}{dt}\right|_b + |B(\boldsymbol{u})|_b + |\boldsymbol{f}|_b + |\eta \boldsymbol{u}|_b \\ &\le \left|\frac{d\boldsymbol{u}}{dt}\right|_b + C\chi_2 |\boldsymbol{u}|_b^{1/2} |A_{b\nu}\boldsymbol{u}|_b^{1/2} \|\boldsymbol{u}\|_b + |\boldsymbol{f}|_b + \sup \eta\, |\boldsymbol{u}|_b, \end{aligned}$$

where we have used (A.10). Applying Young inequality, we have:

$$|A_{b\nu}\boldsymbol{u}|_b \le 2\left|\frac{d\boldsymbol{u}}{dt}\right|_b + C^2\chi_2^2 |\boldsymbol{u}|_b \|\boldsymbol{u}\|_b^2 + 2|\boldsymbol{f}|_b + 2\sup\eta\, |\boldsymbol{u}|_b.$$

Now take $t \ge t_0(r) + 2\alpha$ so that, in the above estimate, both (2.16) and (2.17) can be used. One immediately gets:

$$|A_{b\nu}\boldsymbol{u}|_b \le \frac{4}{\alpha}\rho_1 + C^2\chi_2^2 \rho_0 \rho_1^2 + 2|\boldsymbol{f}|_b + 2\rho_0 \sup\eta \equiv \widetilde{\rho}_2. \tag{A.15}$$

The above bound can be inserted in the Brezis–Gallouet inequality (A.11) and one obtains the desired result (A.14) with ρ_2 given by

$$\rho_2 = \chi_3 \rho_1 \left(1 + \left|\log \frac{\widetilde{\rho}_2^2}{\lambda_1}\right| + \left|\log \rho_1^2\right|\right)^{1/2} \tag{A.16}$$

and $\widetilde{\rho}_2$ defined in (A.15). $\square$

Lemma 12. *There exist two constants M_0 and M_1 (the explicit expressions are given in the proof) such that for every $\boldsymbol{u}, \boldsymbol{v} \in V$ results:*

$$|B_\theta \boldsymbol{u}|_b \leq M_0 \tag{A.17}$$

$$|B_\theta \boldsymbol{u} - B_\theta \boldsymbol{v}|_b \leq M_1 \|\boldsymbol{u} - \boldsymbol{v}\|_b. \tag{A.18}$$

Proof. Suppose that $\|\boldsymbol{u}\|_b \leq \rho_1$ (if $\|\boldsymbol{u}\|_b \geq \rho_1$ then $B_\theta \boldsymbol{u} = 0$) and consider

$$|B(\boldsymbol{u}, \boldsymbol{u})|_b \leq C\|\boldsymbol{u}\|_b |\boldsymbol{u}|_{L^\infty(\Omega)} \leq C\rho_1(|P_n \boldsymbol{u}|_{L^\infty(\Omega)} + |Q_n \boldsymbol{u}|_{L^\infty(\Omega)}).$$

Applying (A.11) and (A.14), we obtain (A.17) with M_0 given by:

$$M_0 = \rho_1 C \left(\rho_1 \chi_3 b_i^{-1/2} \left(1 + \log \frac{C'\lambda_n}{\lambda_1}\right)^{1/2} + \rho_2 \right),$$

where the constant C' expresses the continuity of the operator $A_{bv}^{1/2}$ from V to H, i.e. $|A_{bv}^{1/2}\boldsymbol{u}|_b^2 \leq C'\|\boldsymbol{u}\|_V^2$.

Passing to the proof of (A.18), we denote by L_θ the Lipschitz constant of the function θ.

$$\begin{aligned}
&|B_\theta(\boldsymbol{u}) - B_\theta(\boldsymbol{v})|_b \\
&\leq \left|\theta\left(\frac{\|\boldsymbol{u}\|_b}{\rho_1}\right) - \theta\left(\frac{\|\boldsymbol{v}\|_b}{\rho_1}\right)\right| |B(\boldsymbol{u})|_b + \left|\theta\left(\frac{\|\boldsymbol{v}\|_b}{\rho_1}\right)\right| |B(\boldsymbol{u}) - B(\boldsymbol{v})|_b \\
&\leq \frac{M_0}{\rho_1} L_\theta \,|\|\boldsymbol{u}\|_b - \|\boldsymbol{v}\|_b| + |B(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v})|_b + |B(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{v})|_b \\
&\leq \frac{M_0}{\rho_1} L_\theta \|\boldsymbol{u} - \boldsymbol{v}\|_b + |B(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v})|_b + |B(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{v})|_b \\
&\leq \frac{M_0}{\rho_1} L_\theta \|\boldsymbol{u} - \boldsymbol{v}\|_b + C\|\boldsymbol{u} - \boldsymbol{v}\|_b |\boldsymbol{u}|_{L^\infty(\Omega)} + C\|\boldsymbol{v}\|_b |\boldsymbol{u} - \boldsymbol{v}|_{L^\infty(\Omega)}.
\end{aligned}$$

Using (A.11), (A.14), (A.15) and (A.16) in the above expression we obtain the desired (A.18) with

$$M_1 \equiv \frac{M_0}{\rho_1} L_\theta + C\rho_2 + C\frac{\chi_3}{b_i^{1/2}} \rho_1 \left(1 + \left|\log \frac{2\widetilde{\rho}_2^2}{\lambda_1}\right| + \left|\log 2\rho_1^2\right|\right)^{1/2}. \qquad \square$$

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