Derivations of a $(n, 2, 1)$-nilpotent Lie algebra. [*][†][‡]

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Abstract

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1 Introduction

The most simple (non-abelian) Lie algebras are the generalized Heisenberg Lie algebras, defined on a $(n+1)$-dimensional vector space $\mathfrak{h} = V \oplus \langle z \rangle$ by a non-degenerate alternating form F on the n-dimensional sub-space V (n even) putting, for any $u, v \in V$, $[u, v] = F(u, v)z$.

In this case, there exist bases z of $\mathfrak{h}' = \mathfrak{z}$, $\{v_1, \ldots, v_{\frac{n}{2}}\}$ of $\mathfrak{v}$ and $\{w_1, \ldots, w_{\frac{n}{2}}\}$ of $\mathfrak{w}$ such that and, for $i, j \geq 1$, $[v_i, w_j] = \delta_{ij} z$.

Complex metabelian $(n+2)$-dimensional Lie algebra $\mathfrak{h} = V \oplus \langle z_1, z_2 \rangle$ defined by a pair of alternating form F_1, F_2 on V putting, for any $u, v \in V$, $[u, v] = F_1(u, v)z_1 + F_2(u, v)z_2$ have been classified firstly by Gauger [4], applying the canonical reduction of the pair F_1, F_2. More recently [2] Belitskii, Lipyanski and Sergeichuk showed that the case of a $(n+3)$-dimensional Lie algebra defined by a triple of alternating form F_1, F_2, F_3 on V putting $[u, v] = F_1(u, v)z_1 + F_2(u, v)z_2 + F_3(u, v)z_3$, for any $u, v \in V$, is hard.

In [1] we have shown that the same argument as in [4] is effective also in the case where the centre of is one-dimensional and the commutator ideal is two-dimensional. It turns out that, while the structure of $\mathfrak{h}$ depends on the field K if $\mathfrak{h}'$ is central, it is independent of K if $\mathfrak{h}'$ is non-central and is uniquely determined by the dimension of $\mathfrak{h}$.

In this case, the nilpotent Lie K-algebra $\mathfrak{h}$ of $(n, 2, 1)$-type with $n < 2|K|$ and 1-dimensional centre $\mathfrak{z}$ decomposes, as a vector space, into $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{v} \oplus \mathfrak{w}$ where

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$\mathfrak{h}_0 = \begin{cases} \langle z, v_0, v_1, w_1 \rangle \text{ if } n \text{ is even} \\ \langle z, v_0, v_1, w_1, w_0 \rangle \text{ if } n \text{ is odd} \end{cases}$, $\mathfrak{v}$ and $\mathfrak{w}$ are Abelian subalgebras of $\mathfrak{h}$. More precisely, we have:

1) if n is odd, then there exist bases z, v_0 of $\mathfrak{h}'$ with $z \in \mathfrak{z}$, $\{v_2, \dots, v_{\frac{n-1}{2}}\}$ of $\mathfrak{v}$ and $\{w_2, \dots, w_{\frac{n-1}{2}}\}$ of $\mathfrak{w}$ such that and, for $i, j \geq 0$, $[v_i, w_j] = \delta_{ij} z$, excepting $[v_1, w_0] = z + v_0$.

2) if n is even, then there exist bases z, v_0 of $\mathfrak{h}$ with $z \in \mathfrak{z}$, $\{v_2, \dots, v_{\frac{n}{2}}\}$ of $\mathfrak{v}$ and $\{w_2, \dots, w_{\frac{n}{2}}\}$ of $\mathfrak{w}$ such that $[v_0, w_1] = z$ and, for $i, j \geq 1$, $[v_i, w_j] = \delta_{ij} z$, excepting $[v_1, w_1] = z + v_0$.

In the present paper we consider the algebra of derivations ∂ of $\mathfrak{h}$ and determine a Borel subalgebra. Since any derivation ∂ of $\mathfrak{h}$ belongs to a Borel subalgebra, this allows us to give a classification of solvable $(n+3)$-dimensional complex Lie algebras, having $\mathfrak{h}$ as an ideal of co-dimension one.

These fall into two classes, namely the algebraic and the non-algebraic ones. The non-algebraic Lie algebras are solvable non-nilpotent such that the outer derivation ∂ is neither semi-simple nor nilpotent.

The algebraic Lie algebras can be nilpotent or not, according to the case that we take a nilpotent derivation ∂ or a semi-simple one. In the first case, we obtain a complete classification considering the possible rank of ∂. In the second case, one has a precise description of the possible cases.

It turns out that, fon n even, the algebra of derivations can represented, with respect to the basis $\{z, v_0, v_1, w_1; v_2, \dots, v_{\frac{n}{2}}; w_2, \dots, w_{\frac{n}{2}}\}$ by the set of matrices of type

$$\left(\begin{array}{cccc|c|c} a_{33}+2a_{44} & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{32} & a_{33}+a_{44} & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{31} & a_{32} & a_{33} & 0 & \mathbf{0} & \mathbf{0} \\ a_{41} & a_{42} & a_{43} & a_{44} & \mathbf{u}_1 & \mathbf{u}_2 \\ \hline \mathbf{u}_3' & -\mathbf{u}_2' & \mathbf{0}' & \mathbf{0}' & A & B \\ \hline \mathbf{u}_4' & \mathbf{u}_1' & \mathbf{0}' & \mathbf{0}' & C & (a_{33}+2a_{44})I - A' \end{array}\right)$$

with B, C symmetric.

The algebra of inner derivations can represented by the set of matrices of type

$$\left(\begin{array}{cccc|c|c} 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{32} & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 0 & a_{32} & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{41} & a_{42} & 0 & 0 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{u}_3' & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{u}_4' & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' & \mathbf{0} & \mathbf{0} \end{array}\right)$$

hence, up to an inner derivation, a derivation can represented, by the matrix

$$\left(\begin{array}{cccc|c|c} a_{33}+2a_{44} & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 0 & a_{33}+a_{44} & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{31} & 0 & a_{33} & 0 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & a_{43} & a_{44} & \mathbf{u}_1 & \mathbf{u}_2 \\ \hline 0' & -\mathbf{u}_2' & \mathbf{0}' & \mathbf{0}' & A & B \\ \hline 0' & \mathbf{u}_1' & \mathbf{0}' & \mathbf{0}' & C & (a_{33}+2a_{44})I - A' \end{array}\right)$$

with B, C symmetric.

Among the diagonal ones, we find matrices with mutually different eigenvalues, hence a flag of invariant subspaces for a maximal toral subalgebra containing the diagonal matrices is built by coordinate subspaces. Thus, the diagonal matrices of the above kind give a maximal toral subalgebra. It turns out that the ideal I generated by $\{z, v_0, v_1, v_2, \ldots, v_{\frac{n}{2}}\}$ is invariant for a Borel subalgebra containing the diagonal matrices of the above kind. In fact, let W be the $\frac{n}{2}+2$-dimensional invariant ideal in a flag of such a Borel subalgebra. If W is not the ideal I, then consider the $\frac{n}{2}+1$-dimensional invariant subspace S in the flag. The shape of a derivation shows that S cannot contain w_i, for any $0 \leq i \leq \frac{n}{2}$. Consequently S is contained in the ideal I. Since the ideal generated by $\{z, v_0, v_1\}$ is invariant under *any* derivation, we conclude anyway that I is a $\frac{n}{2}+2$-dimensional invariant ideal of a Borel subalgebra containing the diagonal matrices. Therefore we can take $B = 0$, that is, a Borel subalgebra containing the maximal toral subalgebra of diagonal derivations is given by the derivations of the kind

$$\left(\begin{array}{cccc|c|c} a_{33}+2a_{44} & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{32} & a_{33}+a_{44} & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{31} & a_{32} & a_{33} & 0 & \mathbf{0} & \mathbf{0} \\ a_{41} & a_{42} & a_{43} & a_{44} & \mathbf{u}_1 & \mathbf{u}_2 \\ \hline \mathbf{u}_3' & -\mathbf{u}_2' & \mathbf{0}' & \mathbf{0}' & A & \mathbf{0} \\ \hline \mathbf{u}_4' & \mathbf{u}_1' & \mathbf{0}' & \mathbf{0}' & C & (a_{33}+2a_{44})I - A' \end{array}\right),$$

with C symmetric. Since two Borel subalgebras are conjugated by an inner automorphism $\exp \operatorname{ad} y = \operatorname{Ad} \exp y$ and $\operatorname{Ad} \exp y$ acts by matrix conjugation, a Borel subalgebra can be taken in the above form.

The solvable radical is given by the set of matrices

$$\left(\begin{array}{cccc|c|c} a_{33}+2a_{44} & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{32} & a_{33}+a_{44} & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{31} & a_{32} & a_{33} & 0 & \mathbf{0} & \mathbf{0} \\ a_{41} & a_{42} & a_{43} & a_{44} & \mathbf{u}_1 & \mathbf{u}_2 \\ \hline \mathbf{u}_3' & -\mathbf{u}_2' & \mathbf{0}' & \mathbf{0}' & A & \mathbf{0} \\ \hline \mathbf{u}_4' & \mathbf{u}_1' & \mathbf{0}' & \mathbf{0}' & \mathbf{0} & (a_{33}+2a_{44})I - A' \end{array}\right)$$

with A diagonal, and a Levi complement by the set of matrices

$$\left(\begin{array}{cccc|c|c} 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0}' & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' & A & B \\ \hline \mathbf{0}' & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' & C & -A' \end{array}\right)$$

with B, C symmetric, that is, the symplectic Lie algebra $\mathfrak{sp}$.

Fon n odd, the algebra of derivations can represented, with respect to the basis $\{z, v_0, v_1, w_1, w_0; v_2, \ldots, v_{\frac{n}{2}}; w_2, \ldots, w_{\frac{n}{2}}\}$ by the set of matrices of type

$$\left(\begin{array}{ccccc|c|c} a_{44}+2a_{55} & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{42}-a_{53} & a_{44}+a_{55} & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{31} & a_{32} & 2a_{55} & 0 & 0 & \mathbf{0} & \mathbf{0} \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & \mathbf{u}_1 & \mathbf{u}_2 \\ a_{51} & a_{52} & a_{53} & -a_{32} & a_{55} & \mathbf{u}_3 & \mathbf{u}_4 \\ \hline \mathbf{u}_5' & \mathbf{u}_4' & -\mathbf{u}_2' & \mathbf{0}' & \mathbf{0}' & A & B \\ \hline \mathbf{u}_6' & -\mathbf{u}_3' & \mathbf{u}_1' & \mathbf{0}' & \mathbf{0}' & C & (a_{44}+2a_{55})I - A' \end{array}\right)$$

with B, C symmetric. The structure of the algebra of derivations can be deduced from the case of n even.

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