Admissible perturbations of α-ψ-pseudocontractive operators: convergence theorems[‡]

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We prove some convergence theorems for α-ψ-pseudocontractive operators in real Hilbert spaces, by using the concept of admissible perturbation. Our results extend and complement some theorems in the existing literature.

Keywords: α-admissible mapping; α-ψ-pseudocontractive operator; Krasnoselskij type fixed point iterative scheme

1. Introduction

Samet et al. [20] introduced the notion of α-ψ-contractive mapping and established some fixed point results in the setting of complete metric spaces. Thereafter, based on the ideas in [20], many authors established a variety of results and applied these results in solving different practical situations involving differential equations and other mathematical problems [1, 9, 10, 15, 16]. Thus, from a theoretical point of view, the usefulness of the new fixed point theorems is recognized. On the other hand, over the last decades, the study of convergence of fixed point iterative methods has received an increasing attention, due to their performance as tools for solving numerical problems. As a consequence of this fact, the reader can access to a wide literature on iterative schemes involving different types of mappings and operators, see for example [4, 5] and the references therein. We point out that fixed point iterative approximation methods have been largely applied in dealing with stability and convergence problems. In particular, we refer to various control and optimization questions arising in pure and applied sciences involving dynamical systems, where the problem in study can be easily arranged as a fixed point problem. Let $\mathbb{N}$ be the set of natural numbers starting from 1. We recall that, given a non-empty set X and a mapping $T : K \to X$, with $K \subset X$, solving a fixed point problem means to get a point $x \in K$ such that $x = Tx$. Also, at the basis of the proof of metric theorems for existence and uniqueness of fixed point there is an iterative scheme, known as Picard iteration at starting point x_0, which constructs a sequence $\{x_n\}$ of successive approximations of above point x, by assuming $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$, with given x_0. Then, under suitable hypotheses on mapping T and space X, it is possible to prove interesting results on convergence, stability and give a priori and a posteriori estimation errors, see [4, 5, 17]. Here, we merge the potentiality of the ideas in [20] with the constructive development of iterative schemes to obtain new theoretical results. More precisely, this study is based on the recent papers [20] (concerning fixed point theory) and [2] (concerning iterative schemes). Throughout the paper, standard notations and terminologies in nonlinear analysis are used. Also, we follow the general organization of Berinde in [2]. Then, after necessary preliminaries, we prove some convergence theorems for α-ψ-pseudocontractive operators in real Hilbert spaces. Precisely, by using the concept of admissible perturbations of α-ψ-pseudocontractive operators in Hilbert spaces, we prove results for Krasnoselskij type fixed point iterative schemes. Our theorems complement, generalize and unify some existing results, see for instance [2, 3, 4].

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2. Preliminaries

2.1. Fixed point theory

We consider the framework of complete metric spaces and recall some useful notions on mappings.

Definition 1 ([20]) *Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. We say that T is α-admissible if*

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Denote by Ψ the family of non-decreasing functions $\psi : [0, +\infty[\to [0, +\infty[$ such that $\psi(t) > 0$ and $\lim_{n \to +\infty} \psi^n(t) = 0$ for each $t > 0$, where ψ^n is the n-th iterate of ψ.

Lemma 1 ([20]) *For every function $\psi \in \Psi$, we have $\psi(t) < t$ for each $t > 0$.*

Definition 2 ([20]) *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$.

Also, we consider the following properties of regularity, see also [11]. Let (X, d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Then

(i) X is α-regular if for each sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, we have that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$,

(ii) X has the property (C) with respect to α if for each sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $\alpha(x_m, x_n) \geq 1$ for all $n > m \geq n_0$.

In the sequel, we will use the following theorem, with and without continuity hypothesis on mapping T, see [11, 20].

Theorem 1 *Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *X has the property (C) with respect to α;*
(iv) *T is continuous or X is α-regular.*

Then, T has a fixed point, that is, there exists $x^* \in X$ such that $x^* = Tx^*$.

Now, we get some additional conditions and notions for a pair of mappings.

Definition 3 *Let $S, T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. We say that S is α-T-admissible if*

$$x \in X, \quad \alpha(x, Sx) \geq 1 \Longrightarrow \alpha(Sx, TSx) \geq 1 \text{ and } \alpha(TSx, STSx) \geq 1.$$

Definition 4 *Let (X, d) be a metric space and $S, T : X \to X$ be two given mappings. We say that (S, T) is an α-ψ-contractive pair of mappings if there exist two functions $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that*

$$d(Tx, Sy) \leq \psi(d(x, y)),$$

for all $x, y \in X$ with $\max\{\alpha(x, y), \alpha(y, x)\} \geq 1$.

Then, we prove the following theorem, with and without continuity hypothesis on mappings S and T, see also [14].

Theorem 2 *Let (X, d) be a complete metric space and $S, T : X \to X$ be such that (S, T) is an α-ψ-contractive pair of mappings satisfying the following conditions:*

(i) *S is α-T-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*
(iii) *X has the property (C) with respect to α;*
(iv) *S and T are continuous or X is α-regular.*

Then, S and T have a common fixed point, that is, there exists $x^* \in X$ such that $x^* = Tx^* = Sx^*$.

Proof Let x_0 be an arbitrary point in X satisfying condition (ii). If $x_0 = Sx_0$, then $d(Tx_0, SSx_0) \leq \psi(d(x_0, Sx_0)) = 0$ and hence $d(Tx_0, SSx_0) = d(Tx_0, x_0) = 0$ so that the proof is finished. Thus, we assume that $x_0 \neq Sx_0$. Then, we define a sequence $\{x_n\} \subseteq X$ as

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}, \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Since S is α-T-admissible, by using condition (ii), we inductively obtain $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Now, we suppose that the successive terms of sequence $\{x_n\}$ are distinct; otherwise the proof is finished. Next, from $\alpha(x_{2n-1}, x_{2n}) \geq 1$, we obtain

$$d(x_{2n}, x_{2n+1}) = d(Tx_{2n-1}, Sx_{2n}) \leq \psi(d(x_{2n-1}, x_{2n})).$$

Also, from $\alpha(x_{2n}, x_{2n+1}) \geq 1$, we get

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(x_{2n+2}, x_{2n+1}) \\ &= d(Tx_{2n+1}, Sx_{2n}) \\ &\leq \psi(d(x_{2n+1}, x_{2n})) \\ &= \psi(d(x_{2n}, x_{2n+1})). \end{aligned}$$

From above inequalities, we have

$$d(x_{2n}, x_{2n+1}) \leq \psi(d(x_{2n-1}, x_{2n})) \leq \cdots \leq \psi^{2n}(d(x_0, x_1))$$

and

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n}, x_{2n+1})) \leq \cdots \leq \psi^{2n+1}(d(x_0, x_1)).$$

Consequently, we write

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)), \quad \text{for all } n \in \mathbb{N},$$

and hence we conclude that $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$. Now, we prove that $\{x_n\}$ is a Cauchy sequence. Given $\varepsilon > 0$, there exists $n(\varepsilon) > 0$ such that

$$\max\{d(x_{2m}, x_{2m+1}), d(x_{2m}, x_{2m+2})\} < \varepsilon - \psi^2(\varepsilon), \quad \text{for all } m \geq n(\varepsilon),$$

where $\varepsilon - \psi^2(\varepsilon) > 0$ by Lemma 1. We claim that $d(x_{2m}, x_{2n+1}) < \varepsilon$ for all $n \geq m \geq n(\varepsilon)$. If $n = m$ the statement is obvious. We can assume that for some $n \geq m$, we have $d(x_{2m}, x_{2n+1}) < \varepsilon$. Then, we get

$$\begin{aligned} d(x_{2m}, x_{2n+3}) &\leq d(x_{2m}, x_{2m+2}) + d(x_{2m+2}, x_{2n+3}) \\ &= d(x_{2m}, x_{2m+2}) + d(Tx_{2m+1}, Sx_{2n+2}) \\ &< \varepsilon - \psi^2(\varepsilon) + \psi(d(x_{2m+1}, x_{2n+2})) \\ &\leq \varepsilon - \psi^2(\varepsilon) + \psi(d(Tx_{2n+1}, Sx_{2m})) \\ &\leq \varepsilon - \psi^2(\varepsilon) + \psi(\psi(d(x_{2m}, x_{2n+1}))) \\ &\leq \varepsilon - \psi^2(\varepsilon) + \psi^2(\varepsilon) = \varepsilon. \end{aligned}$$

This ensures that $\{x_{2n}\}$ is a Cauchy sequence and so $\{x_n\}$ is also Cauchy.

From the completeness of (X, d), there exists $z \in X$ such that $x_n \to z$ as $n \to +\infty$.

If S and T are continuous, from $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$, we obtain $z = Sz = Tz$.

On the other hand, if X is α-regular, then $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N}$.

From

$$\begin{aligned} d(z, Sz) &= \lim_{n\to+\infty} d(x_{2n+2}, Sz) \\ &= \lim_{n\to+\infty} d(Tx_{2n+1}, Sz) \\ &\leq \lim_{n\to+\infty} \psi(d(x_{2n+1}, z)) \\ &\leq \lim_{n\to+\infty} d(x_{2n+1}, z) = 0, \end{aligned}$$

we deduce that $d(z, Sz) = 0$.

From

$$\begin{aligned} d(Tz, x_{2n+1}) &= d(Tz, Sx_{2n}) \\ &\leq \psi(d(z, x_{2n})) < d(z, x_{2n}), \end{aligned}$$

passing to the limit as $n \to +\infty$, we deduce $Tz = z$. We conclude that S and T have a common fixed point.

2.2. *Iterative approximation schemes*

Fixed point iterative methods were studied to approximate the solutions of fixed point problems involving mappings with specific properties. Here we consider pseudocontractive type operators. Precisely, we recall the following concepts, see [12, 19, 21].

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

An operator $T : K \to H$, with $K \subset H$, is called

(i) pseudocontractive if, for all $x, y \in K$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - Ty - (x - y)\|^2,$$

or equivalently

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2.$$

(ii) strongly pseudocontractive if there exists a constant $k \in]0, 1[$ such that, for $x, y \in K$,

$$\langle Tx - Ty, x - y \rangle \leq k\|x - y\|^2.$$

Successively, many researchers generalized above concepts to get classes of pseudocontractive type operators, with unifying power over the existing ones, see for instance [3, 12]. Also, Rus [18] proposed a new approach to fixed point iterative schemes, by giving the concept of admissible perturbation, see [5, 6, 7].

We recall three fundamental iterative schemes from existing literature [4, 5, 17]. In fact, starting from the Picard iteration scheme, many researchers introduced various fixed point iterative methods for solving the basic fixed point problem in Introduction, under weaker hypotheses, see again [4] and references therein.

Let E be a real vector space, $x_0 \in E$ be an arbitrary starting point and $T : E \to E$ be a given operator. First, we consider Krasnoselskij iteration scheme, that is, an approximation sequence $\{x_n\} \subset E$ given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{1}$$

Clearly, by putting $\lambda = 1$ in (1), we get the Picard iteration scheme.

Secondly, let $\{\alpha_n\} \subset [0, 1]$ be a sequence of real numbers. We cite Mann iterative scheme [13], that is, an approximation sequence $\{x_n\} \subset E$ given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2}$$

Finally, Ishikawa iterative scheme [8] is a sequence $\{x_n\} \subset E$ given by

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \end{cases} \tag{3}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

Obviously, for $\beta_n = 0$, (3) reduces to (2), while, for $\alpha_n = \lambda$, (2) reduces to (1).

Now, according to Rus [18], we give some useful concepts.

Definition 5 ([18]) *Let X be a nonempty set. A mapping $G : X \times X \to X$ is called admissible if it satisfies the following two conditions:*

(i) *$G(x, x) = x$, for all $x \in X$;*
(ii) *$G(x, y) = x$ implies $x = y$.*

Definition 6 ([18]) *Let X be a nonempty set. If $T : X \to X$ is a given mapping and $G : X \times X \to X$ is an admissible mapping, then the mapping $T_G : X \to X$, defined by*

$$T_G x = G(x, Tx), \quad \text{for all } x \in X,$$

is called the admissible perturbation of T.

Remark 1 ([2]) *Let $Fix(T) := \{x \in X : x = Tx\}$ denote the set of all fixed points of mapping $T : X \to X$. Notice that, if $T_G : X \to X$ is the admissible perturbation of T, then T_G and T have the same set of fixed points, that is, $Fix(T_G) = Fix(T)$.*

We give some significant examples from existing literature.

Example 1 ([18]) *Let $(V, +, \mathbb{R})$ be a real vector space, $K \subset V$ a convex subset, $\lambda \in]0, 1[$, $T : K \to K$ and $G : K \times K \to K$ be defined by*

$$G(x, y) = (1 - \lambda)x + \lambda y, \quad x, y \in K.$$

Then G is an admissible mapping and $T_G x = G(x, Tx)$ is the corresponding admissible perturbation of T. In this case, T_G is also known as Krasnoselskij perturbation of T.

Definition 7 ([18]) *Let $T : K \to K$ be a nonlinear mapping and $G : K \times K \to K$ an admissible operator. Then the iterative scheme $\{x_n\}$ given by $x_0 \in K$ and*

$$x_{n+1} = G(x_n, Tx_n), \qquad \text{for all } n \in \mathbb{N} \cup \{0\},$$

is called the Krasnoselskij iterative scheme corresponding to G, say GK-iterative scheme for short.

Definition 8 *Let $G : E \times E \to E$ be an admissible operator on a normed space E. We say that G is affine Lipschitzian if there exists a constant $\mu \in [0, 1]$ such that*

$$\|G(x_1, y_1) - G(x_2, y_2)\| \leq \|\mu(x_1 - x_2) + (1 - \mu)(y_1 - y_2)\|,$$

for all $x_1, x_2, y_1, y_2 \in E$.

3. Convergence theorems of α-ψ-pseudcontractive operators

We build the fundamental concept of this section on the following definition.

Definition 9 ([3]) *Let H be a real Hilbert space. An operator $T : K \subset H \to H$ is said to be strictly ψ-pseudocontractive if, for given $a, b, c \in [0, 1]$ with $a + b + c = 1$, there exists a function $\psi \in \Psi$ such that*

$$a\|x - y\|^2 + b\langle Tx - Ty, x - y\rangle + c\|Tx - Ty\|^2 \leq \psi^2(\|x - y\|),$$

holds, for all $x, y \in K$.

On this basis, we introduce the following class of α-ψ-pseudocontractions.

Definition 10 *Let H be a real Hilbert space. Two operators $T_1, T_2 : K \subset H \to H$ are said to be an α-ψ-pseudocontractive pair if, for given $a, b, c \in [0, 1]$ with $a + b + c = 1$, there exist two functions $\psi \in \Psi$ and $\alpha : K \times K \to \mathbb{R}$ such that*

$$a\|x - y\|^2 + b\langle T_1x - T_2y, x - y\rangle + c\|T_1x - T_2y\|^2 \leq \psi^2(\|x - y\|),$$

holds, for all $x, y \in K$ with $\max\{\alpha(x, y), \alpha(y, x)\} \geq 1$. Clearly, if $T_1 = T_2$, we get the definition of α-ψ-pseudocontractive operator. In addition, if $\alpha(x, y) = 1$ for all $x, y \in K$, then we retrieve Definition 9.

Now, we present the main result of this paper, which is a convergence theorem of GK-iterative scheme.

Theorem 3 *Let K be a nonempty closed and convex subset of a real Hilbert space H, $T_1, T_2 : K \to K$ an α-ψ-pseudocontractive pair of operators and $G : K \times K \to K$ an affine Lipschitzian admissible operator with constant $\lambda \in]0, 1[$. Assume that the following conditions hold:*

(i) *T_{1_G} is α-T_{2_G}-admissible;*
(ii) *there exists $x_0 \in K$ such that $\alpha(x_0, T_{1_G}x_0) \geq 1$;*
(iii) *K has the property (C) with respect to α;*
(iv) *T_{1_G}, T_{2_G} are continuous or K is α-regular.*

Then T_1 and T_2 have a common fixed point in K and the GK-iterative scheme $\{x_n\}$, given by $x_0 \in K$ and

$$x_{2n+1} = G(x_{2n}, T_1x_{2n}) \text{ and } x_{2n+2} = G(x_{2n+1}, T_2x_{2n+1}), \qquad \text{for all } n \in \mathbb{N} \cup \{0\},$$

converges to a common fixed point of T_1 and T_2, for any $x_0 \in K$ such that (ii) holds.

Proof Let $T_{1_G}, T_{2_G} : K \to K$ be the admissible perturbation operators associated with operators T_1 and T_2, that is, $T_{1_G}x = G(x, T_1x)$ and $T_{2_G}y = G(y, T_2y)$, for all $x, y \in K$.
Now, since G is affine Lipschitzian admissible, then there exists a constant $\lambda \in]0, 1[$ such that

$$\|G(x, T_1x) - G(y, T_2y)\| \leq \|(1 - \lambda)(x - y) + \lambda(T_1x - T_2y)\|,$$

for all $x, y \in K$. From above facts, we can write

$$\begin{aligned} \|T_{1_G}x - T_{2_G}y\|^2 &= \|G(x, T_1x) - G(y, T_2y)\|^2 \\ &\leq \|(1 - \lambda)(x - y) + \lambda(T_1x - T_2y)\|^2 \\ &= (1 - \lambda)^2\|x - y\|^2 + 2\lambda(1 - \lambda)\langle T_1x - T_2y, x - y\rangle + \lambda^2\|T_1x - T_2y\|^2. \end{aligned}$$

Thus, from the last inequality, by denoting $a = (1-\lambda)^2$, $b = 2\lambda(1-\lambda)$ and $c = \lambda^2$ so that $a+b+c = 1$ and since T_1 and T_2 are an α-ψ-pseudocontractive pair of operators, we deduce that there exist two functions $\psi \in \Psi$ and $\alpha : K \times K \to \mathbb{R}$ such that $x, y \in K$ and $\max\{\alpha(x,y), \alpha(y,x)\} \geq 1$ imply $\|T_{1_G}x - T_{2_G}y\|^2 \leq \psi^2(\|x-y\|)$. Thus, we have

$$\|T_{1_G}x - T_{2_G}y\| \leq \psi(\|x-y\|),$$

for all $x, y \in K$, with $\max\{\alpha(x,y), \alpha(y,x)\} \geq 1$.

This means that the pair (T_{1_G}, T_{2_G}) is α-ψ-contractive and hence T_{1_G} and T_{2_G} have a common fixed point, by an application of Theorem 2. Notice that $Fix(T_{1_G}) = Fix(T_1)$ and $Fix(T_{2_G}) = Fix(T_2)$, therefore $Fix(T_{1_G}) \cap Fix(T_{2_G}) = Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Thus, the conclusions of Theorem 3 hold true.

Remark 2 *In solving some problems involving differential equations, it is natural to consider a partial order on the setting space. In this context, the function $\alpha : K \times K \to \mathbb{R}$ defined by*

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise,} \end{cases}$$

is a typical example of function suitable for Theorem 3.

We give an easy example, which shows that the function α really enlarges applicability of Theorem 3.

Example 2 *Let $H = \mathbb{R}$, $K = [0,1]$, $G : K \times K \to K$ as in Example 1 and $T_1, T_2 : K \to K$ be given by $T_1x = x^2$ and $T_2x = x^3$. Take $\lambda = 1/2$ and consider the starting point $x_0 = 1$ so that $T_1x_0 = 1$. Let $\alpha : K \times K \to \mathbb{R}$ be defined by*

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x = y \in \{0,1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the conditions (i)-(iv) of Theorem 3 hold true. Clearly, $G : K \times K \to K$ is affine Lipschitzian admissible and $T_1, T_2 : K \to K$ are an α-ψ-pseudocontractive pair of operators, for all $a, b, c \in \mathbb{R}_+$ with $a+b+c = 1$, and for each $\psi \in \Psi$. Here, we obtain the constant GK-iterative scheme $\{x_n\}$ given by $x_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$. Note that $Fix(T_1) \cap Fix(T_2) = \{0,1\}$.
On the other hand, define $\alpha : K \times K \to [0, +\infty[$ by $\alpha(x,y) = 1$ for all $x, y \in K$ and retaining the rest (which is equivalent to say that we do not consider the function α in Definition 10 and Theorem 3). In this case, there do not exist $a, b, c \in \mathbb{R}_+$, with $a+b+c = 1$, such that T_1, T_2 are an α-ψ-pseudocontractive pair of operators. Thus, Theorem 3 (without the function α) does not apply.

Clearly, in the case of Krasnoselskij perturbation with $T_1 = T_2 = T$, we have the following result, see also Theorem 2 of [2].

Theorem 4 *Let K be a nonempty closed and convex subset of a real Hilbert space H, $T : K \to K$ an α-ψ-pseudocontractive operator and $G : K \times K \to K$ an affine Lipschitzian admissible operator with constant $\lambda \in]0,1[$. Assume that the following conditions hold:*

- (i) *T_G is α-admissible;*
- (ii) *there exists $x_0 \in K$ such that $\alpha(x_0, T_Gx_0) \geq 1$;*
- (iii) *K has the property (C) with respect to α;*
- (iv) *T_G is continuous or K is α-regular.*

Then T has a fixed point in K and the GK-iterative scheme $\{x_n\}$, given by $x_0 \in K$ and

$$x_{n+1} = G(x_n, Tx_n), \qquad \text{for all } n \in \mathbb{N} \cup \{0\},$$

converges to a fixed point of T, for any $x_0 \in K$ such that (ii) holds.

Example 3 *Let $H = l^2(\mathbb{R})$, $K = \{\overline{x} \in H : \sum_{i=1}^{\infty} x_i^2 \leq 1\}$, $G : K \times K \to K$ as in Example 1 and $T : K \to K$ be given by*

$$T(\overline{x}) = T(x_1, x_2, \cdots, x_n, \cdots) = \left(-\frac{x_1}{2}, \frac{x_2}{4}, \cdots, \frac{x_n}{4}, \cdots\right).$$

Take $\lambda = 1/2$ and consider the starting point $\overline{x}_0 = (-1, 0, \cdots, 0, \cdots)$ so that $T_G\overline{x}_0 = (-\frac{1}{4}, 0, \cdots, 0, \cdots)$.
Let $\alpha : K \times K \to \mathbb{R}$ be defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x_i \leq y_i, \quad i \in \mathbb{N} \setminus \{1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the conditions (i)-(iv) of Theorem 4 hold true. Clearly, $G : K \times K \to K$ is affine Lipschitzian admissible and $T : K \to K$ is α-ψ-pseudocontractive, with $a = b = 0$, $c = 1$ and $\psi(t) = t/2$. Here, we obtain the GK-iterative scheme $\{\overline{x}_n\}$ given by $\overline{x}_n = (-\frac{1}{4^n}, 0, \cdots, 0, \cdots)$ for all $n \in \mathbb{N} \cup \{0\}$, which converges to $\overline{x}^* = (0, 0, \cdots, 0, \cdots)$. Also, x^* is a unique fixed point of T.

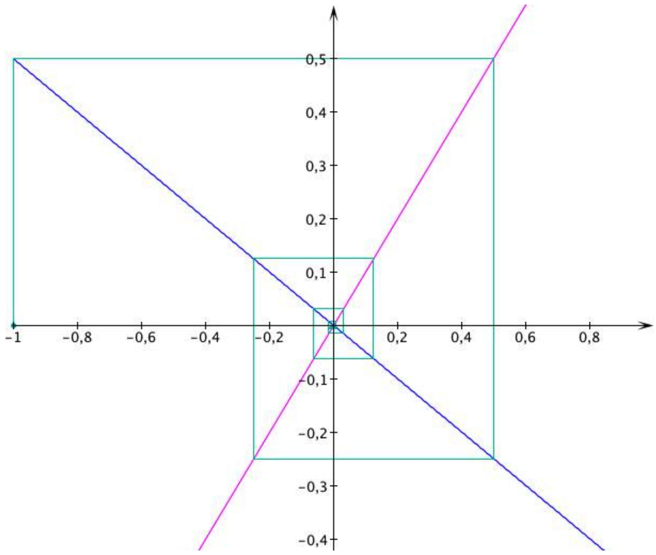

Figure 1. $x_{0_1} = -1$, $x_1^* = 0$, $\lambda = 1$, $toll = 0.0001$

Table 1. Comparison between different values of λ

Iteration	$\lambda = 1/2$	$\lambda = 3/4$	$\lambda = 1$
0	-1	-1	-1
1	- 0.25000	0.12500	0.50000
2	- 0.06250	- 0.01563	-0.25000
3	- 0.01563	0.00195	0.12500
4	- 0.00391	- 0.00024	-0.06250
5	- 0.00098	0.00003	0.03125
6	- 0.00024	0.00000	-0.01563
7	- 0.00006	0.00000	0.00781
8	- 0.00002	0.00000	-0.00391
9	0.00000	0.00000	0.00195
10	0.00000	0.00000	0.00098

Example 4 *Let $H = \mathbb{R}$, $K = [0, 1]$, $G : K \times K \to K$ as in Example 1 and $T : K \to K$ be given by $Tx = x^2$ for all $x \in [0, 1]$. Let $\alpha : K \times K \to \mathbb{R}$ be defined by*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1/4] \text{ or } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $T : K \to K$ is α-ψ-pseudocontractive, with $a = b = 0$, $c = 1$ and $\psi(t) = kt$, with $k \in [1/2, 1[$. Take $\lambda = 1/2$ and consider the starting point $x_0 = 1/4$ so that $T_G x_0 = 5/32$.
Again, all the hypotheses of Theorem 4 hold true. Here, the GK-iterative scheme $\{x_n\}$ converges to 0, which is a fixed point of T. Note that $Fix(T) = \{0, 1\}$.

4. Parallel algorithm

In this section, by following the research line in Zhang and Guo [22], we study the convergence of a parallel algorithm. Precisely, let K be a nonempty closed and convex subset of a real Hilbert space H, $N \in \mathbb{N}$ and $\{T_i\}_{i=1}^N$ a finite family of self-operators on K, satisfying certain properties. We consider the parallel GK-iterative scheme $\{x_n\}$ given by $x_0 \in K$ and

$$x_{n+1} = G(x_n, \sum_{i=1}^{N} \gamma_i T_i x_n), \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{4}$$

Thus, we give sufficient conditions to prove the convergence of (4) to a common fixed point x^* of $\{T_i\}_{i=1}^N$, that is $x^* \in \cap_{i=1}^N Fix(T_i)$.

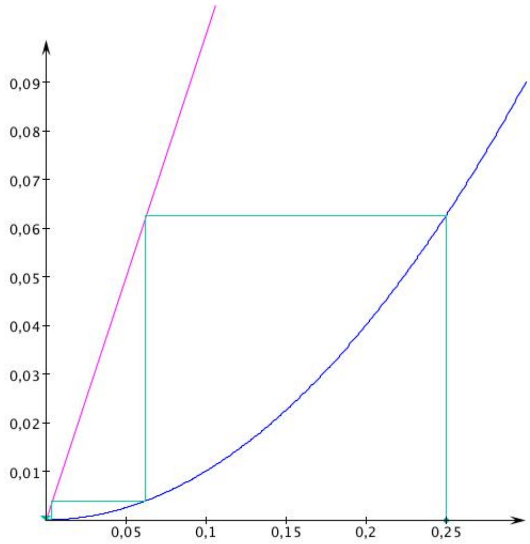

Figure 2. $x_0 = 1/4$, $x^* = 0$, $\lambda = 1$, $toll = 0.0001$

Table 2. Comparison between different values of λ

Iteration	$\lambda = 1/2$	$\lambda = 3/4$	$\lambda = 1$
0	0.25000	0.25000	0.25000
1	0.15625	0.10938	0.62500
2	0.09033	0.03632	0.00391
3	0.04925	0.01007	0.00002
4	0.02584	0.00259	0.00000
5	0.01325	0.00065	0.00000
6	0.00671	0.00016	0.00000
7	0.00338	0.00004	0.00000
8	0.00170	0.00001	0.00000
9	0.00085	0.00000	0.00000
10	0.00042	0.00000	0.00000
11	0.00021	0.00000	0.00000
12	0.00011	0.00000	0.00000
13	0.00005	0.00000	0.00000

Theorem 5 *Let K be a nonempty closed and convex subset of a real Hilbert space H, $\{T_i\}_{i=1}^N$ a finite family of operators such that $T_i : K \to K$ is α-ψ_i-pseudocontractive, where $\psi_i(t) = \delta_i t$ with $\delta_i \in]0, 1[$. Let $G : K \times K \to K$ be an affine Lipschitzian admissible operator with constant $\lambda \in]0, 1[$. Denote $T := \sum_{i=1}^N \gamma_i T_i$, where $\{\gamma_i\}_{i=1}^N \subset]0, +\infty[$ is a finite sequence such that $\sum_{i=1}^N \gamma_i = 1$, and assume that the following conditions hold:*

(i) *T_G is α-admissible;*
(ii) *there exists $x_0 \in K$ such that $\alpha(x_0, T_G x_0) \geq 1$;*
(iii) *K has the property (C) with respect to α;*
(iv) *T_G is continuous or K is α-regular;*
(v) *$Fix(T) = \cap_{i=1}^N Fix(T_i) \neq \emptyset$.*

Then the GK-iterative scheme $\{x_n\}$, given by $x_0 \in K$ and (4), converges to a common fixed point in $\cap_{i=1}^N Fix(T_i)$, for any $x_0 \in K$ such that (ii) holds.

Proof Since $T = \sum_{i=1}^N \gamma_i T_i$ and $T_i : K \to K$ is α-ψ_i-pseudocontractive operator, where $\psi_i(t) = \delta_i t$ with $\delta_i \in]0, 1[$, then T is an α-ψ-pseudocontractive operator, where $\psi(t) = \delta t$ with $\delta = \max_i \delta_i < 1$.
Now, let $T_G : K \to K$ be the admissible perturbation operator associated with operator T, that is, $T_G x = G(x, Tx)$, for all $x \in K$.
Also, since $G : K \times K \to K$ is an affine Lipschitzian admissible operator, then there exists a constant $\lambda \in]0, 1[$ such that

$$\|G(x, Tx) - G(y, Ty)\| \leq \|(1 - \lambda)(x - y) + \lambda(Tx - Ty)\|,$$

for all $x, y \in K$. From above facts, we can write

$$\begin{aligned} \|T_G x - T_G y\|^2 &= \|G(x, Tx) - G(y, Ty)\|^2 \\ &\leq \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\|^2 \\ &= (1-\lambda)^2\|x-y\|^2 + 2\lambda(1-\lambda)\langle Tx - Ty, x - y\rangle + \lambda^2\|Tx - Ty\|^2. \end{aligned}$$

From last inequality, by denoting $a = (1-\lambda)^2$, $b = 2\lambda(1-\lambda)$ and $c = \lambda^2$ so that $a + b + c = 1$ and since T is α-ψ-pseudocontractive with $\psi(t) = \delta t$, we deduce that there exist $\delta = \max_i \delta_i$ and a function $\alpha : K \times K \to \mathbb{R}$ such that

$$x, y \in K \text{ and } \alpha(x, y) \geq 1 \text{ imply } \|T_G x - T_G y\|^2 \leq \delta^2\|x - y\|^2.$$

Thus, we have

$$\|T_G x - T_G y\| \leq \delta\|x - y\|,$$

for all $x, y \in K$ with $\alpha(x, y) \geq 1$.

This means that T_G is an α-ψ-contractive operator with $\psi(t) = \delta t$, and hence by an application of Theorem 1, T_G has a fixed point. Finally, from Remark 1, we recall that $Fix(T_G) = Fix(T)$ and hence the conclusion of Theorem 5 holds true since $Fix(T) = \cap_{i=1}^N Fix(T_i)$.

Example 5 *Let $H = \mathbb{R}$, $K = [0, +\infty[$, $G : K \times K \to K$ as in Example 1 and $\{T_i\}_{i=1}^N$ be a finite family of operators such that $T_i : K \to K$ is given by $T_i x = \frac{\gamma_i x}{x+1}$ with $\gamma_i \in]0, 1/4[$ and $\sum_{i=1}^N \gamma_i = 1$. Take $\lambda = 1/2$ and consider the starting point $x_0 = 1/4$ so that $T_G x_0 = 1/8 + \sum_{i=1}^N {\gamma_i}^2/10 < 1/4$ for all $\gamma_i \in]0, 1/4[$. Let $\alpha : K \times K \to \mathbb{R}$ be defined by*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, \frac{1}{4}], \\ 0 & \text{otherwise.} \end{cases}$$

Thus the conditions (i)-(iv) of Theorem 5 hold true. Clearly, $T_i : K \to K$ is α-ψ_i-pseudocontractive, with $a = b = 0$, $c = 1$ and $\psi(t) = \gamma_i t$, with $\gamma_i \in]0, 1/4[$. Here, the GK-iterative scheme $\{x_n\}$ converges to 0, which is a common fixed point of $\{T_i\}_{i=1}^N$. Note that $\cap_{i=1}^N Fix(T_i) = \{0\}$.

5. Conclusions

The recent literature is rich in iterative methods useful to solve numerical problems involving linear and nonlinear equations. Thus, we gave some new convergence theorems for iterative approximation schemes, via fixed point theory. Our results use an implicit formulation of iterative schemes and hence unify and shorten the proofs of previously existing theorems. Some examples illustrate the new theory and show as new concepts may enlarge applicability of iterative schemes.

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