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Two positive solutions for a nonlinear parameter-depending algebraic system

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Abstract

The existence of two positive solutions for a nonlinear parameter-depending algebraic system is investigated. The main tools are a finite dimensional version of a two critical point theorem and a recent weak-strong discrete maximum principle.

1 Introduction

Let N be a positive integer. Consider the following parameter-depending system of nonlinear algebraic equations

$$Au = \lambda \underline{f}(u) \qquad (A_{\lambda,\underline{f}})$$

where $u = (u(1), \ldots, u(N))^t$, $\underline{f}(u) := (f_1(u(1)), f_2(u(2)), \ldots, f_N(u(N)))^t \in \mathbb{R}^N$ are two column vectors, $f_k : \mathbb{R} \to \mathbb{R}$ is a continuous function for every $k = 1, 2, \ldots, N$, λ is a positive parameter and $A = [a_{ij}]_{N\times N}$ is a positive definite symmetric $Z-$matrix. As special case, we consider the tridiagonal nonlinear symmetric systems

$$T_N(a, b, b) = \lambda \underline{f}(u), \qquad (T_{\lambda,\underline{f}})$$

where the matrix A takes the shape of a tridiagonal matrix

$$T_N(a,b,b) := \begin{pmatrix} a & b & 0 & \ldots & 0 \\ b & a & b & \ldots & 0 \\ & \ldots & \ldots & \ldots & \\ 0 & \ldots & b & a & b \\ 0 & \ldots & 0 & b & a \end{pmatrix}_{N\times N}$$

where $a, b \in \mathbb{R}$ with $b < 0$ and

$$a > 2|b| \cos\left(\frac{\pi}{N+1}\right), \qquad (1)$$

which plays an important role to develop numerical schemes to find approximations of solutions of differential boundary value problems, as the finite element method or the finite difference method, see for instance [15] and the therein references. For instance, we can reduce to our setting the following second order nonlinear discrete Dirichlet boundary value problem, namely

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f_k(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \qquad (2)$$

where $[1, N]$ denotes the discrete interval $\{1, \ldots, N\}$, for every $k \in [1, N]$, $\Delta u(k) := u(k+1) - u(k)$ is the forward difference operator, $\Delta^2 u(k-1) := u(k+1) - 2u(k) + u(k-1)$ is the second order difference operator and $f_k(u(k)) = f(k, u(k))$, being $f : [1, N] \times \mathbb{R} \to \mathbb{R}$ a continuous function. Indeed, by computations we can show that problem (2) is a particular case of system $(T_{\lambda,\underline{f}})$ where the matrix A is given by $T_N(2, -1, -1)$.

It is worth noticing that, in general, in the right hand-side of (2) as well as in that of $(A_{\lambda,\underline{f}})$, the function $f_k(s)$ are not restrictions of the same function $f : [1, N] \times \mathbb{R} \to \mathbb{R}$.

To investigate the existence of two positive solutions, we combine variational methods with truncation techniques. Roughly speaking, we solve the algebraic system $(A_{\lambda,\underline{f}})$ looking for nontrivial critical points of the so called energy function $I_\lambda : \mathbb{R}^N \to \mathbb{R}$ defined by putting

$$I_\lambda(u) := \frac{1}{2} u^t A u - \lambda \sum_{k=1}^{N} \int_0^{u(k)} f_k^+(t)\,dt, \quad \forall\, u \in \mathbb{R}^N,$$

where, for all $k \in [1, N]$ and for all $s \in \mathbb{R}$,

$$f_k^+(s) = \begin{cases} f_k(s), & \text{if } s \geq 0; \\ f_k(0), & \text{if } s < 0. \end{cases} \qquad (3)$$

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Clearly, standard arguments ensure that I_λ is a C^1 functional in $\mathbb{R}^N$ with gradient given by

$$\nabla I_\lambda u = Au - \lambda \underline{f}^+(u), \quad \forall\, u \in \mathbb{R}^N,$$

being $\underline{f}^+(u) := (f_1^+(u(1)), f_2^+(u(2)), ..., f_N^+(u(N)))^t \in \mathbb{R}^N$. Hence, the directional derivative of I_λ at the point $u \in \mathbb{R}^N$ in the direction $v \in \mathbb{R}^N$ is given by

$$\frac{\partial I_\lambda(u)}{\partial v} = \langle \nabla I_\lambda u, v\rangle = v^t Au - \lambda \sum_{k=1}^N f_k^+(u(k))v(k), \quad \forall u,\ v \in \mathbb{R}^N. \tag{4}$$

Therefore, we have that $\nabla I_\lambda u \equiv 0$ if and only if

$$v^t Au - \lambda \sum_{k=1}^N f_k^+(u(k))v(k) = 0, \quad \forall v \in \mathbb{R}^N. \tag{5}$$

So, it is by now evident that (5) can be considered as the weak formulation of problem ($A_{\lambda,\underline{f}}$) and it is the key to study the nonlinear system ($A_{\lambda,\underline{f}}$) via variational methods. More precisely, we have that *the critical points of I_λ are nonnegative solutions of problem* ($A_{\lambda,\underline{f}}$) (see the proof of Theorem 3.1).

Finally, to guarantee that such solutions are positive, we apply a discrete strong maximum principle for problem ($A_{\lambda,\underline{f}}$) contained in [8]. However, with respect to [8], here we are able to obtain the existence of two positive solutions without requiring the additional assumption

$$f_k(0) \neq 0, \text{ for some } k \in [1, N].$$

In other words, we assume that

(j_1): $f_k(0) \geq 0$ *for every* $k \in [1, N]$,

hence the system ($A_{\lambda,\underline{f}}$) can admit the trivial solution.

In particular, our aim is to describe suitable intervals of parameters for which the system ($A_{\lambda,\underline{f}}$) admits two positive solutions (Theorem 3.1). To this end, we use a finite dimensional version of a two critical point theorem established in [9], see Theorem 2.2 below.

Arguing in a similar way, we can see that other difference boundary value problems, as for instance, Neumann problem, three-point problem, etc., can be considered as special cases of system ($A_{\lambda,\underline{f}}$), for more details we refer to [1, 2, 17, 24].

Variational methods are used to study algebraic nonlinear equations and nonlinear difference problem in many directions, as for instance: the existence of at least three solutions for systems with indefinite coefficient matrices [19]; positive and negative solutions in [27]; existence and multiplicity solutions for difference equations with different boundary conditions [4-8], [10-12] and difference equations with discontinuous nonlinearities in [13]. For general references on nonlinear algebraic systems we refer the reader to [20-26]. In particular, in [24] and in therein references, among the other results, you can find a review on many problems related to nonlinear algebraic systems of type ($A_{\lambda,\underline{f}}$) which includes also compartmental systems, strongly damped lattice system and the discrete periodic boundary value problems.

2 Mathematical Background

In the N-dimensional Banach space $\mathbb{R}^N$, we consider the two equivalent norms

$$\|u\|_2 := \Big(\sum_{k=1}^N u(k)^2\Big)^{1/2} \quad \text{and} \quad \|u\|_\infty := \max_{k\in[1,N]} |u(k)|,$$

for which we have

$$\|u\|_\infty \leq \|u\|_2 \leq \sqrt{N}\|u\|_\infty. \tag{6}$$

Let be $u \in \mathbb{R}^N$, we say that u is nonnegative ($u \geq 0$), if $u(k) \geq 0$ for every $k \in [1, N]$, while we say that u is positive ($u > 0$), if $u(k) > 0$ for every $k \in [1, N]$. We recall that a matrix $A = [a_{ij}]_{N\times N}$ is said: *positive definite*, if $u^t Au > 0$ for all $u \neq 0$; *positive semidefinite*, if $u^t Au \geq 0$ for all $u \in \mathbb{R}^N$. It is easy to show that the diagonal entries of any positive semidefinite matrix are nonnegative. Moreover, if $A = [a_{ij}]_{N\times N}$ denotes a positive semidefinite matrix with eigenvalues $\lambda_1, ..., \lambda_N$ ordered as $\lambda_1 \leq \ldots \leq \lambda_N$, we know that

$$\lambda_1\|u\|_2^2 \leq u^t Au \leq \lambda_N\|u\|_2^2, \quad \forall u \in \mathbb{R}^N, \tag{7}$$

from which we have that a real matrix A is positive definite if and only if its eigenvalues are all positive.

We say that a matrix $A = [a_{ij}]_{N\times N}$ is *a Z−matrix*, if $a_{ij} \leq 0$ for every $i \neq j$; a Z−matrix is a *strongly Z−matrix* iff for each $k \in [2, N]$, one has

- there exists $j_k < k$ such that $a_{kj_k} < 0$;
- there exists $i_k < k$ such that $a_{i_k k} < 0$.

For more details on these topics see also [16]. Putting together Theorems 2.1 and 2.2 of [8], we have the following weak-strong maximum principle for problem ($A_{\lambda,\underline{f}}$)

Theorem 2.1. *Let $A=[a_{ij}]_{N\times N}$ be a positive definite real $Z-$matrix. If $u\in\mathbb{R}^N$ satisfies the following condition:*

(i) *either* $u(k)>0$ *or* $(Au)(k)\geq 0$, *for each* $k\in[1,N]$.

Then, one has $u\geq 0$. If in addition, A is a strongly $Z-$matrix, then, either $u\equiv 0$ or $u>0$.

Our main tool is a two non-zero critical points theorem established in [9], that we recall here for the reader's convenience. To introduce such result, we need the definition of the well known Palais-Smale condition, in brief (PS). If X is a real Banach space, we say that $I_\lambda : X\to\mathbb{R}$ satisfies the (PS)-condition whenever one has that any sequence $\{u_n\}$ such that

1. $\{I_\lambda(u_n)\}$ is bounded;
2. $\{I'_\lambda(u_n)\}$ is convergent at 0 in X^*

admits a subsequence which is convergent in X.

Theorem 2.2. *Let X be a real Banach space and let Φ, $\Psi : X\to\mathbb{R}$ be two functionals of class C^1 such that $\inf_X\Phi=\Phi(0)=\Psi(0)=0$. Assume that there are $r\in\mathbb{R}$ and $\tilde{u}\in X$, with $0<\Phi(\tilde{u})<r$, such that*

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r}<\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{8}$$

and, for each

$$\lambda\in\Lambda=\left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})},\frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[,$$

the functional $I_\lambda=\Phi-\lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below.

Then, for each $\lambda\in\Lambda$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I(u_{\lambda,1})<0<I(u_{\lambda,2})$.

Remark 1. It is worth noticing that the previous result guaranties the existence of two non-zero critical points for an appropriate class of differentiable functionals. In particular, a careful reading of its proof shows that $u_{\lambda,1}$ is a local minimum for I_λ, while $u_{\lambda,2}$ is a mountain pass critical point, see also [3].

Next proposition is dedicated to study the (PS)-condition for the energy functional I_λ. To this end, we put

$$L_\infty(k):=\liminf_{s\to+\infty}\frac{F_k(s)}{s^2}\qquad L_\infty:=\min_{1\leq k\leq N}L_\infty(k),$$

$$L^\infty(k):=\limsup_{s\to+\infty}\frac{F_k(s)}{s^2}\qquad L^\infty:=\max_{1\leq k\leq N}L^\infty(k),$$

$$\Psi(u):=\sum_{k=1}^N F_k(u(k)),\quad \forall\, u\in\mathbb{R}^N, \tag{9}$$

where $F_k(s):=\int_0^s f_k^+(t)dt$ for all $s\in\mathbb{R}$ and for all $k\in[1,N]$. We read $\frac{1}{+\infty}=0$ whenever this case occurs.

Proposition 2.3. *Let $A=[a_{ij}]_{N\times N}$ be a positive definite, symmetric real Z-matrix. Assume that (j_1) hold and either $\lambda<\frac{\lambda_1}{2L^\infty}$ or $\frac{\lambda_N}{2L_\infty}<\lambda$. Then, the energy functional I_λ satisfies the $(PS)-$ condition. Moreover,*

(ps_1) *if $\frac{\lambda_N}{2L_\infty}<\lambda$, then I_λ is unbounded from below;*

(ps_2) *if $\lambda<\frac{\lambda_1}{2L^\infty}$, then I_λ is coercive, i.e. $\lim\limits_{\|u\|_2\to+\infty}I_\lambda(u)=+\infty$;*

Proof. Fix a positive λ as in the assumptions. Clearly, it is enough to show that any (PS) sequence of I_λ is bounded in $\mathbb{R}^N$. Let $\{u_n\}$ be a (PS) sequence of I_λ, that is

$$\lim_{n\to+\infty}I_\lambda(u_n)=c,\ c\in\mathbb{R}\qquad\lim_{n\to+\infty}\sup_{\|v\|_2\leq 1}\langle\nabla I_\lambda(u_n),v\rangle=0. \tag{10}$$

Consider the vectors $u_n^\pm$ defined by putting $u_n^\pm(k):=\max\{\pm u_n(k),0\}$, for every $n\in\mathbb{N}$ and $k\in[1,N]$ and, first, let us verify that $\{u_n^-\}$ is bounded. By (6), (7), (j_1), using the decomposition $u_n=u_n^+-u_n^-$ and recalling that A is a Z-matrix, we can estimate the derivative of I_λ at u_n, in the direction of $-u_n^-$

$$\begin{aligned}\langle\nabla I_\lambda(u_n),-u_n^-\rangle &= (-u_n^-)^tAu_n+\lambda\sum_{k=1}^N f_k(0)u_n^-(k)\\ &\geq (-u_n^-)^tAu_n^+ + (u_n^-)^tAu_n^-\\ &\geq \sum_{i,j=1}^N(-a_{ij})u_n^-(i)u_n^+(j)+\lambda_1\|u_n^-\|_2^2\\ &\geq \lambda_1\|u_n^-\|_2^2,\end{aligned}$$

that is,

$$\lambda_1 \|u_n^-\|_2 \le \langle \nabla I_\lambda(u_n), \frac{-u_n^-}{\|u_n^-\|_2} \rangle, \quad \forall n \in \mathbb{N}. \tag{11}$$

Thus, by (10), we get $\lim_{n\to+\infty} \|u_n^-\|_2 = 0$, which implies that $\{u_n^-\}$ is bounded in $\mathbb{R}^n$. In addition, by (6), there exists $M > 0$ such that

$$0 \le u_n^-(k) \le M, \text{ for all } k \in [1, N] \text{ and } n \in \mathbb{N}. \tag{12}$$

Now, we also prove that $\{u_n^+\}$ is bounded. Distinguish the cases:

a) $\lambda > \frac{\lambda_N}{2L_\infty}$

b) $\lambda < \frac{\lambda_1}{2L^\infty}$

Suppose a) holds. We only consider the case $0 < L_\infty < +\infty$; if $L_\infty = +\infty$ one can work in analogy. Fix $\rho = \rho(\lambda) > 0$ such that

$$\frac{\lambda_N}{2\lambda} < \rho < L_\infty. \tag{13}$$

For every $k \in [1, N]$, there is $\delta_k > 0$ such that

$$F_k(s) > \rho s^2, \quad \forall\, s > \delta_k.$$

A direct computation shows that for every $k \in [1, N]$ there exists $\eta_k > 0$ such that

$$F_k(s) > \rho s^2 - \eta_k, \quad \forall\, s \in \mathbb{R}^+. \tag{14}$$

Fix $n \in \mathbb{N}$. Clearly, the previous inequality ensures

$$\Psi(u_n^+) = \sum_{k=1}^N F_k(u_n^+(k)) \ge \rho \sum_{k=1}^N |u_n^+(k)|^2 - \sum_{k=1}^N \eta_k = \rho \|u_n^+\|_2^2 - \eta.$$

On the other hand, from (12) one has

$$\Psi(-u_n^-) = \sum_{k=1}^N F_k(-u_n^-(k)) = -\sum_{k=1}^N f_k(0) u_n^-(k) \ge -M \sum_{k=1}^N f_k(0).$$

Hence, since $\|u_n\|_2^2 = \|u_n^+\|_2^2 + \|u_n^-\|_2^2$, bearing in mind also (6), (7) and (12), one has

$$\begin{aligned} I_\lambda(u_n) &= \frac{1}{2} u_n^t A u_n - \lambda\left(\Psi(u_n^+) + \Psi(-u_n^-)\right) \\ &\le \frac{\lambda_N}{2} \|u_n^+\|_2^2 - \lambda\rho \|u_n^+\|_2^2 + \lambda\left(\eta + M \sum_{k=1}^N f_k(0)\right) + \frac{\lambda_N}{2} N M^2 \\ &= \left(\frac{\lambda_N}{2} - \lambda\rho\right) \|u_n^+\|_2^2 + \lambda\left(\eta + M \sum_{k=1}^N f_k(0)\right) + \frac{\lambda_N}{2} N M^2. \end{aligned}$$

Therefore, by contradiction, if $\|u_n^+\|_2 \to +\infty$, then one would have that $\lim_{n\to+\infty} I_\lambda(u_n) = -\infty$, against (10). Hence, $\{u_n^+\}$ is bounded and our conclusion follows.

Suppose b) holds. Fix $\rho = \rho(\lambda) > 0$ such that

$$L^\infty < \rho < \frac{\lambda_1}{2\lambda}. \tag{15}$$

For every $k \in [1, N]$, there is $\delta_k > 0$ such that

$$F_k(s) < \rho s^2, \quad \forall\, s > \delta_k.$$

Observing that $F_k(s) \le 0$ for every $s \le 0$, we can find some $\eta > 0$ such that for every $k \in [1, N]$

$$F_k(s) \le \rho s^2 + \eta, \quad \forall\, s \in \mathbb{R}. \tag{16}$$

Therefore, for every $u \in \mathbb{R}^N$, by (6) and the previous inequality, we have that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2} u^t A u - \lambda(\Psi(u^+) + \Psi(-u^-)) \\ &\ge \frac{\lambda_1}{2} \|u\|_2^2 - \lambda \Psi(u^+) \\ &\ge \left(\frac{\lambda_1}{2} - \lambda\rho\right) \|u^+\|_2^2 + \frac{\lambda_1}{2} \|u^-\|_2^2 - \lambda N \eta. \end{aligned}$$

Obviously, if $\|u\|_2 \to +\infty$ at least one between $\|u^+\|_2$ and $\|u^-\|_2$ tends to $+\infty$. Hence, I_λ is coercive and, in view of (10), it is obvious that any (PS) sequence is bounded. In particular, (ps_2) holds.

We conclude the proof verifying (ps_1). Fix $\{u_n\}$ in $\mathbb{R}^N$ such that $u_n = u_n^+$ for every $n \in \mathbb{N}$ and $\|u_n\|_2 \to +\infty$. Reasoning as in case a), one has

$$I_\lambda(u_n) \le \left(\frac{\lambda_N}{2} - \lambda\rho\right) \|u_n\|_2^2 + \lambda\eta.$$

Namely, I_λ is unbounded from below. □

3 Main results

In this section, we present our main results, where we obtain the existence of two positive solutions for problem $(A_{\lambda,\underline{f}})$ provided that A is a positive symmetric real strongly $Z-$matrix and the continuous vector field $\underline{f}$ satisfies condition (j_1).

Theorem 3.1. *Let A be a positive definite symmetric real strongly $Z-$matrix and let $\underline{f}$ be a continuous vector field fulfilling condition (j_1). Let c be a positive constant and let $w \in \mathbb{R}^N$ be a vector with $0 < w^t A w < \lambda_1 c^2$. Assume that*

$$(j_2)\quad \frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2} < \lambda_1 \min\left\{ \frac{\sum_{k=1}^{N} F_k(w(k))}{w^t A w}, \frac{L_\infty}{\lambda_N} \right\}.$$

Then, for each $\lambda \in \Lambda_1 := \left] \frac{1}{2} \max\left\{ \frac{w^t A w}{\sum_{k=1}^{N} F_k(w(k))}, \frac{\lambda_N}{L_\infty} \right\}, \frac{\lambda_1}{2} \frac{c^2}{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)} \right[$, *problem $(A_{\lambda,\underline{f}})$ admits at least two positive solutions.*

Proof. Obviously, by (j_2) the interval Λ_1 is well-posed. We apply Theorem 2.2 by putting

$$X = \mathbb{R}^N, \quad \tilde{u} = w, \quad \Phi(u) := \frac{1}{2} u^t A u, \quad \forall u \in \mathbb{R}^N, \tag{17}$$

and $I_\lambda := \Phi - \lambda\Psi$, where Ψ is the function introduced in (9). Clearly, Φ and Ψ are two functions of class C^1 with $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Taking $r = \frac{\lambda_1}{2} c^2$, by (6) and (7), we observe that

$$\Phi(u) \le r \Longrightarrow \|u\|_\infty \le c. \tag{18}$$

Therefore, we have that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} \le \frac{2}{\lambda_1} \frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2}. \tag{19}$$

On the other hand, we observe that,

$$\frac{\Psi(w)}{\Phi(w)} = 2\frac{\sum_{k=1}^{N} F_k(w(k))}{\sum_{i,j=1}^{N} a_{ij} w(i) w(j)}. \tag{20}$$

Hence, owing to (j_2), combining (19) and (20), we get

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(w)}{\Phi(w)},$$

being in particular $\Lambda_1 \subset \Lambda$.

Clearly, one has $0 < \Phi(w) < r$. Thus, for every $\lambda \in \Lambda_1$, owing to (ps_1) of Proposition 2.3, we get that the function $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below. Therefore, I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$.

Fixed $k \in [1, N]$, one has that either $u_{\lambda,i}(k) > 0$ or $(Au_{\lambda,i})(k) \ge 0$, $i = 1, 2$, owing to (j_1). So also condition (i) of Theorem 2.1 is verified and this implies that such solutions are positive. So, the proof is completed. □

Remark 2. In the proof of Theorem 3.1, exploiting that A is a positive Z-matrix, we can obtain that $u_{\lambda,1}$, $u_{\lambda,2}$ are two non-negative solutions of problem $(A_{\lambda,\underline{f}})$, testing the weak formulation (5) with $-u^-_{\lambda,i}$, $i = 1, 2$, without using condition (i) of Theorem 2.1.

Let A be a positive definite symmetric real strongly $Z-$matrix and let $\underline{f}$ be a continuous vector field fulfilling condition (j_1). Put,

$$\sigma(A) := \sum_{i,j=1}^{N} a_{ij},$$

some useful consequences of Theorem 3.1 are the following results.

Corollary 3.2. *Assume that $\sigma(A) > 0$. Let c and d be two positive constants with $d < c$ such that*

$$(j_3)\quad \frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2} < \lambda_1 \min\left\{ \frac{\sum_{k=1}^{N} F_k(d)}{\sigma(A) d^2}, \frac{L_\infty}{\lambda_N} \right\}.$$

Then, for each $\lambda \in \Lambda_2 := \left] \frac{1}{2} \max\left\{ \frac{\sigma(A)d^2}{\sum_{k=1}^{N} F_k(d)}, \frac{\lambda_N}{L_\infty} \right\}, \frac{\lambda_1}{2} \frac{c^2}{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)} \right[$*, problem* $(A_{\lambda,\underline{f}})$ *admits at least two positive solutions.*

Proof. We apply Theorem 3.1 by choosing $w(k) = d$ for every $k \in [1, N]$. Clearly, to get our conclusion it is enough to verify that $w^t Aw < \lambda_1 c^2$, that is $d < \sqrt{\frac{\lambda_1}{\sigma(A)}}c$. Arguing, by contradiction, we have that $c > d \geq \sqrt{\frac{\lambda_1}{\sigma(A)}}c$, from which it follows that

$$\frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2} \geq \frac{\sum_{k=1}^{N} F_k(d)}{c^2} \geq \frac{\lambda_1}{d^2} \frac{\sum_{k=1}^{N} F_k(d)}{\sigma(A)},$$

which contradicts our assumption (j_3). □

Corollary 3.3. *Let c and d be two positive constant with* $d < c$ *such that*

(j_4) $\frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2} < \lambda_1 \min\left\{ \frac{F_{\bar{k}}(d)}{a_{\bar{k}\bar{k}} d^2}, \frac{L_\infty}{\lambda_N} \right\}$*, for some* $\bar{k} \in [1, N]$.

Then, for each $\lambda \in \Lambda_3 := \left] \frac{1}{2} \max\left\{ \frac{a_{\bar{k}\bar{k}} d^2}{F_{\bar{k}}(d)}, \frac{\lambda_N}{L_\infty} \right\}, \frac{\lambda_1}{2} \frac{c^2}{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)} \right[$*, problem* $(A_{\lambda,\underline{f}})$ *admits at least two positive solutions.*

Proof. We apply Theorem 3.1 arguing as in the proof of Corollary 3.2, by choosing $w(\bar{k}) = d$ and $w(k) = 0$ for every $k \in [1, N]$ with $k \neq \bar{k}$. □

Corollary 3.4. *Assume that*

(j_5) $\inf_{c>0} \frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2} < \frac{\lambda_1}{\lambda_N} L_\infty$.

(j_6) *There exists* $\bar{k} \in [1, N]$ *such that* $\limsup_{s\to 0^+} \frac{F_{\bar{k}}(s)}{s^2} = +\infty$.

Then, for each $\lambda \in \Lambda_4 := \left] \frac{1}{2} \frac{\lambda_N}{L_\infty}, \frac{\lambda_1}{2} \sup_{c>0} \frac{c^2}{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)} \right[$*, problem* $(A_{\lambda,\underline{f}})$ *admits at least two positive solutions.*

Proof. We apply Corollary 3.3. For simplicity, we give the proof only for $L_\infty < +\infty$. If $L_\infty = +\infty$, the proof is analogous. By (j_5) there exists $c > 0$ such that $\frac{\sum_{k=1}^{N} \max_{s\in[0,c]} F_k(s)}{c^2} < \frac{\lambda_1}{\lambda_N} L_\infty$. In force of (j_6), there exists $d < c$ such that,

$$\frac{F_{\bar{k}}(d)}{d^2} > a_{\bar{k}\bar{k}} \frac{L_\infty}{\lambda_N}.$$

Thus, condition (j_4) of Corollary 3.3 is verified. So, the proof is completed. □

Now, we point out some consequences of the previous results for the tridiagonal system $(T_{\lambda,\underline{f}})$ when the diagonal field $\underline{f}$ is super-linear at $+\infty$ and it is with separable variables, i.e. $f_k : [1, N] \times \mathbb{R} \to \mathbb{R}$ is defined by putting, for all $k \in [1, N]$ and $s \in \mathbb{R}$,

$$f_k(s) := \alpha(k) g(s), \quad \lim_{s\to+\infty} \frac{g(s)}{s} = +\infty, \tag{21}$$

where $\alpha : [1, N] \to \mathbb{R}^+$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. To simplify notations we put

$$\Sigma(\alpha) := \sum_{k=1}^{N} \alpha(k), \quad G(s) = \int_0^s g(t)dt, \quad s \in \mathbb{R}.$$

Corollary 3.5. *Let a, b, c and d be four constants with $a > 0$, $b < 0$, $c > 0$ and $0 < d < c$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function fulfilling (21) with $g(s) \geq 0$ for every $s \in [0, c]$. Assume that (1) holds. In addition, suppose that*

(γ_1) $\dfrac{G(c)}{c^2} < \dfrac{a + 2b\cos(\pi/(N+1))}{aN + 2(N-1)b}\dfrac{G(d)}{d^2}$.

Then, for every $\lambda \in \left]\dfrac{aN + 2(N-1)b}{2\Sigma(\alpha)}\dfrac{d^2}{G(d)}, \dfrac{a + 2b\cos(\pi/(N+1))}{2\Sigma(\alpha)}\dfrac{c^2}{G(c)}\right[$, system $(T_{\lambda,\underline{f}})$ admits at least two positive solutions.

Proof. Since the tridiagonal matrix $T_N(a, b, b)$ has eigenvalues given by

$$\lambda_k = a + 2b\cos\left(\frac{k\pi}{N+1}\right), \quad k = 1, 2, \dots, N, \tag{22}$$

as you can see, for instance in [18, Theorem 2.2], by (1) it turns out to be a positive definite symmetric strongly Z−matrix being $b < 0$. By (21), we have $L_\infty = +\infty$ and our conclusion follows at once by applying Corollary 3.2. □

Corollary 3.6. *Let a, b, c and d be four constants with $a > 0$, $b < 0$, $c > 0$ and $0 < d < c$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function fulfilling (21) with $g(s) \geq 0$ for every $s \in [0, c]$. Assume that (1) holds. In addition, suppose that*

(γ_2) $\dfrac{G(c)}{c^2} < \dfrac{\alpha(\bar{k})(a + 2b\cos(\pi/(N+1)))}{a\Sigma(\alpha)}\dfrac{G(d)}{d^2}$.

Then for every $\lambda \in \left]\dfrac{a}{2\alpha(\bar{k})}\dfrac{d^2}{G(d)}, \dfrac{a + 2b\cos(\pi/(N+1))}{2\Sigma(\alpha)}\dfrac{c^2}{G(c)}\right[$, system $(T_{\lambda,\underline{f}})$ admits at least two positive solutions.

Proof. Arguing as in the proof of Corollary 3.5, our goal is achieved by applying Corollary 3.3. □

An interesting consequence of Corollary 3.4 is the following

Corollary 3.7. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function fulfilling (21) with $g(s) \geq 0$ for every $s \in [0, c]$. Assume that (1) holds. In addition, suppose that*

(γ_3) $\lim\limits_{s \to 0^+} \dfrac{g(s)}{s} = +\infty$.

Then, for every $\lambda \in \left]0, \dfrac{a + 2b\cos(\pi/(N+1))}{2\Sigma(\alpha)} \sup\limits_{c>0} \dfrac{c^2}{G(c)}\right[$, system $(T_{\lambda,\underline{f}})$ admits at least two positive solutions.

Remark 3. We highlight that a careful reading of the proofs of Corollaries 3.5, 3.6 and 3.7 shows that the sign condition on the function g can be removed just replacing $G(c)$ with $\max_{s\in[0,c]} G(s)$. Indeed, it is useful only to guarantee that $\max_{s\in[0,c]} G(s) = G(c)$, however in this way the typical behaviour of the functions that could satisfy the assumptions (γ_1) and (γ_2) should be more clear. Roughly speaking, the function $s \to \frac{G(s)}{s^2}$ has a peak near the point d.

Moreover, we would like to observe that we obtain at least two positive solutions, even though the algebraic system investigated admits the trivial solution, i.e. if $g(0) = 0$. In particular, if $g(0) > 0$, then is evident that (γ_3) is verified and we obtain the same interval of parameter described in [8, Theorem 3.3].

Finally, we give an application of Corollary 3.7 to the difference Dirichlet boundary value problem (2). See, also [7, Theorem 1.1] where at least one positive solution is obtained when $g(0) > 0$.

Example 3.1. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function fulfilling (21) with $g(s) \geq 0$ for every $t \in [0, c]$. Assume that

(γ_3) $\lim\limits_{s \to 0^+} \dfrac{g(s)}{s} = +\infty$.

Then, applying Corollary 3.7 with the tridiagonal matrix $T(2, -1, -1)$, for every $\lambda \in \left]0, \dfrac{1 - \cos(\pi/(N+1))}{N} \sup\limits_{c>0} \dfrac{c^2}{G(c)}\right[$, problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda g(u(k)), & k \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \tag{23}$$

admits at least two positive solutions.

Remark 4. We remark that [14, Theorem 1.1] gives a larger interval of parameters for the existence of two solutions for problem (23) where the energy functional I_λ is constructed exploiting an equivalent norm in $\mathbb{R}^N$ involving the forward difference operator $\Delta u(k) := u(k+1) - u(k)$.

In the one dimensional case, a nice application of Corollary 3.7 is contained in the following

Example 3.2. Let $g : \mathbb{R} \to \mathbb{R}$ be a positive continuous function fulfilling (21). Then, one has that the equation

$$x = \lambda g(x), \quad x \in \mathbb{R},$$

admits at least two positive solutions for every $\lambda \in \left]0, \dfrac{1}{2} \sup\limits_{c>0} \dfrac{c^2}{G(c)}\right[$, provided that condition (γ_3) holds.

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