Chaotic Motion of an Elastic-Plastic Beam[3]

P. S. Symonds[4], G. Borino[5], and U. Perego[6]. The authors looked for evidence of chaotic behavior both under impulsive loading and under periodic excitation, in the pin ended elastic-plastic Shanley model. Their results and conclusions in the impulsive loading ("free vibration") case disagree with ours, in recent studies. The disagreement seems to arise from the different ways of treating damping. Here we demonstrate this briefly. Details are given in a forthcoming paper by Borino et al. (1988).

The authors purport to show in their Figs. 3 and 4 how certain initial conditions lead to final displacements of negative sign—i.e., in the opposite direction to the loading ("anomalous"). In these figures a dot is entered if the prediction is for a negative value, and a blank for a positive outcome. Figure 3 shows a complex pattern. Figure 4 shows a portion of this to expanded scales, and is equally complex. This suggests a fine structure, with possible resemblance to fractal boundaries between attracting basins. They observe that the "slightest change in initial conditions can cause a drastic change in the response, and attempts at obtaining detailed numerical solutions to the problem are meaningless".

Our work on the response due to short pulse loading in the presence of damping (Genna and Symonds (1988); Borino et al. (1988)) leads to the opposite conclusion. For any value of damping ratio ζ, there are regions in the initial condition space such that the final displacement is negative, and outside of which it is positive. For the value $\zeta=0.1$ used by the authors, there are two bands that are loci of pairs (a_0, $\dot{a}_0$) leading to negative final displacements. These are shown in Fig. 1 for the region of the authors' Fig. 3 (we use the authors' notation, a_0 and $\dot{a}_0$ being initial displacement and velocity, respectively). Within each band, e.g., holding $\dot{a}_0$ constant, the final displacement is a piecewise continuous function of the initial displacement a_0. It must be found by a numerical integration scheme, but standard schemes of different types readily furnish essentially identical results. The curves separating the (+) and (−) regions do require more care for their determination, but only that normally demanded in problems involving bifurcations.

To explain why our results and conclusions differ from the authors' is not hard. In our calculations we took the system to be damped from the outset. In contrast, the authors started all their calculations with $\zeta=0$. For each loading they "compute(d) the solution for a certain number of cycles". They then examined the maximum and minimum values of displacement. If they found $a_{max}<0$, they took the final displacement

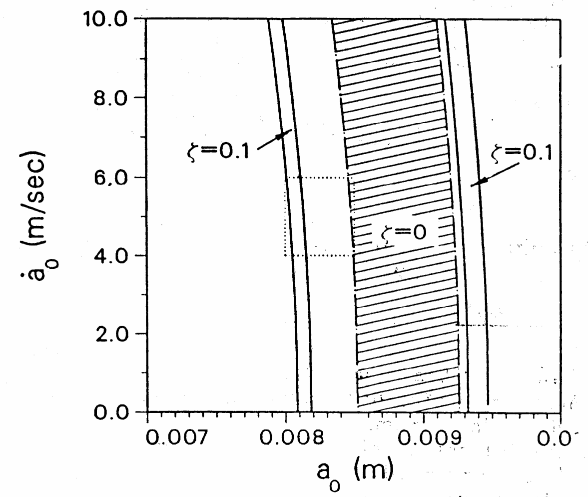

Fig. 1 Regions of initial conditions such that for damping ratio $\zeta=0.1$ the final displacement is negative (shaded) and positive (unshaded). Cross-hatched region defines initial conditions such that the limit cycle of the *undamped* system is negative. Small rectangle enclosed by dotted lines indicates area shown to enlarged scales in authors' Fig. 4. (Our figure would merely increase in size.)

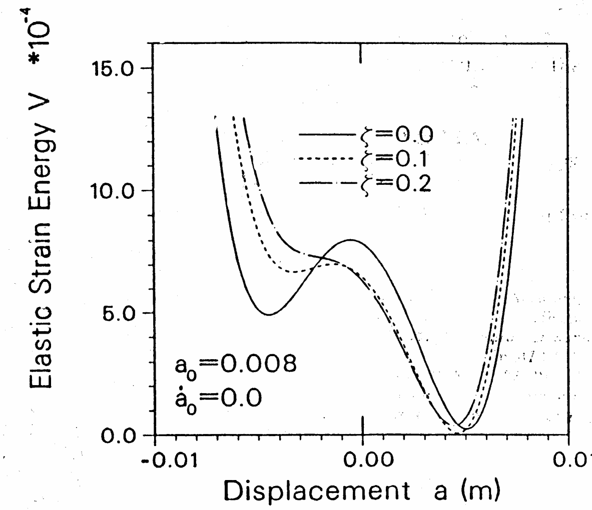

Fig. 2 Curves of elastic strain energy versus displacement for three damping ratios ζ. These curves describe the elastic vibrations; their shapes are determined by the prior plastic flow, and are seen to depend on the magnitude of damping.

[3]By B. Poddar, F. C. Moon, and S. Mukherjee and published in the March, 1988 issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 55, No. 1, pp. 185-189.

[4]Professor of Engineering, Research, Brown University, Division of Engineering, Providence, RI 02912. Fellow ASME.

[5]Visiting Research Associate, Brown University, Division of Engineering, Providence, RI 02912. (On leave from Università di Palermo, Dipartmento di Ingegneria Strutturale e Geotecnica, Viale delle Scienze, 90128 Palermo, Italy.)

[6]Visiting Research Associate, Brown University, Division of Engineering, Providence, RI 02912. (On leave from Politecnico di Milano, Dipartimento di Ingegneria Strutturale, Piazza L. Da Vinci 32, 20133 Milano, Italy.)

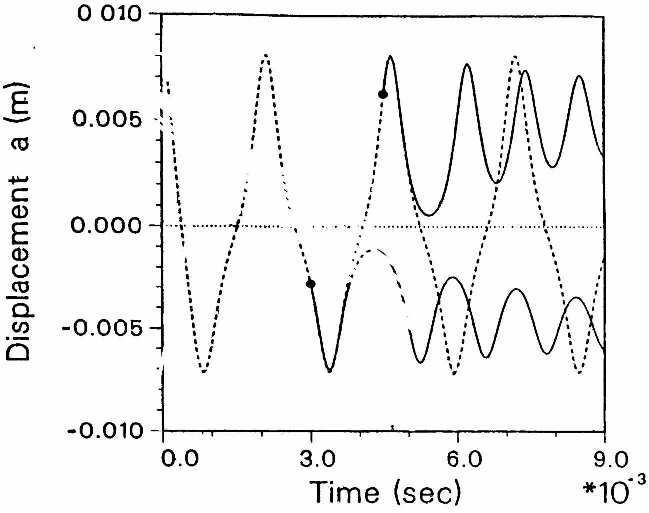

Fig. 3 Dashed curve shows response of undamped system for $a_0 = 0.0083$m, $\dot{a}_0 = 4$m/sec. Solid curves show responses when damping with $\zeta = 0.1$ is inserted at two different times (black circles); the final displacement is positive in one case, negative in the other.

to be negative and entered a dot. If they found $a_{max} > 0$, they then (and only then) set $\zeta = 0.1$ and ran the problem to an equilibrium point a_E. If they found $a_E < 0.1(a_{max} - a_{min})$, they entered a dot. Only by accident does this procedure furnish the sign of the final rest displacement in agreement with that of the structure damped from the outset.

The authors' procedure tacitly involves the assumptions (1) that the plastic deformation are unaffected by damping, and (2) that it is immaterial when damping is inserted. Neither is valid, even approximately, in their calculations. For example, Fig. 2 illustrates the effects of damping on the strain energy function $V(a)$ of the elastic vibration for the initial conditions $a_0 = 0.008$, $\dot{a}_0 = 0$; the shape of this function is determined, of course, by the prior plastic flow. A negative final state requires there to be a stable equilibrium point at a negative a. There is such a state for $\zeta = 0$ and 0.1, but not for $\zeta = 0.2$. However, the authors' method always presumes the curve for $\zeta = 0$. Hence, it can predict anomalous results when none exist. Figure 3 shows that by inserting the damping at different times in the elastic vibration one can obtain either a positive or a negative final state. Exactly how the authors chose the instant to introduce damping is not stated, but their procedure probably accounts for the (spurious) complexity of their Figs. 3 and 4.

In the cases where the authors found $a_{max} < 0$, their procedure furnishes the correct sign of the final state (limit cycle) of the undamped system. Thus, it agrees with early results for this case (Symonds and Yu (1985)). The region in the initial condition space of anomalous outcomes for this case ($\zeta = 0$) is shown in Fig. 1 as the cross-hatched band between dot-dash lines; this agrees with the wide band of dense points in the authors' Fig. 3. However, for the damped system, the loci of points which actually lead to anomalous final displacements are the shaded areas. The authors' procedure predicts no anomalous points correctly in this case, $\zeta = 0.1$. If ζ is quite small (about 0.01 or less—see Genna and Symonds (1988)), the wide band of anomalous outcomes of the damped model approaches that of the undamped system, and the authors' method will predict these points correctly. However, in the cases where $a_{max} > 0$ it will predict the correct sign only by accident, for any ζ.

Contrary to the authors' assertion, the response is in general predictable by standard numerical methods. The exceptional case is that of vanishingly small (nonzero) damping. As ζ is taken smaller, the final state alternates in sign more rapidly, e.g., in a plot against the initial displacement (Genna and Symonds (1988)). Eventually, as ζ is decreased, the widths of the bands become less than the error of the computation (for a given algorithm and device). The outcome is then unpredictable. Chaotic behavior does not seem to be involved in this kind of unpredictability. However, under *periodic* loading the authors have clearly demonstrated that fractal structures and chaos do occur. We conjecture that their existence *depends on the possibility of anomalous response under impulsive loading*. If so, the bounds discussed by Borino et al. (1988) would be relevant, in an appropriate sense, to the question of bounds on chaotic behavior under periodic loading.

References

Borino, G., Perego, U., and Symonds, P. S., 1988, "An Energy Approach to Anomalous Damped Elastic-Plastic Response to Short Pulse Loading," submitted to ASME JOURNAL OF APPLIED MECHANICS, April.

Genna, F., and Symonds, P. S., 1988, "Dynamic Plastic Instabilities in Response to Short Pulse Excitation—Effects of Slenderness Ratio and Damping," *Proc. Royal Society*, Vol. A417, pp. 31-44.

Symonds, P. S., and Yu, T. X., 1985, "Counter-Intuitive Behavior in a Problem of Elastic-Plastic Beam Dynamics," ASME JOURNAL OF APPLIED MECHANICS, Vol. 52, pp. 517-522.