RIESZ-LIKE BASES IN RIGGED HILBERT SPACES

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Abstract. The notions of Bessel sequence, Riesz-Fischer sequence and Riesz basis are generalized to a rigged Hilbert space $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$. A Riesz-like basis, in particular, is obtained by considering a sequence $\{\xi_n\} \subset \mathcal{D}$ which is mapped by a one-to-one continuous operator $T : \mathcal{D}[t] \to \mathcal{H}[\|\cdot\|]$ into an orthonormal basis of the central Hilbert space $\mathcal{H}$ of the triplet. The operator T is, in general, an unbounded operator in $\mathcal{H}$. If T has a bounded inverse then the rigged Hilbert space is shown to be equivalent to a triplet of Hilbert spaces.

1. Introduction

Riesz bases (i.e., sequences of elements $\{\xi_n\}$ of a Hilbert space $\mathcal{H}$ which are transformed into orthonormal bases by some bounded operator with bounded inverse) often appear as eigenvectors of nonself-adjoint operators. The simplest situation is the following one. Let H be a self-adjoint operator with discrete spectrum defined on a subset $D(H)$ of the Hilbert space $\mathcal{H}$. Assume, to be more definite, that each eigenvalue λ_n is simple. Then the corresponding eigenvectors $\{e_n\}$ constitute an orthonormal basis of $\mathcal{H}$. If X is another operator *similar* to H, i.e., there exists a bounded operator T with bounded inverse T^{-1} which intertwines X and H, in the sense that $T : D(H) \to D(X)$ and $XT\xi = TH\xi$, for every $\xi \in D(H)$, then, as it is easily seen, the vectors $\{\varphi_n\}$ with $\varphi_n = Te_n$ are eigenvectors of X and constitute a Riesz basis for $\mathcal{H}$. There are, however, more general situations, mostly coming from physical applications, where the intertwining operator T exists but at least one between T and T^{-1} is unbounded. This is actually the case of the so-called cubic Hamiltonian $X = p^2 + ix^3$ of Pseudo-Hermitian Quantum Mechanics, for which it has been proved that there is no intertwining operator bounded with bounded inverse which makes it similar to a self-adjoint operator [28]. Of course, for studying these cases, one also has to relax the notion of similarity since problems of domain may easily arise (see [6, 7] for a full discussion of the various notions of (quasi-) similarity that one may introduce).

Also, when studying the formal commutation relation $[A, B] = \mathbb{1}$, where B is not the adjoint of A (the so-called *pseudo-bosons* studied by Bagarello [8, 9]), in the most favorable situation, one finds two biorthogonal families of

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vectors $\{\phi_k\}, \{\psi_k\}$, a positive intertwining operator K ($K\varphi_n = \psi_n$, $n \in \mathbb{N}$) and the family $\{e_n\}$ with $e_n = K^{-\frac{1}{2}}\varphi_n$, $n \in \mathbb{N}$, is an orthonormal family of vectors. But, in general both K and K^{-1} are unbounded [10, 11].

These examples motivate, in our opinion, a study of possible generalizations of the notion of Riesz basis that could cover these situations of interest for applications.

Whenever unbounded operators are involved, dealing with discontinuity and with sometimes nontrivial domain problems becomes unavoidable. Both difficulties can be by-passed if one enlarges the set-up from Hilbert spaces to *rigged Hilbert spaces.*

A rigged Hilbert space (RHS) consists of a triplet $(\mathcal{D}, \mathcal{H}, \mathcal{D}^\times)$ where $\mathcal{D}$ is a dense subspace of $\mathcal{H}$ endowed with a topology t, finer than that induced by the Hilbert norm of $\mathcal{H}$, and $\mathcal{D}^\times$ is the conjugate dual of $\mathcal{D}[t]$, endowed with the strong topology $t^\times := \beta(\mathcal{D}^\times, \mathcal{D})$.

Of course, one could also pose the problem of extending the notion of Riesz basis in the more general set-up of locally convex spaces, but the nature itself of the notion of Riesz basis requires also a control of its behavior in the context of duality and, as we shall see, a *Riesz-like* basis on a locally convex space $\mathcal{D}[t]$ will automatically make of $\mathcal{D}$ the *smallest* space of rigged Hilbert space. Thus it appears natural to consider rigged Hilbert spaces from the very beginning.

On the other hand, rigged Hilbert spaces (and their further generalizations like e.g. partial inner product spaces) have plenty of applications. In Analysis they provide the general framework for distribution theory; in Quantum Physics they give a convenient description of the Dirac formalism [5, Chapter 7]. Finally, rigged Hilbert spaces (e.g. those generated by the Feichtinger algebra) or lattices of Hilbert or Banach spaces (mixed-norm spaces, amalgam spaces, modulation spaces) play also an important role in signal analysis (see [5, Chapter 8], for an overview).

As it is known, a Riesz basis $\{\xi_n\}$ in a Hilbert space $\mathcal{H}$ is also a *frame* [15, 17, 19]; i.e., there exist positive numbers c, C such that

$$c\|\xi\|^2 \leq \sum_{n=1}^{\infty} |\langle \xi | \xi_n \rangle|^2 \leq C\|\xi\|^2, \quad \forall \xi \in \mathcal{H}. \tag{1}$$

The peculiarity of a Riesz basis relies in its exactness or minimality: a frame is a Riesz basis if it ceases to be a frame when anyone of its elements is dropped out. The notion of frame is crucial in signal analysis and for coherent states (see e.g. [15] and references therein) and in approximation theory [1, 18, 30]. A further generalization is the notion of *semi-frame* [2] for which one of the above frame bounds is absent (lower or upper semi-frames). For instance a lower semi-frame has an *unbounded* frame operator, with *bounded* inverse.

The paper is organized as follows. In Section 2, after some preliminaries, we discuss shortly the notion of basis in a rigged Hilbert space

$\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$. Then we introduce *Bessel-like* sequences in $\mathcal{D}^\times[t^\times]$ and *Riesz-Fischer-like* sequences in $\mathcal{D}[t]$ and study, in the present context, their interplay in terms of duality. In Section 3, we define $\{\xi_n\}$ to be a *Riesz-like basis* if there exists a one-to-one linear map $T : \mathcal{D} \to \mathcal{H}$, continuous from $\mathcal{D}[t]$ into $\mathcal{H}[\|\cdot\|]$, such that $\{T\xi_n\}$ is an orthonormal basis for the central Hilbert space $\mathcal{H}$. Some characterizations of these bases are given. Finally, we consider the special case where T has also a continuous inverse. This additional assumption, even though natural, reveals to be quite strong, since, as we will see, the rigged Hilbert space $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ is in fact equivalent to a triplet of Hilbert spaces. An application to nonself-adjoint Hamiltonians is briefly discussed in Section 3.2.

2. Preliminaries and basic aspects

2.1. **Rigged Hilbert spaces and operators on them.** Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$. A locally convex topology t on $\mathcal{D}$ finer than the topology induced by the Hilbert norm defines, in standard fashion, a *rigged Hilbert space* (RHS)

$$\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[t^\times], \tag{2}$$

where $\mathcal{D}^\times$ is the vector space of all continuous conjugate linear functionals on $\mathcal{D}[t]$, i.e., the conjugate dual of $\mathcal{D}[t]$, endowed with the *strong dual topology* $t^\times = \beta(\mathcal{D}^\times, \mathcal{D})$ and $\hookrightarrow$ denotes a continuous embedding. Since the Hilbert space $\mathcal{H}$ can be identified with a subspace of $\mathcal{D}^\times[t^\times]$, we will systematically read (2) as a chain of topological inclusions: $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$. These identifications imply that the sesquilinear form $B(\cdot,\cdot)$ that puts $\mathcal{D}$ and $\mathcal{D}^\times$ in duality is an extension of the inner product of $\mathcal{D}$; i.e., $B(\xi,\eta) = \langle \xi | \eta \rangle$, for every $\xi, \eta \in \mathcal{D}$ (to simplify notations we adopt the symbol $\langle \cdot | \cdot \rangle$ for both of them).

Example 2.1. *Let T be a closed densely defined operator with domain $D(T)$ in Hilbert space $\mathcal{H}$. Let us endow $D(T)$ with the graph norm $\|\cdot\|_T$ defined by*

$$\|\xi\|_T = (\|\xi\|^2 + \|T\xi\|^2)^{\frac{1}{2}} = \|(I + T^*T)^{\frac{1}{2}}\xi\|, \quad \xi \in D(T).$$

With this norm $D(T)$ becomes a Hilbert space, denoted by $\mathcal{H}_T$. If $\mathcal{H}_T^\times$ denotes the Hilbert space conjugate dual of $\mathcal{H}_T$, then we get the triplet of Hilbert spaces

$$\mathcal{H}_T \subset \mathcal{H} \subset \mathcal{H}_T^\times$$

which is a particular example of rigged Hilbert space.

Example 2.2. *Let $\mathcal{D}$ be a dense domain in Hilbert space $\mathcal{H}$ and denote by $\mathcal{L}^\dagger(\mathcal{D})$ the *-algebra consisting of all closable operators A with $D(A) = \mathcal{D}$, which together with their adjoints, A^*, leave $\mathcal{D}$ invariant. The involution of $\mathcal{L}^\dagger(\mathcal{D})$ is defined by $A \mapsto A^\dagger$, where $A^\dagger = A^*\!\restriction_{\mathcal{D}}$. The *-algebra $\mathcal{L}^\dagger(\mathcal{D})$ defines in $\mathcal{D}$ the* graph topology $t_\dagger$ *by the family of seminorms*

$$\xi \in \mathcal{D} \to \|\xi\|_A := \|(I + A^*\overline{A})^{\frac{1}{2}}\xi\|, \quad A \in \mathcal{L}^\dagger(\mathcal{D}).$$

Since the topology $t_\dagger$ is finer than the topology induced on $\mathcal{D}$ by the Hilbert norm of $\mathcal{H}$, it defines in natural way a structure of rigged Hilbert space.

Let now $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a rigged Hilbert space, and let $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ denote the vector space of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}^\times[t^\times]$. If $\mathcal{D}[t]$ is barreled (e.g. reflexive), an involution $X \mapsto X^\dagger$ can be introduced in $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ by the equality

$$\left\langle X^\dagger \eta \,|\xi \right\rangle = \overline{\langle X\xi \,|\eta\rangle}, \quad \forall \xi, \eta \in \mathcal{D}.$$

Hence, in this case, $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ is a †-invariant vector space.

If $\mathcal{D}[t]$ is a smooth space (e.g. Fréchet and reflexive), then $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ is a quasi *-algebra over $\mathcal{L}^\dagger(\mathcal{D})$ [3, Definition 2.1.9].

Let $\mathcal{E}, \mathcal{F} \in \{\mathcal{D}, \mathcal{H}, \mathcal{D}^\times\}$ and $\mathfrak{L}(\mathcal{E}, \mathcal{F})$ the space of all continuous linear maps from $\mathcal{E}[t_\mathcal{E}]$ into $\mathcal{F}[t_\mathcal{F}]$. We put

$$\mathcal{C}(\mathcal{E}, \mathcal{F}) := \{X \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times) : \exists Y \in \mathfrak{L}(\mathcal{E}, \mathcal{F}),\, Y\xi = X\xi, \forall \xi \in \mathcal{D}\}.$$

In particular, if $X \in C(\mathcal{D}, \mathcal{H})$ then its adjoint $X^\dagger \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ has an extension from $\mathcal{H}$ into $\mathcal{D}^\times$, which we denote by the same symbol.

The space $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ has been studied at length by several authors (see e.g. [22, 23, 24, 29]) and several pathologies concerning their multiplicative structure have been considered (see also [3, 5] and references therein). Recently some spectral properties of operators of these classes have also been studied [13].

2.2. **Topological bases and Schauder bases.** Let $\mathcal{E}[t_\mathcal{E}]$ be a locally convex space and $\{\xi_n\}$ a sequence of vectors of $\mathcal{E}$. We adopt the following terminology:

(i) the sequence $\{\xi_n\}$ is *complete* or *total* if the linear span of $\{\xi_n\}$ is dense in $\mathcal{E}[t_\mathcal{E}]$;

(ii) the sequence $\{\xi_n\}$ is a *topological basis* for $\mathcal{E}$ if, for every $\phi \in \mathcal{E}$, there exists a *unique* sequence $\{c_n\}$ of complex numbers such that

$$\phi = \sum_{n=1}^{\infty} c_n \xi_n, \tag{3}$$

where the series on the right hand side converges in $\mathcal{E}[t_\mathcal{E}]$.

Every coefficient $c_n = c_n(\phi)$ in (3) can be regarded as a linear functional on $\mathcal{E}$ and, following [20], we say that

(iii) a topological basis $\{\xi_n\}$ of $\mathcal{E}[t_\mathcal{E}]$ is a *Schauder basis* if the coefficient functionals $\{c_n\}$ are $t_\mathcal{E}$-continuous.

Remark 2.3. *We notice the following well-known facts.*

(a) *If $\mathcal{E}$ has a total sequence, then it is a separable space.*

(b) *Every topological basis is a complete sequence; the converse is false, in general.*

(c) *If* $\{\xi_n\}$ *is a topological basis for* $\mathcal{E}$*, then* $\{\xi_n\}$ *is* ω-independent*; i.e., if* $\sum_{n=1}^{\infty} c_n\xi_n = 0$*, then* $c_n = 0$*, for every* $n \in \mathbb{N}$*. This in turn implies that the sequence* $\{\xi_n\}$ *consists of linearly independent vectors.*

(d) *If* $\mathcal{E}[t_\mathcal{E}]$ *is a Fréchet space, then every topological basis is a Schauder basis (*[20, Section 14.2, Theorem 5]*).*

By a slight modification of [20, Section 14.3, Theorem 6] we have

Proposition 2.4. *A complete sequence of vectors* $\{\xi_n\} \subset \mathcal{E}$ *is a Schauder basis of* $\mathcal{E}[t_\mathcal{E}]$ *if, and only if, for every* $n \in \mathbb{N}$ *and every continuous seminorm* p *on* $\mathcal{E}[t_\mathcal{E}]$*, there exists a continuous seminorm* q *on* $\mathcal{E}[t_\mathcal{E}]$ *such that*

$$p\left(\sum_{i=1}^{n} c_i\xi_i\right) \leq q\left(\sum_{i=1}^{n+m} c_i\xi_i\right)$$

where $c_1, \ldots, c_{n+m}$ *are arbitrary complex numbers and* m *is an arbitrary natural number.*

As is known, a Riesz basis $\{\xi_n\}$ in Hilbert space $\mathcal{H}$ is transformed by some bounded operator into an orthonormal basis of $\mathcal{H}$; this is equivalent to saying that a new (and equivalent) inner product can be introduced in $\mathcal{H}$ which makes of $\{\xi_n\}$ an orthonormal basis. A similar notion for locally convex spaces, calls immediately on the stage rigged Hilbert spaces.

Proposition 2.5. *Let* $\{\xi_n\} \subset \mathcal{E}$ *be a Schauder basis of* $\mathcal{E}[t_\mathcal{E}]$ *and assume that there exists a one-to-one continuous linear map* T *from* $\mathcal{E}[t_\mathcal{E}]$ *into some Hilbert space* $\mathcal{K}[\|\cdot\|]$ *such that* $\{T\xi_n\}$ *is an orthonormal basis of* $\mathcal{K}$*. Then there exists an inner product* $\langle \cdot | \cdot \rangle_+$ *on* $\mathcal{E} \times \mathcal{E}$ *such that the topology induced on* $\mathcal{E}$ *by the norm* $\|\cdot\|_+$ *is coarser than* $t_\mathcal{E}$ *and* $\{\xi_n\}$ *is an orthonormal basis.*

Proof. Define $\langle \xi | \eta \rangle_+ := \langle T\xi | T\eta \rangle$, $\xi, \eta \in \mathcal{E}$. Then all the statements follow immediately. □

Then, under the conditions of Proposition 2.5, one can consider $\mathcal{E}$ as a subspace of the Hilbert space completion $\mathcal{H}_+$ of $\mathcal{E}[\|\cdot\|_+]$, so that a rigged Hilbert space can be built in natural way: $\mathcal{E}[t_\mathcal{E}] \subset \mathcal{H}_+ \subset \mathcal{E}^\times[t_\mathcal{E}^\times]$. This is essentially the reason why, as announced in the Introduction, we will confine ourselves within this framework.

Let $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a rigged Hilbert space and $\{\xi_n\}$ a Schauder basis for $\mathcal{D}[t]$. Then, every $f \in \mathcal{D}$ can be written as $\sum_{n=1}^{\infty} c_n(f)\xi_n$, for uniquely determined suitable coefficients $c_n(f)$. Since every c_n is a continuous linear functional on $\mathcal{D}[t]$, there exists a sequence $\{\zeta_n\} \subset \mathcal{D}^\times$ such that

$$c_n(f) = \overline{\langle \zeta_n | f \rangle}, \quad \forall n \in \mathbb{N}, f \in \mathcal{D}.$$

For every $n \in \mathbb{N}$, the vector ζ_n is uniquely determined. If we take $f = \xi_k$, then it is clear that $c_n(\xi_k) = \delta_{n,k}$. Hence $\langle \zeta_n | \xi_k \rangle = \delta_{n,k}$; i.e., the sequences $\{\xi_n\}$ and $\{\zeta_n\}$ are *biorthogonal*. More precisely,

Proposition 2.6. *Let* $\{\xi_n\}$ *be a topological basis for* $\mathcal{D}[t]$*. The following statements are equivalent.*

(i) $\{\xi_n\}$ *is a Schauder basis.*
(ii) $\{\xi_n\}$ *is minimal; i.e.,* $\xi_k \notin \overline{\text{span}\{\xi_m; m \neq k\}}^t$, *for every* $k \in \mathbb{N}$.
(iii) *There exists a unique sequence* $\{\zeta_n\} \subset \mathcal{D}^\times$ *such that* $\{\xi_n\}$ *and* $\{\zeta_n\}$ *are biorthogonal.*

(i) $\Leftrightarrow$ (ii) is proved in [20, Section 14.2, Proposition 3], and (ii) $\Rightarrow$ (iii) in [20, Section 14.2, Proposition 1]; (iii) $\Rightarrow$ (ii) is trivial. See also [17, Theorem 6.1.1].

Proposition 2.7. *Let* $\{\xi_n\}$ *be a Schauder basis for* $\mathcal{D}[t]$. *Then there exists a sequence* $\{\zeta_n\}$ *of vectors of* $\mathcal{D}^\times$ *such that*
(i) *the sequences* $\{\xi_n\}$ *and* $\{\zeta_n\}$ *are biorthogonal;*
(ii) *for every* $f \in \mathcal{D}$,

$$f = \sum_{n=1}^{\infty} \overline{\langle \zeta_n | f \rangle} \xi_n; \tag{4}$$

(iii) *The partial sum operator* S_n, *given by*

$$S_n f = \sum_{k=1}^{n} \overline{\langle \zeta_k | f \rangle} \xi_k, \quad f \in \mathcal{D},$$

is continuous from $\mathcal{D}[t]$ *into* $\mathcal{D}[t]$ *and has an adjoint* $S_n^\dagger$ *everywhere defined in* $\mathcal{D}^\times$ *given by*

$$S_n^\dagger \Psi = \sum_{k=1}^{n} \langle \Psi | \xi_k \rangle \zeta_k, \quad \Psi \in \mathcal{D}^\times.$$

The proof is straightforward.

Proposition 2.8. *Let* $\{\xi_n\}$ *be a Schauder basis for* $\mathcal{D}[t]$. *Then, the following statements hold.*
(i) *The sequence* $\{\zeta_n\}$ *in* (4) *is complete in* $\mathcal{D}^\times[\tau]$, *where* τ *is a topology of the conjugate dual pair* $(\mathcal{D}^\times, \mathcal{D})$. *If* $\mathcal{D}[t]$ *is reflexive,* $\{\zeta_n\}$ *is complete also with respect to* $t^\times$.
(ii) *The sequence* $\{\zeta_n\}$ *is a basis for* $\mathcal{D}^\times$ *with respect to the weak topology; i.e., if* $\Psi \in \mathcal{D}^\times$ *one has*

$$\langle \Psi | f \rangle = \left\langle \sum_{k=1}^{\infty} \langle \Psi | \xi_k \rangle \zeta_k \,|\, f \right\rangle = \sum_{k=1}^{\infty} \langle \Psi | \xi_k \rangle \langle \zeta_k | f \rangle, \quad \forall f \in \mathcal{D}. \tag{5}$$

Proof. (i): Assume that $\{\zeta_n\}$ is not complete. Then there exists $f \neq 0$, $f \in \mathcal{D}$ (regarded as the conjugate dual of $\mathcal{D}^\times[\tau]$) such that $\langle \zeta_n | f \rangle = 0$, for every $n \in \mathbb{N}$. From (4) it follows that $f = 0$, a contradiction. If $\mathcal{D}[t]$ is reflexive, the statement follows from the equality of t and the Mackey topology $\tau(\mathcal{D}^\times, \mathcal{D})$.

(ii): Assume first that $\Phi \in \mathcal{D}^\times$ is of the form $\Phi = \sum_{k=1}^n c_k \zeta_k$. Then it is easily seen that $S_n^\dagger \Phi = \Phi$. Now, if $\Psi \in \mathcal{D}^\times$, for every $f \in \mathcal{D}$ and for every $\epsilon > 0$, there exists $\Phi = \sum_{k=1}^{n} c_k\zeta_k$ such that $|\langle \Psi - \Phi | f \rangle| < \epsilon$. On the other hand, since $S_n f \to f$, there exists $n_\epsilon \in \mathbb{N}$ such that for $n > n_\epsilon$, $|\langle \Psi - \Phi | S_n f - f \rangle| < \epsilon$. Thus we have

$$\begin{aligned}\left|\left\langle S_n^\dagger \Psi - \Psi | f \right\rangle\right| &\leq \left|\left\langle S_n^\dagger \Psi - S_n^\dagger \Phi | f \right\rangle\right| + \left|\left\langle S_n^\dagger \Phi - \Phi | f \right\rangle\right| + |\langle \Phi - \Psi | f \rangle| \\ &= |\langle \Psi - \Phi | S_n f \rangle| + |\langle \Phi - \Psi | f \rangle| \\ &\leq |\langle \Psi - \Phi | S_n f - f \rangle| + 2|\langle \Phi - \Psi | f \rangle| \\ &< 3\epsilon.\end{aligned}$$

Hence $S_n^\dagger \Psi \to \Psi$ weakly, or

$$\langle \Psi | f \rangle = \left\langle \sum_{k=1}^{\infty} \langle \Psi | \xi_k \rangle \zeta_k | f \right\rangle = \sum_{k=1}^{\infty} \langle \Psi | \xi_k \rangle \langle \zeta_k | f \rangle .$$

□

For $f \in \mathcal{D} \subset \mathcal{D}^\times$, (5) gives in particular

$$\|f\|^2 = \sum_{k=1}^{\infty} \langle f | \xi_k \rangle \langle \zeta_k | f \rangle , \quad \forall f \in \mathcal{D};$$

so that the series on the right hand side is convergent, for every $f \in \mathcal{D}$.

Remark 2.9. *There is a wide interest and a rich literature on bases or frames in locally convex spaces (in particular, Banach spaces) and on their existence, see e.g.* [16, 25] *and* [14] *and references therein.*

2.3. **Bessel- and Riesz-Fischer-like sequences.** We assume, from now on, that $\mathcal{D}[t]$ is complete and reflexive.

Definition 2.10. *Let $\{\zeta_n\}$ be a sequence in $\mathcal{D}^\times$. We say that $\{\zeta_n\}$ is a* Bessel-like sequence *if, for every bounded subset $\mathcal{M}$ of $\mathcal{D}[t]$,*

$$\sup_{\eta \in \mathcal{M}} \sum_{k=1}^{\infty} |\langle \zeta_k | \eta \rangle|^2 =: \gamma_{\mathcal{M}} < \infty. \tag{6}$$

Proposition 2.11. *A sequence $\{\zeta_n\}$ of elements of $\mathcal{D}^\times$ is Bessel-like if and only if*

$$\sum_{k=1}^{\infty} |\langle \zeta_k | \eta \rangle|^2 < \infty, \quad \forall \eta \in \mathcal{D}$$

and the analysis operator

$$F : \eta \in \mathcal{D}[t] \to \{\overline{\langle \zeta_k | \eta \rangle}\} \in \ell^2[\| \cdot \|_2]$$

is continuous.

Proof. Let $\{\zeta_n\}$ be Bessel-like. From (6) it is clear that for every $\eta \in \mathcal{D}$, $\sum_{k=1}^{\infty} |\langle \zeta_k | \eta \rangle|^2 < \infty$.

Now we prove that

$$U : \{a_n\} \in \ell^2 \to \sum_{n=1}^{\infty} a_n \zeta_n$$

is a well-defined continuous linear map from $\ell^2[\|\cdot\|_2]$ into $\mathcal{D}^\times[t^\times]$.

We begin with proving that $\sum_{n=1}^{\infty} a_n\zeta_n$ converges in $\mathcal{D}^\times[t^\times]$. Let $\mathcal{M}$ be a bounded subset of $\mathcal{D}[t]$. Then, for $n > m$,

$$\begin{aligned}\sup_{\eta\in\mathcal{M}} \left| \left\langle \sum_{k=1}^{n} a_k\zeta_k - \sum_{k=1}^{m} a_k\zeta_k \,|\eta \right\rangle \right| &= \sup_{\eta\in\mathcal{M}} \left| \left\langle \sum_{k=m+1}^{n} a_k\zeta_k \,|\eta \right\rangle \right| \\ &\leq \sup_{\eta\in\mathcal{M}} \sum_{k=m+1}^{n} |a_k \langle \zeta_k \,|\eta\rangle| \\ &\leq \left(\sum_{k=m+1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \cdot \sup_{\eta\in\mathcal{M}} \left(\sum_{k=1}^{\infty} |\langle \zeta_k \,|\eta\rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \gamma_{\mathcal{M}}^{\frac{1}{2}} \left(\sum_{k=m+1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \to 0, \quad \text{as } n, m \to \infty.\end{aligned}$$

Hence the partial sums $\sum_{k=1}^{n} a_k\zeta_k$ constitute a Cauchy sequence in $\mathcal{D}^\times[t^\times]$ and, since $\mathcal{D}^\times[t^\times]$ being reflexive is[1] quasi-complete [27, Chapter IV, 5.5, Corollary 1], the series converges in $\mathcal{D}^\times$. Moreover, by simple modifications of the previous inequalities it follows also that

$$\sup_{\eta\in\mathcal{M}} \left| \sum_{k=1}^{\infty} a_k \langle \zeta_k \,|\eta\rangle \right| \leq \gamma_{\mathcal{M}}^{\frac{1}{2}} \|\{a_n\}\|_2.$$

Thus, U is continuous from $\ell^2[\|\cdot\|_2]$ into $\mathcal{D}^\times[t^\times]$ and therefore by the reflexivity of $\mathcal{D}[t]$, U has a continuous adjoint map $U^\dagger : \mathcal{D}[t] \to \ell^2[\|\cdot\|_2]$. It is easily checked that

$$U^\dagger\eta = \{\overline{\langle \zeta_n \,|\eta\rangle}\}, \quad \forall \eta \in \mathcal{D}.$$

Thus $U^\dagger = F$ and F is continuous.

Conversely, let us assume that $\{\langle \zeta_k \,|\eta\rangle\} \in \ell^2$ and that F is continuous. This implies that there exists a continuous seminorm p on $\mathcal{D}[t]$ such that

$$\|F\eta\|_2 = \left(\sum_{k=1}^{\infty} |\langle \zeta_k \,|\eta\rangle|^2 \right)^{\frac{1}{2}} \leq p(\eta), \quad \forall \eta \in \mathcal{D}.$$

Thus, if $\mathcal{M}$ is a bounded subset of $\mathcal{D}[t]$, we get

$$\sup_{\eta\in\mathcal{M}} \sum_{k=1}^{\infty} |\langle \zeta_k \,|\eta\rangle|^2 = \sup_{\eta\in\mathcal{M}} \|F\eta\|_2^2 \leq \sup_{\eta\in\mathcal{M}} p(\eta)^2 < \infty, \quad \forall \eta \in \mathcal{D}.$$

[1] A locally convex space is said to be quasi-complete if every closed bounded subset is complete.

Hence $\{\zeta_n\}$ is a Bessel-like sequence. □

As usual, we will call the operator $F^\dagger\colon \{a_n\} \in \ell^2 \to \sum_{n=1}^\infty a_n\zeta_n \in \mathcal{D}^\times$, the *syntesis operator* of the sequence $\{\zeta_n\}$.

From Proposition 2.11 and from the fact that (6) is not affected from a possible reordering of the elements $\{\zeta_n\}$ it follows that if $\{\zeta_n\}$ is a Bessel-like sequence and $\{a_n\} \in \ell^2$ then the series $\sum_{n=1}^\infty a_n\zeta_n$ converges *unconditionally* in $\mathcal{D}^\times[t^\times]$.

If $\{\zeta_n\}$ is a Bessel-like sequence, then the operator $F^\dagger F$ (we keep for it the name of *frame operator*, as usual) is a continuous linear map from $\mathcal{D}[t]$ into $\mathcal{D}^\times[t^\times]$; i.e., $F^\dagger F \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$. Clearly,

$$F^\dagger F\eta = \sum_{k=1}^\infty \langle \zeta_k \,|\eta\rangle\, \zeta_k$$

where the series on the right hand side converges in $\mathcal{D}^\times[t^\times]$. The operator $F^\dagger F$ is positive, in the sense that $\left\langle F^\dagger F\eta \,|\eta\right\rangle \geq 0$, for every $\eta \in \mathcal{D}$.

Remark 2.12. *A sequence $\{\xi_n\}$ of elements of $\mathcal{D}$ can also be considered as a sequence in $\mathcal{D}^\times$. Hence the notion of Bessel-like sequence can be given also in this case, and analysis and synthesis operators act in the very same way as before. Moreover, if both series $\sum_{k=1}^\infty a_k\xi_k$ and $\sum_{k=1}^\infty a_{\sigma(k)}\xi_{\sigma(k)}$ converge in $\mathcal{D}[t]$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection, then they have the same sum, since $\sum_{k=1}^\infty a_k\xi_k$ converges unconditionally in $\mathcal{D}^\times[t^\times]$.*

Proposition 2.13. *A sequence $\{\zeta_n\}$ of elements of $\mathcal{D}^\times$ is Bessel-like if and only if, for every orthonormal basis $\{e_n\}$ in $\mathcal{H}$, there exists $W \in C(\mathcal{H}, \mathcal{D}^\times)$ such that $We_n = \zeta_n$, for every $n \in \mathbb{N}$.*

Proof. Let $\{\zeta_n\}$ be Bessel-like and $\{e_n\}$ an orthonormal basis for $\mathcal{H}$. For $f \in \mathcal{H}$, $f = \sum_{k=1}^\infty \langle f\,|e_k\rangle\, e_k$, we define $Wf = \sum_{k=1}^\infty \langle f\,|e_k\rangle\, \zeta_k$. This series converges in $\mathcal{D}^\times[t^\times]$ as seen in Proposition 2.11 and it is clear that $We_n = \zeta_n$, for every $n \in \mathbb{N}$. We now prove that $W \in C(\mathcal{H}, \mathcal{D}^\times)$. Let us consider a bounded subset $\mathcal{M}$ of $\mathcal{D}[t]$; then,

$$\begin{aligned}
\sup_{\eta\in\mathcal{M}} |\langle Wf\,|\eta\rangle| &= \sup_{\eta\in\mathcal{M}} \left|\left\langle \sum_{k=1}^\infty \langle f\,|e_k\rangle\, \zeta_k \,|\eta \right\rangle\right| \\
&= \sup_{\eta\in\mathcal{M}} \left|\sum_{k=1}^\infty \langle f\,|e_k\rangle \langle \zeta_k\,|\eta\rangle\right| \\
&\leq \left(\sum_{k=1}^\infty |\langle f\,|e_k\rangle|^2\right)^{\frac{1}{2}} \sup_{\eta\in\mathcal{M}} \left(\sum_{k=1}^\infty |\langle \zeta_k\,|\eta\rangle|^2\right)^{\frac{1}{2}} \\
&\leq \gamma_{\mathcal{M}}^{\frac{1}{2}} \|f\|.
\end{aligned}$$

Conversely, assume that, given an orthonormal basis $\{e_n\}$, there exists $W \in C(\mathcal{H}, \mathcal{D}^\times)$ such that $We_n = \zeta_n$. Then, if $\mathcal{M}$ is a bounded subset of $\mathcal{D}[t]$,

$$\sup_{\eta\in\mathcal{M}} \sum_{k=1}^{\infty} |\langle \zeta_k|\eta\rangle|^2 = \sup_{\eta\in\mathcal{M}} \sum_{k=1}^{\infty} |\langle We_k|\eta\rangle|^2 = \sup_{\eta\in\mathcal{M}} \sum_{k=1}^{\infty} \left|\left\langle e_k \middle| W^\dagger \eta \right\rangle\right|^2 = \sup_{\eta\in\mathcal{M}} \|W^\dagger \eta\|^2 < \infty.$$

Hence $\{\zeta_n\}$ is a Bessel-like sequence. □

As in the case of Hilbert spaces, Bessel-like sequences have a dual counterpart.

Definition 2.14. *Let $\{\xi_n\}$ be a sequence in $\mathcal{D}$. We say that $\{\xi_n\}$ is a* Riesz-Fischer-like *sequence, if for every orthonormal basis $\{e_n\}$ of $\mathcal{H}$, there exists $S \in C(\mathcal{D}, \mathcal{H})$ such that $S\xi_n = e_n$, for every $n \in \mathbb{N}$.*

For an arbitrary sequence $\{\xi_n\}$ in $\mathcal{D}$, we define, a second *analysis operator V* as follows:

$$\begin{cases} D(V) = \left\{ \Phi \in \mathcal{D}^\times : \displaystyle\sum_{k=1}^{\infty} |\langle \Phi | \xi_k \rangle|^2 < \infty \right\} \\ V\Phi = \{\langle \Phi | \xi_k \rangle\}, \ \Phi \in D(V) \end{cases} \tag{7}$$

Proposition 2.15. *If $\{\xi_n\}$ is a Riesz-Fischer-like sequence, then $V : D(V) \to \ell^2$ is surjective.*

Proof. Let $\{a_n\} \in \ell^2$ and $\{e_n\}$ an orthonormal basis of $\mathcal{H}$. Put $f = \sum_{k=1}^{\infty} a_k e_k \in \mathcal{H}$. Then,

$$a_n = \langle f | e_n \rangle = \langle f | S\xi_n \rangle = \left\langle S^\dagger f | \xi_n \right\rangle, \quad \forall n \in \mathbb{N}.$$

Then $\Phi = S^\dagger f \in D(V)$ and $V\Phi = \{a_n\}$. □

Let $\{\omega_n\}$ denote the canonical basis in ℓ^2; i.e., $\omega_n = \{\delta_{kn}\}$, for every $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$, there exists $\zeta_n \in \mathcal{D}^\times$ (in general, nonunique) such that $\delta_{kn} = \langle \zeta_n | \xi_k \rangle$, $n, k \in \mathbb{N}$.

The *duality* between Riesz-Fischer-like sequences and Bessel-like ones is then stated by the following

Proposition 2.16. *$\{\xi_n\}$ is a Riesz-Fischer-like sequence in $\mathcal{D}$ if and only if there exists a Bessel-like sequence $\{\zeta_n\}$ in $\mathcal{D}^\times$ such that $\{\xi_n\}$ and $\{\zeta_n\}$ are biorthogonal.*

Proof. Suppose that $\{\xi_n\}$ has a Bessel-like biorthogonal sequence. Then, for every orthonormal basis $\{e_n\}$ in $\mathcal{H}$, there exists $T \in C(\mathcal{H}, \mathcal{D}^\times)$ such that $Te_n = \zeta_n$, for every $n \in \mathbb{N}$. Then,

$$\delta_{kn} = \langle \zeta_n | \xi_k \rangle = \langle Te_n | \xi_k \rangle = \left\langle e_n \middle| T^\dagger \xi_k \right\rangle, \quad n, k \in \mathbb{N}.$$

This easily implies that $T^\dagger \xi_k = e_k$, for every $k \in \mathbb{N}$.

Conversely, suppose that $\{\xi_n\}$ is a Riesz-Fischer-like sequence. Then, for every orthonormal basis $\{e_n\}$ in $\mathcal{H}$, there exists $S \in C(\mathcal{D},\mathcal{H})$ such that $S\xi_n = e_n$, for every $n \in \mathbb{N}$. Hence,

$$\delta_{kn} = \langle S\xi_n | e_k \rangle = \left\langle \xi_n \middle| S^\dagger e_k \right\rangle.$$

Let us define $\zeta_k = S^\dagger e_k$, $k \in \mathbb{N}$. Then $\{\zeta_k\}$ is Bessel-like and $\{\xi_n\}$ and $\{\zeta_n\}$ are biorthogonal. □

For a sequence $\{\zeta_n\} \subseteq \mathcal{D}^\times$, we only get a partial result.

Proposition 2.17. *Let $\{\zeta_n\}$ be a sequence in $\mathcal{D}^\times$. If $\{\zeta_n\}$ possesses a biorthogonal sequence $\{\xi_n\}$ which is total and Riesz-Fischer-like, then $\{\zeta_n\}$ is a Bessel-like sequence.*

Proof. Since $\{\xi_n\}$ is Riesz-Fischer-like, for every orthonormal basis $\{e_n\}$ in $\mathcal{H}$, there exists $S \in C(\mathcal{D},\mathcal{H})$ such that $S\xi_n = e_n$, for every $n \in \mathbb{N}$. Hence,

$$\delta_{kn} = \langle S\xi_n | e_k \rangle = \left\langle \xi_n \middle| \mathcal{S}^\dagger e_k \right\rangle.$$

This implies that $\left\langle \xi_n \middle| \mathcal{S}^\dagger e_k - \zeta_k \right\rangle = 0$, for all $k, n \in \mathbb{N}$. Since $\{\xi_k\}$ is total, we conclude that, for every $k \in \mathbb{N}$, $\zeta_k = S^\dagger e_k$. Clearly, $S^\dagger \in C(\mathcal{H},\mathcal{D}^\times)$; hence, the statement follows from Proposition 2.13. □

Let us now assume that $\{\xi_n\}$ is a Schauder basis for $\mathcal{D}[t]$ and that the dual basis $\{\zeta_n\}$ is a Bessel-like sequence. Then, $\{\xi_n\}$ is a Riesz-Fischer-like sequence and by (5), we have, for every $\Phi \in \mathcal{D}^\times$ and for every bounded subset $\mathcal{M}$ of $\mathcal{D}[t]$,

$$\sup_{f\in\mathcal{M}} |\langle \Phi | f \rangle| \leq \left(\sum_{k=1}^{\infty} |\langle \Phi | \xi_k \rangle|^2 \right)^{\frac{1}{2}} \sup_{f\in\mathcal{M}} \left(\sum_{k=1}^{\infty} |\langle \zeta_k | f \rangle|^2 \right)^{\frac{1}{2}} \leq \gamma_{\mathcal{M}}^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |\langle \Phi | \xi_k \rangle|^2 \right)^{\frac{1}{2}},$$

which, by putting $\Phi = f$ gives a lower estimate of $\left(\sum_{k=1}^{\infty} |\langle f | \xi_k \rangle|^2\right)^{\frac{1}{2}}$, similar to that one gets in the usual formulation in Hilbert spaces.

3. Riesz-like bases

3.1. Basic properties.

Definition 3.1. *A Schauder basis $\{\xi_n\}$ for $\mathcal{D}[t]$ is called a* Riesz-like basis *if there exists an operator $T \in C(\mathcal{D},\mathcal{H})$ such that $\{T\xi_n\}$ is an orthonormal basis for $\mathcal{H}$.*

It is clear that T is automatically one-to-one. It is easy to see that every Riesz-like basis is a Riesz-Fisher-like sequence.

Since T maps $\mathcal{D}[t]$ into $\mathcal{H}[\|\cdot\|]$ continuously, $T^\dagger$ has a continuous extension (which we denote by the same symbol) from $\mathcal{H}[\|\cdot\|]$ into $\mathcal{D}^\times[t^\times]$. The range $R(T)$ of T is dense in $\mathcal{H}$ since it contains the orthonormal basis $\{e_k\}$ with $e_k := T\xi_k$, $k \in \mathbb{N}$. In particular, it may happen that $R(T) = \mathcal{H}$. Hence, the operator T^{-1} is everywhere defined and it is continuous if, and only if

$R(T^\dagger) = \mathcal{D}^\times$. We will name $\{\xi_n\}$ a *strict* Riesz-like basis, in this case. As we shall see in Theorem 3.9, this imposes severe constraints on the topology t of $\mathcal{D}$.

If $\{\xi_n\}$ is a Riesz-like basis, we can find explicitly the sequence $\{\zeta_n\} \subset \mathcal{D}^\times$ of Proposition 2.7. The continuity of T and (4), in fact, imply

$$Tf = \sum_{n=1}^{\infty} \overline{\langle \zeta_n | f \rangle} T\xi_n = \sum_{n=1}^{\infty} \overline{\langle \zeta_n | f \rangle} e_n, \quad \forall f \in \mathcal{D}.$$

This, in turn, implies that $\overline{\langle \zeta_n | f \rangle} = \langle Tf | e_n \rangle$, for every $f \in \mathcal{D}$. Hence $\zeta_n = T^\dagger e_n$, for every $n \in \mathbb{N}$.

Clearly, for every $n, k \in \mathbb{N}$,

$$\langle \zeta_k | \xi_n \rangle = \left\langle T^\dagger e_k | \xi_n \right\rangle = \langle e_k | T\xi_n \rangle = \langle e_k | e_n \rangle = \delta_{k,n},$$

and $T^\dagger T \xi_n = \zeta_n$, for every $n \in \mathbb{N}$.

Moreover, $\{\zeta_n\}$ is a Bessel-like sequence (Proposition 2.16). Indeed, one has, for every bounded subset $\mathcal{M}$ of $\mathcal{D}[t]$,

$$\sup_{f \in \mathcal{M}} \sum_{n=1}^{\infty} |\langle \zeta_n | f \rangle|^2 = \sup_{f \in \mathcal{M}} \sum_{n=1}^{\infty} \left| \left\langle T^\dagger e_n | f \right\rangle \right|^2 = \sup_{f \in \mathcal{M}} \sum_{n=1}^{\infty} |\langle e_n | Tf \rangle|^2 = \sup_{f \in \mathcal{M}} \|Tf\|^2 < \infty.$$

An easy computation shows that

$$T^\dagger g = \sum_{k=1}^{\infty} d_k \zeta_k \quad \text{if} \quad g = \sum_{n=1}^{\infty} d_n e_n \in \mathcal{H}.$$

Finally, we have

$$T^\dagger(\mathcal{H}) = \left\{ \Psi \in \mathcal{D}^\times : \sum_{k=1}^{\infty} | \langle \Psi | \xi_k \rangle |^2 < \infty \right\},$$

that is, $T^\dagger(\mathcal{H}) = D(V)$, where V is the operator defined in (7).

Indeed, if $\Psi \in T^\dagger(\mathcal{H})$, then $\Psi = T^\dagger h$, for some $h \in \mathcal{H}$. Let $h = \sum_{k=1}^\infty c_k e_k$. Then, using the continuity of $T^\dagger$,

$$T^\dagger h = T^\dagger \left(\sum_{k=1}^{\infty} c_k e_k \right) = \sum_{k=1}^{\infty} c_k T^\dagger e_k = \sum_{k=1}^{\infty} c_k \zeta_k.$$

This implies that $c_k = \langle \Psi | \xi_k \rangle$ and $\sum_{k=1}^\infty | \langle \Psi | \xi_k \rangle |^2 < \infty$.

Conversely, let $\sum_{k=1}^\infty \langle \Psi | \xi_k \rangle \zeta_k \in \mathcal{D}^\times$ with $\sum_{k=1}^\infty | \langle \Psi | \xi_k \rangle |^2 < \infty$. Define $h = \sum_{k=1}^\infty \langle \Psi | \xi_k \rangle e_k \in \mathcal{H}$. Then

$$T^\dagger h = \sum_{k=1}^{\infty} \langle \Psi | \xi_k \rangle T^\dagger e_k = \sum_{k=1}^{\infty} \langle \Psi | \xi_k \rangle \zeta_k.$$

The operator T can also be regarded as an Hilbertian operator (by assumption it maps $\mathcal{D}$ into $\mathcal{H}$). This operator is closable in $\mathcal{H}$ if, and only if, the subspace

$$D(T^*) = \{ g \in \mathcal{H} : T^\dagger g \in \mathcal{H} \}$$

is dense in $\mathcal{H}$. In this case, the operator T^*, the adjoint of T, is defined by $T^*g = T^\dagger g$, $g \in D(T^*)$.

Remark 3.2. *If $\{\xi_n\}$ is a Riesz-like basis for $\mathcal{D}[t]$ and $\{c_n\} \in \ell^2$ with $\sum_{k=1}^\infty c_k\xi_k = 0$, then $c_n = 0$, for every $n \in \mathbb{N}$.*

Theorem 3.3. *Let $\mathcal{D}[t]$ be complete and reflexive and $\mathcal{D}^\times$ quasi-complete. Let $\{\xi_n\}$ be a topological basis of $\mathcal{D}$. The following statements are equivalent.*

(i) *$\{\xi_n\}$ is a Riesz-like basis.*
(ii) *There exists a unique sequence $\{\zeta_n\} \subset \mathcal{D}^\times$ such that*
  (ii.a) *$\{\xi_n\}$ and $\{\zeta_n\}$ are biorthogonal;*
  (ii.b) *for every $f \in \mathcal{D}$, $\sum_{n=1}^\infty |\langle \zeta_n | f\rangle|^2 < \infty$;*
  (ii.c) *the seminorm p_ζ defined by*

$$p_\zeta(f) = \left(\sum_{k=1}^\infty |\langle \zeta_n | f\rangle|^2\right)^{\frac{1}{2}}$$

is continuous on $\mathcal{D}[t]$.
(iii) *There exists $S \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$, $S \geq 0$, such that $\{\xi_n\}$ and $\{S\xi_n\}$ are biorthogonal.*

Proof. (i) $\Rightarrow$ (ii): Let $\{\xi_n\}$ be a Riesz-like basis for $\mathcal{D}$. Then there exists $T \in C(\mathcal{D}, \mathcal{H})$ such that $\{T\xi_n\}$ is an orthonormal basis for $\mathcal{H}$. Put $e_n = T\xi_n$ and $\zeta_n = T^\dagger e_n$. Then,

$$\langle \zeta_n | \xi_k\rangle = \left\langle T^\dagger e_n | \xi_k \right\rangle = \langle e_n | T\xi_k\rangle = \langle e_n | e_k\rangle = \delta_{n,k}, \quad n, k \in \mathbb{N}.$$

It is easily seen that, if $f = \sum_{n=1}^\infty a_n\xi_n$, then $a_n = \overline{\langle \zeta_n | f\rangle}$ and

$$Tf = \sum_{n=1}^\infty \overline{\langle \zeta_n | f\rangle} e_n\,.$$

Hence $\sum_{n=1}^\infty |\langle \zeta_n | f\rangle|^2 < \infty$. Moreover, since $T \in C(\mathcal{D}, \mathcal{H})$, there exists a continuous seminorm p on $\mathcal{D}[t]$ such that $\|Tf\| \leq p(f)$, for every $f \in \mathcal{D}$. Hence,

$$p_\zeta(f) := \left(\sum_{k=1}^\infty |\langle \zeta_n | f\rangle|^2\right)^{\frac{1}{2}} = \|Tf\| \leq p(f), \quad \forall f \in \mathcal{D}.$$

This implies that p_ζ, which is a seminorm on $\mathcal{D}$, is continuous.

(ii) $\Rightarrow$ (iii): First, let us define $S\xi_k = \zeta_k$ and extend S by linearity to $\mathcal{D}_0 := \operatorname{span}\{\xi_m; m \in \mathbb{N}\}$. Thus $S : \mathcal{D}_0 \to \mathcal{D}^\times$. If $f = \sum_{k=1}^n \overline{\langle \zeta_k | f\rangle}\xi_k \in \mathcal{D}_0$ and $g = \sum_{h=1}^\infty \overline{\langle \zeta_h | f\rangle}\xi_h \in \mathcal{D}$, we get

$$|\langle Sf | g\rangle| = \left|\sum_{k=1}^\infty \overline{\langle \zeta_k | f\rangle}\langle \zeta_k | g\rangle\right| \leq \left(\sum_{n=1}^\infty |\langle \zeta_n | f\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^\infty |\langle \zeta_n | g\rangle|^2\right)^{\frac{1}{2}} = p_\zeta(f)p_\zeta(g).$$

Hence, if $\mathcal{M}$ is a bounded subset of $\mathcal{D}[t]$, we obtain

$$\sup_{g\in\mathcal{M}} |\langle Sf\,|g\rangle| \le p_\zeta(f) \sup_{g\in\mathcal{M}} p_\zeta(g).$$

This proves that S is continuous from $\mathcal{D}_0[t]$ into $\mathcal{D}^\times[t^\times]$. Thus S has an extension (denoted by the same symbol) to a continuous linear map from the quasi-completion of $\mathcal{D}_0[t]$, which is $\mathcal{D}$, to the quasi-completion of $\mathcal{D}^\times$, which coincides with $\mathcal{D}^\times$. Hence, $S \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$. It is easily seen that $\langle Sf\,|f\rangle \ge 0$, for every $f \in \mathcal{D}$.

(iii) $\Rightarrow$ (i): Since $\{\xi_n\}$ is a topological basis, every $f \in \mathcal{D}$ can be represented, in unique way, as $f = \sum_{k=1}^\infty a_k\xi_k$. Let now $S \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ be such that $\{\xi_n\}$ and $\{S\xi_n\}$ are biorthogonal. Then the following equality holds

$$\langle Sf\,|f\rangle = \sum_{k=1}^\infty |a_k|^2 \;\text{ if } f = \sum_{k=1}^\infty a_k\xi_k \in \mathcal{D}. \tag{8}$$

This implies that $S \ge 0$ and $\{a_k\} \in \ell^2$. Then, if $\{e_k\}$ is any orthonormal basis in $\mathcal{H}$ the series $\sum_{k=1}^\infty a_k e_k$ converges in $\mathcal{H}$. Let us fix one of these bases $\{e_n\}$. We define

$$T_n : f = \sum_{k=1}^\infty a_k\xi_k \in \mathcal{D} \to T_n f = \sum_{k=1}^n a_k e_k \in \mathcal{H}$$

and

$$T : f = \sum_{k=1}^\infty a_k\xi_k \in \mathcal{D} \to Tf = \sum_{k=1}^\infty a_k e_k \in \mathcal{H}.$$

Using (8) it is easily seen that $T_n \in C(\mathcal{D}, \mathcal{H})$. Clearly, $T_n f \to Tf$ in $\mathcal{H}$. Since $\mathcal{D}[t]$ is reflexive, it is barreled and then, by the Banach-Steinhaus theorem (see e.g. [20, Theorem 11.1.3]), it follows that $T \in C(\mathcal{D}, \mathcal{H})$. Moreover if $f = \sum_{k=1}^\infty a_k\xi_k \in \mathcal{D}$, then $\|Tf\|^2 = \sum_{k=1}^\infty |a_k|^2$ whence it follows immediately that T is injective. By the definition itself, $T\xi_k = e_k$. Therefore $\{\xi_n\}$ is a Riesz-like basis. □

Example 3.4. *Suppose that $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$ whose elements belong to $\mathcal{D}$. If $\{e_n\}$ is also a basis for $\mathcal{D}[t]$, then it is automatically a Schauder basis and since the identity is continuous from $\mathcal{D}[t]$ into $\mathcal{H}[\|\cdot\|]$, it is clear that $\{e_n\}$ is a Riesz-like basis for $\mathcal{D}[t]$. The dual sequence in $\mathcal{D}^\times$ is clearly $\{e_n\}$ itself. This is a familiar situation. Let us consider, in fact, the triplet $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing C^∞-functions on the real line and $\mathcal{S}^\times(\mathbb{R})$ the space of (conjugate) tempered distributions. Then, it is well known that the set $\{\phi_n\}$ of Hermite functions is not only an orthonormal basis for $L^2(\mathbb{R})$, but also a basis for $\mathcal{S}(\mathbb{R})$ in its own topology (see* [26, Theorem V.13]*).*

Example 3.5. *Let $\mathcal{H}$ be a separable Hilbert space and $\{e_n\}$ an orthonormal basis of $\mathcal{H}$. Let N denote the* number *operator defined on the basis vectors* *by $Ne_k = ke_k$, $k \in \mathbb{N}$. Then as it is well-known N is self-adjoint on its natural domain*

$$D(N) = \left\{ f \in \mathcal{H} : \sum_{k=1}^{\infty} k^2 | \langle f | e_k \rangle |^2 < \infty \right\}.$$

Let $\mathcal{D} := \mathcal{D}^\infty(N) = \bigcap_{k=1}^\infty D(N^k)$ be endowed with the topology t_N defined by the seminorms $p_k(\cdot) = \|N^k \cdot\|$, $k = 0, 1, 2, \ldots$. Then $\mathcal{D}$ is a Fréchet and reflexive space. Define $\xi_k = \frac{1}{k} e_k$, $k \in \mathbb{N}$. Clearly, $N\xi_k = e_k$, for every $k \in \mathbb{N}$ and N maps continuously $\mathcal{D}[t_N]$ into $\mathcal{H}$. Moreover, $\{\xi_k\}$ is a basis for $\mathcal{D}[t_N]$. Indeed, for every $p \in \mathbb{N}$ we have

$$\left\| N^p \left(f - \sum_{k=1}^{n} k \langle f | e_k \rangle \xi_k \right) \right\| = \left\| N^p f - \sum_{k=1}^{n} k \langle f | e_k \rangle N^p \xi_k \right\| = \left\| N^p f - \sum_{k=1}^{n} k^p \langle f | e_k \rangle e_k \right\|$$

and the latter tends obviously to zero as $n \to \infty$. Since $\{\xi_k\}$ is a Schauder basis, it is a Riesz-like basis for $\mathcal{D}[t_N]$.

Example 3.6. *Let $\{\xi_n\}$ be a Schauder basis for $\mathcal{D}[t]$. Assume that there exists a continuous seminorm p on $\mathcal{D}[t]$ such that*

$$\left(\sum_{k=1}^{\infty} |c_k|^2 \right)^{\frac{1}{2}} \leq p \left(\sum_{k=1}^{\infty} c_k \xi_k \right),$$

whenever $\sum_{k=1}^\infty c_k \xi_k$ converges in $\mathcal{D}[t]$.

Let $\{e_k\}$ be any orthonormal basis in $\mathcal{H}$. Then the operator

$$T : f = \sum_{k=1}^{\infty} c_k \xi_k \to Tf = \sum_{k=1}^{\infty} c_k e_k$$

is one-to-one and continuous from $\mathcal{D}[t]$ into $\mathcal{H}[\| \cdot \|]$. Clearly $T\xi_k = e_k$, for every $k \in \mathbb{N}$. Hence, $\{\xi_n\}$ is a Riesz-like basis for $\mathcal{D}[t]$.

Example 3.7. *Let $\{\xi_n\}$ be a Schauder basis for $\mathcal{D}[t]$ and $\{\zeta_n\}$ the corresponding sequence in $\mathcal{D}^\times$ such that $\langle \xi_n | \zeta_m \rangle = \delta_{nm}$. Define $S\xi_n = \zeta_n$, $n \in \mathbb{N}$, and assume that S extends to a positive operator (denoted by the same symbol) of $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$. If $S = T^\dagger T$, with $T \in C(\mathcal{D}, \mathcal{H})$ and $T^\dagger \in C(\mathcal{H}, \mathcal{D}^\times)$, then, as it is easily seen, the sequence $\{e_n\}$ with $e_n = T\xi_n$ is orthonormal and, if T is surjective, it is an orthonormal basis for $\mathcal{H}$. Thus $\{\xi_n\}$ is a Riesz-like basis for $\mathcal{D}[t]$.*

Remark 3.8. *It is worth considering the case where, so to say, the rigged Hilbert space collapses into one Hilbert space only, as it happens if the topology t of $\mathcal{D}$ is equivalent to the Hilbert norm. Then $\{\xi_n\}$ is Riesz-like if there exists an invertible bounded operator T mapping $\{\xi_n\}$ into an orthonormal basis of $\mathcal{H}$. However, the inverse T^{-1} need not be bounded. Nevertheless the discussion made so far shows that the essential features of (usual) Riesz bases in Hilbert space are preserved also in this more general set-up.*

In the usual definition of Riesz basis in Hilbert space $\mathcal{H}$ one requires that $\{\xi_n\}$ is mapped into an orthonormal basis of $\mathcal{H}$ by a bounded operator with bounded inverse. In Definition 3.1, we only required the continuity of the operator T; i.e $T \in C(\mathcal{D}, \mathcal{H})$. In fact, there is no room for the continuity of T^{-1} from $\mathcal{H}$ into $\mathcal{D}[t]$, unless $\mathcal{D}[t]$ (and, then also $\mathcal{D}^\times[t^\times]$) is *equivalent* (in topological sense) to a Hilbert space. We maintain the basic assumption that $\mathcal{D}[t]$ is complete and reflexive.

Theorem 3.9. *Let $\{\xi_n\}$ be a sequence of elements of $\mathcal{D}$. The following statements are equivalent.*

(i) *$\{\xi_n\}$ is a Riesz-like basis and the one-to-one operator $T \in C(\mathcal{D}, \mathcal{H})$ for which $\{T\xi_n\}$ is an orthonormal basis of $\mathcal{H}$, has a continuous inverse; i.e., $T^{-1} \in C(\mathcal{H}, \mathcal{D})$.*

(ii) *The space $\mathcal{D}$ can be endowed with an inner product $\langle \cdot | \cdot \rangle_{+1}$ such that the topology induced by the corresponding norm $\|\cdot\|_{+1}$ is equivalent to t, $\mathcal{D}[\|\cdot\|_{+1}]$ is a Hilbert space and the sequence $\{\xi_n\}$ is an orthonormal basis for $\mathcal{D}[\|\cdot\|_{+1}]$.*

(iii) *The sequence $\{\xi_n\}$ is complete in $\mathcal{D}[t]$ and there exists a continuous seminorm p such that for every $n \in \mathbb{N}$ and complex numbers $\{c_1, \dots, c_n\}$*
$$\sum_{i=1}^n |c_i|^2 \le p\left(\sum_{i=1}^n c_i\xi_i\right)^2$$
and for every continuous seminorm q there exists $C_q > 0$ such that
$$q\left(\sum_{i=1}^n c_i\xi_i\right)^2 \le C_q \sum_{i=1}^n |c_i|^2,$$
for every $n \in \mathbb{N}$ and complex numbers $\{c_1, \dots, c_n\}$.

Proof. (i) $\Rightarrow$ (ii): Let T be the continuous operator with continuous inverse such that $\{T\xi_n\}$ is an orthonormal basis of $\mathcal{H}$ and define
$$\langle \xi | \eta \rangle_{+1} := \langle T\xi | T\eta \rangle, \quad \forall \xi, \eta \in \mathcal{D}.$$
Then, $\langle \cdot | \cdot \rangle_{+1}$ is an inner product on $\mathcal{D}$. Let $\|\cdot\|_{+1}$ be the corresponding norm. Clearly, $\|\xi\|_{+1} = \|T\xi\|$, for every $\xi \in \mathcal{D}$. Since T is continuous from $\mathcal{D}[t]$ to $\mathcal{H}$, there exists a continuous seminorm p such that
$$\|\xi\|_{+1} = \|T\xi\| \le p(\xi), \quad \forall \xi \in \mathcal{D}. \tag{9}$$
On the other hand, T^{-1} is continuous from $\mathcal{H}$ onto $\mathcal{D}[t]$, then for every seminorm q on $\mathcal{D}$ there exists $\gamma_q > 0$ such that
$$q(T^{-1}\zeta) \le \gamma_q \|\zeta\|, \quad \forall \zeta \in \mathcal{H}.$$
If $\xi \in \mathcal{D}$, then $\xi = T^{-1}\zeta$ for some $\zeta \in \mathcal{H}$, hence
$$q(\xi) \le q(T^{-1}\zeta) \le \gamma_q \|T\xi\| = \gamma_q \|\xi\|_{+1}. \tag{10}$$

The equivalence of the topology defined by $\|\cdot\|_{+1}$ and t implies that $\mathcal{D}[\|\cdot\|_{+1}]$ is a Hilbert space.

Finally, the sequence $\{\xi_n\}$ is a basis consisting of orthonormal vectors in $\mathcal{D}[\|\cdot\|_{+1}]$; indeed,

$$\langle \xi_i | \xi_j \rangle_{+1} = \langle T\xi_i | T\xi_j \rangle = \langle e_i | e_j \rangle = \delta_{ij}, \quad i, j \in \mathbb{N}.$$

(ii) $\Rightarrow$ (iii): Since $\|\cdot\|_{+1}$ defines a topology equivalent to t, then there exists a continuous seminorm p on $\mathcal{D}[t]$ such that (9) holds and for every continuous seminorm q there exists $\gamma_q > 0$ such that (10) holds.

Now, consider any fixed $n \in \mathbb{N}$ and complex numbers $\{c_1, \dots, c_n\}$ and consider the orthonormal basis $\{\xi_n\}$ for $\mathcal{D}[\|\cdot\|_{+1}]$. If $\xi = \sum_{i=1}^n c_i\xi_i \in \mathcal{D}$, then $\|\xi\|_{+1}^2 = \sum_{i=1}^n |c_i|^2$and the statement follows by applying (9) and (10) to ξ. Of course the linear span of $\{\xi_n\}$ is dense in $\mathcal{D}[\|\cdot\|_{+1}]$, since $\{\xi_n\}$ is an orthonormal basis for $\mathcal{D}[\|\cdot\|_{+1}]$; hence the sequence $\{\xi_n\}$ is complete in $\mathcal{D}[\|\cdot\|_{+1}]$ and then, by the equivalence of t and of the topology generated by $\|\cdot\|_{+1}$, $\{\xi_n\}$ is complete in $\mathcal{D}[t]$.

(iii) $\Rightarrow$ (i): Let $\{e_n\}$ be any orthonormal basis for $\mathcal{H}$ and define two linear operators $T : \mathcal{D} \to \mathcal{H}$ and $S : \mathcal{H} \to \mathcal{D}$ as follows: for any fixed $n \in \mathbb{N}$ $T\left(\sum_{i=1}^n c_i\xi_i\right) := \sum_{i=1}^n c_ie_i$ and $S\left(\sum_{i=1}^n c_ie_i\right) := \sum_{i=1}^n c_i\xi_i$ with $c_i \in \mathbb{C}$; T and S are continuous; moreover, $T\xi_n = e_n$ and $Se_n = \xi_n$, for every $n \in \mathbb{N}$. Certainly $TS = I$ and, since $\{\xi_n\}$ is complete in $\mathcal{D}[\|\cdot\|_{+1}]$, $ST = I_{\restriction\mathcal{D}}$. Hence, T is a continuous invertible linear operator with continuous inverse and $\{\xi_n\}$ is a strict Riesz-like basis for $\mathcal{D}[t]$. □

The condition given in (iii) is clearly the natural substitute for the inequalities in (1) in this setting.

Let us call, for short, *strict Riesz-like basis* a basis for which (i) of Theorem 3.9 holds.

Proposition 3.10. *If the rigged Hilbert space $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$, with $\mathcal{D}[t]$ complete and reflexive, has a strict Riesz-like basis $\{\xi_n\}$ then it is (equivalent to) a triplet of Hilbert spaces $\mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_{-1}$. Moreover, $\{\xi_n\}$ is an orthonormal basis for $\mathcal{H}_{+1}$ and the dual sequence $\{\zeta_n\}$ is an orthonormal basis for $\mathcal{H}_{-1}$.*

In fact, from the previous discussion, it follows also that $\mathcal{H}_{+1} = \mathcal{D}$ with norm $\|\xi\|_{+1} = \|T\xi\|$, $\xi \in \mathcal{D}$, where $T \in C(\mathcal{D}, \mathcal{H})$ is an operator such that $\{T\xi_n\}$ is an orthonormal basis for $\mathcal{H}$. This operator T, regarded as an operator in $\mathcal{H}$ is, in general, an unbounded operator with domain $D(T) = \mathcal{D}$ and bounded inverse.

Strict Riesz-like bases have an interest in their own since Riesz bases in triplets of Hilbert spaces are useful for some applications [21]. A more detailed analysis will be given in [12].

Example 3.11. *Let A be a closed operator in a separable Hilbert space $\mathcal{H}$, with domain $D(A)$. Then $D(A)$ can be made into a Banach space, denoted by $\mathcal{B}_A$, if a new norm is defined by $\|\varphi\|_A := \|\varphi\| + \|A\varphi\|$.*

Let $\mathcal{B}_A^\times$ be the conjugate dual of $\mathcal{B}_A$ w.r.t. $\|\cdot\|_A$. The operator $(I+A^*A)^{\frac{1}{2}}$ is continuous from $\mathcal{B}_A$ into $\mathcal{H}$ and its continuous extension to $\mathcal{H}$, denoted by the same symbol, is continuous from $\mathcal{H}$ into $\mathcal{B}_A^\times$ and has continuous inverse. If $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$, then the sequence $\{\xi_n\}$ defined by $\xi_n := (I+A^*A)^{-\frac{1}{2}}e_n$ is a strict Riesz-like basis for $\mathcal{B}_A$.

A concrete example can be constructed as follows. Consider the triplet of Sobolev spaces $W^{1,2}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset W^{-1,2}(\mathbb{R})$. As it is well-known, $W^{1,2}(\mathbb{R})$ is a Banach space under the norm $\|f\|_{1,2} = \|f\|_2 + \|\mathsf{D}f\|_2$, D denoting the weak derivative.

Let $\{\xi_n\}$ be the family of functions of $W^{1,2}(\mathbb{R})$ defined by

$$\xi_n(x) = \frac{(-i)^n}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\phi_n(y)e^{ixy}}{(1+y^2)^{\frac{1}{2}}}dy,$$

where $\phi_n(x) = H_n(x)e^{-\frac{x^2}{2}}$ denotes the n-th Hermite function. The family $\{\xi_n\}$ is a strict Riesz-like basis of $W^{1,2}(\mathbb{R})[\|\cdot\|_{1,2}]$. In fact it is not difficult to show by standard techniques of Fourier transform that $(I-\mathsf{D}^2)^{\frac{1}{2}}\xi_n = \phi_n$. Moreover the operator $(I-\mathsf{D}^2)^{\frac{1}{2}}$ is continuous from $W^{1,2}(\mathbb{R})$ into $L^2(\mathbb{R})$ and has continuous inverse. The result of (ii) of Theorem 3.9 is not surprising at all. Indeed, as it is well know, the space $W^{1,2}(\mathbb{R})$ can be made into a Hilbert space with inner product

$$\langle \varphi | \psi \rangle'_{1,2} = \left\langle (I-\mathsf{D}^2)^{\frac{1}{2}}\varphi \,\middle|\, (I-\mathsf{D}^2)^{\frac{1}{2}}\psi \right\rangle, \quad \varphi, \psi \in W^{1,2}(\mathbb{R})$$

which endows $W^{1,2}(\mathbb{R})$ with a topology equivalent to that defined by $\|f\|_{1,2}$.

3.2. **An application.** As mentioned in the Introduction, an important problem of Pseudo-Hermitian Quantum Mechanics is the following: given a nonself-adjoint Hamiltonian H, with real spectrum, one tries to find a well-behaved (bounded and with bounded inverse) intertwining operator T which transforms H is a self-adjoint operator H_{sa}. When this happens one can get of course a large amount of information on H making use of the spectral theory of self-adjoint operators. The situation becomes more involved in cases (like the cubic oscillator) where a so regular operator does not exist and one has to deal with *unbounded* intertwining operators. Even the notion of similarity must be relaxed, with a certain loss in the preservation of spectra (see e.g. [4, 6, 7]). In this section, we will show how the approach in rigged Hilbert space can be helpful in these cases.

Let H be a closed operator in Hilbert space. As already mentioned in Example 2.1, its domain $D(\mathsf{H})$ can be made into a Hilbert space $\mathcal{H}_{\mathsf{H}}$ with the graph norm $\|\cdot\|_{\mathsf{H}}$. Let $\mathcal{H}_{\mathsf{H}}^\times$ be its conjugate dual and consider the triplet of Hilbert spaces $\mathcal{H}_{\mathsf{H}} \subset \mathcal{H} \subset \mathcal{H}_{\mathsf{H}}^\times$. Assume that H_{sa} is a self-adjoint operator in $\mathcal{H}$ with discrete spectrum and, for simplicity, that every eigenvalue $\lambda_k \in \mathbb{R}$ has multiplicity 1. Let ψ_k be an eigenvector corresponding to λ_k. Then $\{\psi_k\}$ is an orthonormal basis for $\mathcal{H}$. Assume that there exists $T \in C(\mathcal{H}_{\mathsf{H}}, \mathcal{H})$, invertible and with continuous inverse $T^{-1}:\mathcal{H}\to\mathcal{H}_{\mathsf{H}}$ such that

$$\left\langle \mathsf{H}\xi \middle| T^\dagger\eta \right\rangle = \left\langle T\xi \middle| \mathsf{H}_{sa}\eta \right\rangle, \quad \forall \xi\in\mathcal{H}_{\mathsf{H}},\, \eta\in D(\mathsf{H}_{sa}) \text{ s.t. } T^\dagger\eta\in\mathcal{H}. \tag{11}$$

Let us define $\xi_k = T^{-1}\psi_k$, $k\in\mathbb{N}$. Then, the set $\{\xi_k\}$ is complete and it is a Schauder basis of $\mathcal{H}_{\mathsf{H}}[\|\cdot\|_{\mathsf{H}}]$(Remark 2.3(d)). Hence it is a strict Riesz-like basis. From (11), for every $\eta\in D(\mathsf{H}_{sa})$, we get $\left\langle \mathsf{H}\xi_n \middle| T^\dagger\eta\right\rangle = \left\langle T\xi_n \middle| \mathsf{H}_{sa}\eta\right\rangle = \left\langle \psi_n \middle| \mathsf{H}_{sa}\eta\right\rangle = \left\langle \mathsf{H}_{sa}\psi_n \middle| \eta\right\rangle = \lambda_n \left\langle \psi_n \middle| \eta\right\rangle = \lambda_n\left\langle T\xi_n \middle| \eta\right\rangle = \lambda_n\left\langle \xi_n \middle| T^\dagger\eta\right\rangle$. Thus, if $T^\dagger D(\mathsf{H}_{sa})\cap\mathcal{H}$ is dense in $\mathcal{H}$, we get $\mathsf{H}\xi_n = \lambda_n\xi_n$, for every $n\in\mathbb{N}$.

Conversely, assume that a sequence $\{\xi_n\}$ is a strict Riesz-like basis for $\mathcal{H}_{\mathsf{H}}$ and that $\mathsf{H}\xi_n=\lambda_n\xi_n$, $\lambda_n\in\mathbb{R}$, for every $n\in\mathbb{N}$. Since there exists an operator $T\in C(\mathcal{H}_{\mathsf{H}},\mathcal{H})$, invertible and with continuous inverse $T^{-1}:\mathcal{H}\to\mathcal{H}_{\mathsf{H}}$, such that the vectors $\psi_n = T\xi_n$ constitute an orthonormal basis for $\mathcal{H}$, we can construct a self-adjoint operator H_{sa}, in standard way; i.e.,

$$D(\mathsf{H}_{sa}) = \left\{\xi\in\mathcal{H} : \sum_{k=1}^\infty \lambda_k^2 |\left\langle \xi \middle| \psi_k\right\rangle|^2 < \infty\right\}$$

$$\mathsf{H}_{sa}\xi = \sum_{k=1}^\infty \lambda_k \left\langle \xi \middle| \psi_k\right\rangle \psi_k, \quad \xi\in D(\mathsf{H}_{sa}).$$

If $\xi\in\mathcal{H}_{\mathsf{H}}$, then $\xi = \sum_{k=1}^\infty \left\langle \xi \middle| \zeta_k\right\rangle \xi_k$ w.r.t. $\|\cdot\|_{\mathsf{H}}$. This, in particular, implies that $\mathsf{H}\xi = \sum_{k=1}^\infty \lambda_k\left\langle \xi \middle| \zeta_k\right\rangle \xi_k$, in the norm of $\mathcal{H}$. Then, taking into account that $\xi_k\in\mathcal{H}_{\mathsf{H}}$, for every $k\in\mathbb{N}$ and that $T^\dagger\in C(\mathcal{H},\mathcal{H}_{\mathsf{H}}^\times)$, we have, for every $\eta\in D(\mathsf{H}_{sa})$ s.t. $T^\dagger\eta\in\mathcal{H}$,

$$\begin{aligned}
\left\langle \mathsf{H}\xi \middle| T^\dagger\eta\right\rangle &= \left\langle \lim_{N\to\infty}\sum_{k=1}^N \lambda_k\left\langle \xi \middle| \zeta_k\right\rangle \xi_k \middle| T^\dagger \sum_{r=1}^\infty \left\langle \eta \middle| \psi_r\right\rangle \psi_r\right\rangle \\
&= \lim_{N\to\infty}\left\langle \sum_{k=1}^N \lambda_k\left\langle \xi \middle| \zeta_k\right\rangle \xi_k \middle| T^\dagger \sum_{r=1}^\infty \left\langle \eta \middle| \psi_r\right\rangle \psi_r\right\rangle \\
&= \lim_{N\to\infty}\left\langle \sum_{k=1}^N \lambda_k\left\langle \xi \middle| \zeta_k\right\rangle T\xi_k \middle| \sum_{r=1}^\infty \left\langle \eta \middle| \psi_r\right\rangle \psi_r\right\rangle \\
&= \left\langle \sum_{k=1}^\infty \lambda_k\left\langle \xi \middle| \zeta_k\right\rangle \psi_k \middle| \sum_{r=1}^\infty \left\langle \eta \middle| \psi_r\right\rangle \psi_r\right\rangle \\
&= \sum_{k=1}^\infty \lambda_k\left\langle \xi \middle| \zeta_k\right\rangle \overline{\left\langle \eta \middle| \psi_k\right\rangle}.
\end{aligned}$$

On the other hand,

$$\left\langle T\xi \middle| \mathsf{H}_{sa}\eta\right\rangle = \left\langle \sum_{k=1}^\infty \left\langle \xi \middle| \zeta_k\right\rangle \psi_k \middle| \sum_{r=1}^\infty \lambda_r \left\langle \eta \middle| \psi_r\right\rangle \psi_r\right\rangle = \sum_{k=1}^\infty \lambda_k\left\langle \xi \middle| \zeta_k\right\rangle \overline{\left\langle \eta \middle| \psi_k\right\rangle}.$$

Hence the *weak* similarity condition (11) is fulfilled. It is clear that in what we have done a crucial role is played by the continuity of both H and T as linear maps from $\mathcal{H}_{\mathsf{H}}$ into $\mathcal{H}$, even though they are in general unbounded operators when regarded in $\mathcal{H}$. It is worth pointing out that the assumption $T \in C(\mathcal{H}_{\mathsf{H}}, \mathcal{H})$ does not imply that T is a closable operator in $\mathcal{H}$. But, requiring that $\{\eta \in D(\mathsf{H}_{sa})$ s.t. $T^\dagger \eta \in \mathcal{H}\}$ is dense in $\mathcal{H}$, implies that T has a densely defined *hilbertian* adjoint T^* and so it is automatically closable.

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