POLYNOMIAL GROWTH AND STAR-VARIETIES

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Abstract. Let $\mathcal{V}$ be a variety of associative algebras with involution over a field F of characteristic zero and let $c_n^*(\mathcal{V})$, $n = 1, 2, \dots$, be its $*$-codimension sequence. Such a sequence is polynomially bounded if and only if $\mathcal{V}$ does not contain the commutative algebra $F \oplus F$, endowed with the exchange involution, and M, a suitable 4-dimensional subalgebra of the algebra of 4×4 upper triangular matrices. Such algebras generate the only varieties of $*$-algebras of almost polynomial growth, i.e., varieties of exponential growth such that any proper subvariety is polynomially bounded. In this paper we completely classify all subvarieties of the $*$-varieties of almost polynomial growth by giving a complete list of finite dimensional $*$-algebras generating them.

1. Introduction

Let A be an associative algebra with involution ($*$-algebra) over a field F of characteristic zero and let $c_n^*(A)$, $n = 1, 2, \dots$, be its sequence of $*$-codimensions.

Recall that $c_n^*(A)$, $n = 1, 2, \dots$, is the dimension of the space of multilinear polynomials in n $*$-variables in the corresponding relatively free algebra with involution of countable rank. In case A satisfies a nontrivial identity, it was proved in [9] that, as in the ordinary case, $c_n^*(A)$ is exponentially bounded.

Given a variety of $*$-algebras $\mathcal{V}$, the growth of $\mathcal{V}$ is the growth of the sequence of $*$-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^*(A)$.

In this paper we are interested in varieties of polynomial growth, i.e., varieties of $*$-algebras such that $c_n^*(\mathcal{V}) = c_n^*(A)$ is polynomially bounded.

In such a case, if A is an algebra with 1, in [21] it was proved that

$$c_n^*(A) = qn^k + O(n^{k-1})$$

is a polynomial with rational coefficients. Moreover its leading term satisfies the inequalities

$$\frac{1}{k!} \le q \le \sum_{i=0}^{k} 2^{k-i} \frac{(-1)^i}{i!}.$$

In case of polynomial growth, the following characterization was given in [8]: a variety $\mathcal{V}$ has polynomial growth if and only if $\mathcal{V}$ does not contain the commutative algebra $F \oplus F$, endowed with the exchange involution, and M, a suitable 4-dimensional subalgebra of the algebra of 4×4 upper triangular matrices.

Hence $\mathrm{var}^*(F \oplus F)$ and $\mathrm{var}^*(M)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety is polynomially bounded.

From their description it follows that there exists no variety with intermediate growth of the $*$-codimensions between polynomial and exponential, i.e, either $c_n^*(\mathcal{V})$ is polynomially bounded or $c_n^*(\mathcal{V})$ grows exponentially. The above 2 algebras play the role of the infinite-dimensional Grassmann algebra and the algebra of 2×2 upper triangular matrices in the ordinary case ([12], [13]).

Recently, much interest was put into the study of varieties of polynomial growth (see for instance [3, 4, 5, 6, 15, 16, 14]) and different characterizations were given.

In this paper we completely classify all subvarieties of the varieties of $*$-algebras of almost polynomial growth by giving a complete list of finite dimensional $*$-algebras generating them.

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Moreover we classify all their minimal subvarieties of polynomial growth, i.e., varieties $\mathcal{V}$ satisfying the property: $c_n^*(\mathcal{V}) \approx qn^k$ for some $k \geq 1, q > 0$, and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$, $c_n^*(\mathcal{U}) \approx q'n^t$ with $t < k$.

2. On star-algebras with polynomial codimension growth

Throughout this paper F will denote a field of characteristic zero and A an associative F-algebra with involution $*$. Let us write $A = A^+ + A^-$, where $A^+ = \{a \in A | \ a^* = a\}$ and $A^- = \{a \in A | \ a^* = -a\}$ denote the sets of symmetric and skew elements of A, respectively. Let $X = \{x_1, x_2, \ldots\}$ be a countable set and let $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ be the free associative algebra with involution on X over F. It is useful to regard to $F\langle X, *\rangle$ as generated by symmetric and skew variables: if for $i = 1, 2, \ldots$, we let $y_i = x_i + x_i^*$ and $z_i = x_i - x_i^*$, then $F\langle X, *\rangle = F\langle y_1, z_1, y_2, z_2, \ldots\rangle$. Recall that a polynomial $f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F\langle X, *\rangle$ is a $*$-polynomial identity of A (or simply a $*$-identity), and we write $f \equiv 0$, if $f(s_1, \ldots, s_n, k_1, \ldots, k_m) = 0$ for all $s_1, \ldots, s_n \in A^+$, $k_1, \ldots, k_m \in A^-$.

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle X, *\rangle | f \equiv 0 \text{ on } A\}$ the T^*-ideal of $*$-identities of A, i.e., $\mathrm{Id}^*(A)$ is an ideal of $F\langle X, *\rangle$ invariant under all endomorphisms of the free algebra commuting with the involution $*$.

It is well known that in characteristic zero, every $*$-identity is equivalent to a system of multilinear $*$-identities. We denote by

$$P_n^* = \mathrm{span}_F\big\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, \quad w_i = y_i \quad or \quad w_i = z_i, \quad i = 1, \ldots, n\big\}$$

the vector space of multilinear polynomials of degree n in the variables $y_1, z_1, \ldots, y_n, z_n$. Hence for every $i = 1, \ldots, n$ either y_i or z_i appears in every monomial of P_n^* at degree 1 (but not both).

The study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^* \cap \mathrm{Id}^*(A)$ for all $n \geq 1$ and we denote by

$$c_n^*(A) = \dim_F \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)}, \quad n \geq 1,$$

the n-th $*$-codimension of A.

If A is an algebra with 1, by [2] $\mathrm{Id}^*(A)$ is completely determined by its multilinear proper polynomials. Recall that $f(y_1, z_1, \ldots, y_n, z_n) \in P_n^*$ is a proper polynomial if it is a linear combination of elements of the type

$$z_{i_1} \cdots z_{i_k} w_1 \cdots w_m$$

where $w_1, \ldots, w_m$ are left normed (long) Lie commutators in the y_is and z_is.

Let Γ_n^* denote the subspace of P_n^* of proper polynomials in $y_1, z_1, \ldots, y_n, z_n$ and $\Gamma_0^* = \mathrm{span}\{1\}$.

The sequence of proper $*$-codimensions is defined as

$$\gamma_n^*(A) = \dim \frac{\Gamma_n^*}{\Gamma_n^* \cap \mathrm{Id}^*(A)}, n = 0, 1, 2, \ldots.$$

For a unitary algebra A, the relation between ordinary $*$-codimensions and proper $*$-codimensions (see for instance [2]), is given by the following:

$$c_n^*(A) = \sum_{i=0}^{n} \binom{n}{i} \gamma_i^*(A), \ n = 0, 1, 2, \ldots. \tag{1}$$

One of the main tool in the study of the T*-ideals is provided by the representation theory of the hyperoctahedral group $\mathbb{Z}_2 \wr S_n$.

Recall that the group $\mathbb{Z}_2 \wr S_n$ acts on the space P_n^* as follows: for $h = (a_1, \ldots, a_n; \sigma) \in \mathbb{Z}_2 \wr S_n$, $hy_i = y_{\sigma(i)}$ and $hz_i = z_{\sigma(i)}^{a_{\sigma(i)}} = z_{\sigma(i)}$ or $-z_{\sigma(i)}$ according as $a_{\sigma(i)} = 1$ or -1, respectively.

Since $P_n^* \cap \mathrm{Id}^*(A)$ is invariant under this action, the space $\frac{P_n^*}{P_n^* \cap Id^*(A)}$ has a structure of left $\mathbb{Z}_2 \wr S_n$ module and its character, denoted by $\chi_n^*(A)$, is called the n-th $*$-cocharacter of A. By complete reducibility we can write

$$\chi_n^*(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu} \chi_{\lambda,\mu},$$

where λ and μ are partitions of r and $n-r$ respectively, $\chi_{\lambda,\mu}$ is the irreducible $\mathbb{Z}_2 \wr S_n$-character associated to the pair (λ, μ), and $m_{\lambda,\mu} \geq 0$ is the corresponding multiplicity.

Similarly $\frac{\Gamma_n^*}{\Gamma_n^* \cap Id^*(A)}$ is a $\mathbb{Z}_2 \wr S_n$-module under the induced action and we denote by $\psi_n^*(A)$ its character which is called the n-th proper $*$-cocharacter of A.

By complete reducibility it decomposes into irreducibles as follows

$$\psi_n^*(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m'_{\lambda,\mu} \chi_{\lambda,\mu},$$

where $\chi_{\lambda,\mu}$ is the irreducible $\mathbb{Z}_2 \wr S_n$-character associated to the pair of partitions (λ, μ) and $m'_{\lambda,\mu}$ is the corresponding multiplicity.

We are going to prove that, in case A generates a variety of polynomial growth, then A satisfies the same $*$-identities as a finite dimensional $*$-algebra.

We start with the following.

Theorem 1. *Let $\mathcal{V}$ be a variety of $*$-algebras. If $c_n^*(\mathcal{V}) \leq \alpha n^t$, for some constants α, t then $\mathcal{V} = var^*(A)$, for some finitely generated $*$-algebra A.*

Proof. Since $c_n^*(\mathcal{V}) \leq \alpha n^t$, for some α, t, then by [7, Theorem 3] there exists a constant d such that

$$\chi_n^*(\mathcal{V}) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu} \chi_{\lambda,\mu}$$

and $m_{\lambda,\mu} = 0$ whenever either $|\lambda| - \lambda_1 > d-1$ or $|\mu| > d$. This also says that $m_{\lambda,\mu} = 0$ whenever either $h(\lambda) > d$ or $h(\mu) > d$, where $h(\lambda)$ and $h(\mu)$ denote the height of λ and μ, respectively. Hence, as in the proof of Theorem 11.4.3 in [11], it is proved that $\mathcal{V} = \mathrm{var}^*(A)$, where A is the relatively free algebra of $\mathcal{V}$ generated by d symmetric and d skew variables. $\square$

In order to characterize the varieties of polynomial growth we need to apply the following result.

Theorem 2. [23, Theorem 1]. *If A is a PI-finitely generated associative algebra with involution over a field F of characteristic zero then A satisfies the the same $*$-identities as a finite dimensional associative algebra over F.*

Given two $*$-algebras A and B, we say that A is T^*-equivalent to B, and we write $A \sim_{T^*} B$, if $\mathrm{Id}^*(A) = \mathrm{Id}^*(B)$.

Theorem 3. *Let A be an algebra with involution over a field of characteristic zero and suppose that $c_n^*(A)$, $n = 1, 2, \ldots,$ is polynomially bounded. Then A is T^*-equivalent to a finite direct sum of algebras $B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are finite dimensional algebras with involution over F and $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \ldots, m$.*

Proof. By Theorems 1 and 2, since $c_n^*(A) \leq \alpha n^t$ for some α, t, we may assume that A is a finite dimensional algebra. Hence the result follows by applying the Proposition 7 in [20]. $\square$

Now let us focus our attention to the algebra $UT_n(F)$ of $n \times n$ upper triangular matrices over the field F. One can define an involution on $UT_n(F)$, that we shall denote by $*$, in the following way: if $a \in UT_n(F)$, then $a^* = ba^t b^{-1}$, where a^t denotes the usual transpose and b is the following permutation matrix:

$$b = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \vdots & \ddots & & \vdots \\ 1 & \ldots & 0 & 0 \end{pmatrix}.$$

Clearly, a^* is the matrix obtained from a by reflecting a along its secondary diagonal. Hence, if $a = (a_{ij})$ then $a^* = (a^*_{ij})$ where $a^*_{ij} = a_{n+1-j,n+1-i}$. This involution on $UT_n(F)$ is called the canonical reflection involution.

Given polynomials $f_1, \ldots, f_n \in F\langle X, * \rangle$ let us denote by $\langle f_1, \ldots, f_n \rangle_{T^*}$ the T*-ideal generated by $f_1, \ldots, f_n$.

The purpuse of this paper is to classify the subvarieties of the varieties of $*$-algebras of almost polynomial growth. Such varieties are generated by the following two algebras [7]:

1) $F \oplus F$, the two-dimensional commutative algebra, endowed with the exchange involution $(a, b)^* = (b, a)$;
2) $M = \left\{ \begin{pmatrix} u & r & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & v \\ 0 & 0 & 0 & u \end{pmatrix} \mid u, r, s, v \in F \right\}$, the subalgebra of $UT_4(F)$ endowed with the reflection involution.

Such algebras were extensively studied in [7] and [22]; there it was proved that $\mathrm{Id}^*(F \oplus F) = \langle [y_1, y_2], [y, z], [z_1, z_2] \rangle_{T^*}$ and $\mathrm{Id}^*(M) = \langle z_1 z_2 \rangle_{T^*}$.

3. Constructing $*$-algebras in $\mathrm{var}^*(M)$

The purpose of this section is to construct finite dimensional $*$-algebras belonging to the variety generated by M whose $*$-codimension sequence grows polynomially.

For $k \geq 2$, let

$$A_k = \mathrm{span}_F \{ e_{11} + e_{2k,2k}, E, \ldots, E^{k-2}, e_{12}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-1,2k} \}$$

be the subalgebra of $UT_{2k}(F)$ equipped with the reflection involution, where $E = \sum_{i=2}^{k-1} e_{i,i+1} + e_{2k-i,2k-i+1}$ and the e_{ij}s denote the usual matrix units.

The following result characterizes the $*$-identities and the $*$-codimensions of A_k.

Lemma 1. *Let $k \geq 2$. Then:*

1) *$Id^*(A_k) = \langle y_1 \cdots y_{k-2} st_3(y_{k-1}, y_k, y_{k+1}) y_{k+2} \cdots y_{2k-1},\ y_1 \cdots y_{k-1} z y_k \cdots y_{2k-2},\ z_1 z_2 \rangle_{T^*}$, where $st_3(y_{k-1}, y_k, y_{k+1}) = \sum_{\sigma \in S_3} sgn(\sigma) y_{\sigma(k-1)} y_{\sigma(k)} y_{\sigma(k+1)}$ denotes the standard polynomial of degree 3.*
2) *For $n < 2k-1$, $c_n^*(A_k) = 1 + \sum_{\substack{t<k-1 \\ or \\ n-t<k}} \binom{n}{t}(n-t) + \sum_{\substack{t<k-1 \\ or \\ n-t<k}} \binom{n}{t}(n-t-1)$.*

   For $n \geq 2k-1$, $c_n^*(A_k) = 1 + \sum_{\substack{t<k-1 \\ or \\ n-t<k}} \binom{n}{t}(n-t) + \sum_{\substack{t<k-1 \\ or \\ n-t<k-1}} \binom{n}{t}(n-t-1) + \binom{n-1}{k-2}(n-k+1) \approx q n^{k-1}$, for some $q > 0$.

Proof. Write $I = \langle y_1 \cdot \ldots \cdot y_{k-2} st_3(y_{k-1}, y_k, y_{k+1}) y_{k+2} \cdot \ldots \cdot y_{2k-1},\ y_1 \cdots y_{k-1} z y_k \cdots y_{2k-2},\ z_1 z_2 \rangle_{T^*}$. It is clear that $I \subseteq \mathrm{Id}^*(A_k)$. In order to prove the opposite inclusion, first we find a set of generators of P_n^*, modulo $P_n^* \cap I$, for every $n \geq 1$.

Let $f \in P_n^*$ be a multilinear polynomial of degree n. By the Poincaré-Birkhoff-Witt theorem f can be written as a linear combination of products of the type

$$y_{j_1} \cdots y_{j_r} z_{k_1} \cdots z_{k_t} w_1 \cdots w_m,$$

where $w_1, \ldots, w_m$ are left normed commutators in the y_is and z_is, $j_1 \leq \cdots \leq j_r$ and $k_1 \leq \cdots \leq k_t$.

Because of $z_1 z_2 \in I$ ([20, Remark 8]), modulo I, f is a linear combination of the polynomials

$$y_1 \cdots y_n,\ y_{i_1} \cdots y_{i_{t'}} z_l y_{j_1} \cdots y_{j_{s'}},\ y_{p_1} \cdots y_{p_t} [y_r, y_m] y_{q_1} \cdots y_{q_s} \tag{2}$$

where $i_1 < \ldots < i_{t'}$, $j_1 < \ldots < j_{s'}$, $p_1 < \ldots < p_t$, $r > m < q_1 < \ldots < q_s$.

Notice that in case $n < 2k-1$ we have $t' < k-1$ or $s' < k-1$, $t < k-1$ or $s < k-1$. That is also true for $n \geq 2k-1$, because of $y_1 \cdots y_{k-1} z y_k \cdots y_{2k-2} \in I$. Moreover, always in case $n \geq 2k-1$, we can write

$$y_{p_1} \cdots y_{p_{n-k}} [y_r, y_m] y_{q_1} \cdots y_{q_{k-2}}$$

modulo $\langle y_1 \cdots y_{k-2} st_3(y_{k-1}, y_k, y_{k+1}) y_{k+2} \cdots y_{2k-1} \rangle_{T^*}$, as a linear combination of polynomials of the type

$$y_{r_1} \cdots y_{r_{n-k}} [y_1, y_i] y_{u_1} \cdots y_{u_{k-2}} \tag{3}$$

with $r_1 < \ldots < r_{n-k}$, $u_1 < \ldots < u_{k-2}$.

It follows that the space P_n^* is generated, modulo $P_n^* \cap I$, by the polynomials in (2) , in case $n < 2k-1$ and, by the polynomials

$$\text{(4)} \qquad y_1 \cdots y_n,\ y_{i_1} \cdots y_{i_{t'}} z_l y_{j_1} \cdots y_{j_{s'}},\ y_{p_1} \cdots y_{p_t} [y_r, y_m] y_{q_1} \cdots y_{q_s},\ y_{r_1} \cdots y_{r_{n-k}} [y_1, y_i] y_{u_1} \cdots y_{u_{k-2}}$$

where $i_1 < \ldots < i_{t'}$, $j_1 < \ldots < j_{s'}$, $t' < k-1$ or $s' < k-1$, $p_1 < \ldots < p_t$, $r > m < q_1 < \ldots < q_s$ $t < k-1$ or $s < k-2$, $r_1 < \ldots < r_{n-k}$, $u_1 < \ldots < u_{k-2}$, in case $n \geq 2k-1$.

We next show that the above polynomials are linearly independent modulo $\mathrm{Id}^*(A_k)$.

We assume that $n \geq 2k-1$ (the case $n < 2k-1$ is proved in a similar way). To this end, let $f \in \mathrm{Id}^*(A_k)$ be a linear combination of the above polynomials and write

$$\begin{aligned} f = \delta y_1 \cdots y_n &+ \sum_{\substack{t' < k-1 \\ \text{or} \\ s' < k-1}} \sum_{l,I,J} \alpha_{l,I,J} y_{i_1} \cdots y_{i_{t'}} z_l y_{j_1} \cdots y_{j_{s'}} \\ &+ \sum_{\substack{t < k-1 \\ \text{or} \\ s < k-2}} \sum_{r,P,Q} \beta_{r,P,Q} y_{p_1} \cdots y_{p_t} [y_r, y_m] y_{q_1} \cdots y_{q_s} \\ &+ \sum_{i,R,U} \gamma_{i,R,U} y_{r_1} \cdots y_{r_{n-k}} [y_1, y_i] y_{u_1} \cdots y_{u_{k-2}} \end{aligned}$$

where $t' + s' = n-1$, $t + s = n-2$. Moreover, for any fixed t, s, t' and s', $i_1 < \ldots < i_{t'}$, $j_1 < \ldots < j_{s'}$, $p_1 < \ldots < p_t$, $m < q_1 < \ldots < q_s$, $r_1 < \ldots < r_{n-k}$, $u_1 < \ldots < u_{k-2}$ and $I = \{i_1, \ldots, i_{t'}\}$, $J = \{j_1, \ldots, j_{s'}\}$, $P = \{p_1, \ldots, p_t\}$, $Q = \{q_1, \ldots, q_s\}$, $R = \{r_1, \ldots, r_{n-k}\}$ and $U = \{u_1, \ldots, u_{k-2}\}$.

First suppose that $\delta \neq 0$, then by making the evaluation $y_1 = \ldots = y_n = e_{11} + e_{2k,2k}$ and $z_l = 0$ for all $l = 1, \ldots, n$, one gets $\delta(e_{11} + e_{2k,2k}) = 0$ and so $\delta = 0$, a contradiction.
Suppose that there exists $\beta_{r,P,Q} \neq 0$ for some $t < k-1$, r, P and Q, then by making the evaluation $y_{p_1} = \ldots = y_{p_t} = E$, $y_r = e_{12} + e_{2k-1,2k}$, $y_m = y_{q_1} = \ldots = y_{q_s} = e_{11} + e_{2k,2k}$ and $z_l = 0$ for all $l = 1, \ldots, n$, one gets $\beta_{r,P,Q} e_{2k-t-1,2k} - \beta_{r,Q,P} e_{1,2+t} = 0$. Thus $\beta_{r,P,Q} = \beta_{r,Q,P} = 0$, a contradiction. Suppose now $\beta_{r,P,Q} \neq 0$ for some $t \geq k+1$, r, P and Q. By making the evaluation $y_{p_1} = \ldots = y_{p_t} = e_{11} + e_{2k,2k}$, $y_r = e_{12} + e_{2k-1,2k}$, $y_m = y_{q_1} = \ldots = y_{q_s} = E$ and $z_l = 0$ for all $l = 1, \ldots, n$, one gets $\beta_{r,P,Q} = \beta_{r,Q,P} = 0$ as before.

Suppose now $\gamma_{i,R,U} \neq 0$ for some R and U. The evaluation $y_1 = y_{r_1} = \ldots = y_{r_{n-k}} = e_{11} + e_{2k,2k}$, $y_i = e_{12} + e_{2k-1,2k}$, $y_{u_1} = \ldots = y_{u_{k-2}} = E$ and $z_l = 0$ for all $l = 1, \ldots, n$ gives $\gamma_{i,R,U} = 0$, a contradiction.

Let now $\alpha_{l,I,J} \neq 0$ for some $t' < k-1$, l, I and J. By making the evaluation $z_l = e_{12} - e_{2k-1,2k}$, $y_{i_1} = \ldots = y_{i_{t'}} = E$ and $y_{j_1} = \ldots = y_{j_{s'}} = e_{11} + e_{2k,2k}$ one gets $-\alpha_{l,I,J} e_{2k-t'-1} + \alpha_{l,J,I} e_{1,2+t'} = 0$, thus $\alpha_{l,I,J} = \alpha_{l,J,I} = 0$, a contradiction. Similarly, if $0 \leq s' < k-1$, let $\alpha_{l,I,J} \neq 0$ for some $t' \geq k-1$ l, I and J. Then the evaluation $z_l = e_{12} - e_{2k-1,2k}$, $y_{i_1} = \ldots = y_{i_{t'}} = e_{11} + e_{2k,2k}$ and $y_{j_1} = \ldots = y_{j_{s'}} = E$ gives $\alpha_{l,I,J} = 0$, a contradiction.

Therefore the polynomials in (4) are linearly independent modulo $P_n^* \cap \mathrm{Id}^*(A_k)$ and, since $P_n^* \cap \mathrm{Id}^*(A_k) \supseteq P_n^* \cap I$, the form a basis of P_n^* (mod $P_n^* \cap \mathrm{Id}^*(A_k)$) and $\mathrm{Id}^*(A_k) = I$.

Thus, by counting we obtain

$$c_n^*(A_k) = 1 + \sum_{\substack{t < k-1 \\ \text{or} \\ n-t < k}} \binom{n}{t} (n-t) + \sum_{\substack{t < k-1 \\ \text{or} \\ n-t < k-1}} \binom{n}{t} (n-t-1) + \binom{n-1}{k-2} (n-k+1) \approx q n^{k-1},$$

for some $q > 0$ and we are done.

In case $n < 2k-1$, $c_n^*(A_k) = 1 + \displaystyle\sum_{\substack{t < k-1 \\ \text{or} \\ n-t < k}} \binom{n}{t} (n-t) + \sum_{\substack{t < k-1 \\ \text{or} \\ n-t < k}} \binom{n}{t} (n-t-1).$

□

Next we construct, for any fixed $k \geq 2$, two $*$-algebras with unity in the variety generated by M whose codimension sequences grow as n^{k-1}.

For any $k \geq 2$, let

$$N_k = \text{span}_F\{I, E, \ldots, E^{k-2}, e_{12} - e_{2k-1,2k}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-2,2k}\}$$

be the subalgebra of $UT_{2k}(F)$ equipped with the reflection involution, where I denotes the $2k \times 2k$ identity matrix and $E = \sum_{i=2}^{k-1} e_{i,i+1} + e_{2k-i,2k-i+1}$.

Lemma 2. *The T^*-ideal $Id^*(N_k)$ is generated by the polynomials $[y_1, y_2], [y, z], z_1 z_2$, in case $k = 2$ and by $[y_1, \ldots, y_{k-1}], z_1 z_2$, in case $k \geq 3$.*

Moreover

$$c_n^*(N_k) = 1 + \sum_{i=1}^{k-2} \binom{n}{i}(2i-1) + \binom{n}{k-1}(k-1) \approx qn^{k-1}, \text{ for some } q > 0.$$

Proof. If $k = 2$ then clearly $\langle [y_1, y_2], [y, z], z_1 z_2 \rangle \subseteq \text{Id}^*(N_2)$. The opposite inclusion is a direct consequence of [20, Lemma 10].
Let now $k \geq 3$ and let $I = \langle [y_1, \ldots, y_{k-1}], z_1 z_2 \rangle_{T^*}$. It is easily proved that $I \subseteq \text{Id}^*(N_k)$.

Let now f be a $*$-identity of N_k. We may clearly assume that f is multilinear, and since N_k is an algebra with 1 we may take f proper. After reducing the polynomial f modulo the polynomials in I we obtain that f is the zero polynomial if $\deg f \geq k$ and f is a linear combination of commutators

$$[z_i, y_{i_1}, \ldots, y_{i_{k-2}}], i_1 < \ldots < i_{k-2}$$

in case $\deg f = k - 1$ and is a linear combination of commutators

$$[z_i, y_{i_1}, \ldots, y_{i_{s-1}}], [y_{j_1}, y_{j_2}, \ldots, y_{j_s}], i_1 < \ldots < i_{s-1}, j_1 > j_2 < \ldots < j_s$$

in case $\deg f < k - 1$. Hence

$$f = \sum_{i=1}^{s} \alpha_i [z_i, y_{i_1}, \ldots, y_{i_{s-1}}] + \sum_{j_1=2}^{s} \beta_{j_1} [y_{j_1}, \ldots, y_{j_s}], \text{ for some } 1 < s \leq k-1.$$

Suppose that there exists j_1 such that $\beta_{j_1} \neq 0$. By making the evaluation $y_{j_1} = e_{13} + e_{2k-2,2k}$, $y_{j_2} = \ldots = y_{j_s} = E$ and $z_i = 0$ for all $i = 1, \ldots, s$, we get $\beta_{j_1} = 0$, a contradiction. So $\beta_{j_1} = 0$ for all $j_1 = 2, \ldots, s$. Now suppose that there exists i such that $\alpha_i \neq 0$. By making the evaluation $z_i = e_{12} - e_{2k-1,2k}$, $z_j = 0$ for all $j \neq i$ and $y_{i_1} = \ldots = y_{i_{s-1}} = E$ we get $\alpha_i = 0$, a contradiction. Thus $\alpha_i = 0$ for all $i = 1, \ldots, s$. This says that $f \in I$, and so, $\text{Id}^*(N_k) \subseteq I$.

The arguments above also prove that

$$\gamma_s^*(N_k) = \begin{cases} 2s - 1 & \text{if } s < k - 1 \\ s & \text{if } s = k - 1 \\ 0 & \text{if } s \geq k. \end{cases}$$

Hence, by (1) we obtain that

$$c_n^*(N_k) = 1 + \sum_{i=1}^{k-2} \binom{n}{i}(2i-1) + \binom{n}{k-1}(k-1) \approx qn^{k-1}, \text{ for some } q > 0.$$

□

Let now, for any fixed $k \geq 2$,

$$U_k = \text{span}_F\{I, E, \ldots, E^{k-2}, e_{12} + e_{2k-1,2k}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-2,2k}\}$$

be the subalgebra of UT_{2k} equipped with the reflection involution.

The following lemma holds and it can be proved in a similar way as the previous lemma.

Lemma 3. *The T^*-ideal $Id^*(U_k)$ is generated by the polynomials $[y_1, y_2], z_1$ in case $k = 2$ and by $[z, y_1, \ldots, y_{k-2}], z_1 z_2$ in case $k \geq 3$. Moreover*

$$c_n^*(U_2) = 1 \text{ and } c_n^*(U_k) = 1 + \sum_{i=1}^{k-2} \binom{n}{i}(2i-1) + \binom{n}{k-1}(k-2) \approx qn^{k-1}, \text{ for some } q > 0, \text{ for } k \geq 3.$$

Remark 1. *Notice that $U_2 \sim_{T^*} F$ is a commutative $*$-algebra with trivial involution.*

Finally we give a description of the $*$-identities of the direct sum among U_k and N_k.

Here we remark that if $t \neq k$ than $U_t \oplus N_k \sim_{T^*} U_t$ if $t > k$ and $U_t \oplus N_k \sim_{T^*} N_k$ if $t < k$. Moreover, if $k = t = 2$, then $N_2 \oplus U_2 \sim_{T^*} N_2$.

Lemma 4. *If $k \geq 3$ then*

(1) $Id^*(N_k \oplus U_k) = \langle [y_1, \ldots, y_k], [z, y_1, \ldots, y_{k-1}], z_1 z_2 \rangle_{T^*}$

(2) $c_n^*(N_k \oplus U_k) = 1 + \sum_{i=1}^{k-1} \binom{n}{i}(2i-1) \approx qn^{k-1}$, *for some $q > 0$.*

Proof. Let $I = \langle [y_1, \ldots, y_k], [z, y_1, \ldots, y_{k-1}], z_1 z_2 \rangle_{T^*}$. It is clear that $I \subseteq \mathrm{Id}^*(N_k \oplus U_k)$. Now, if $f \in \mathrm{Id}^*(U_k \oplus N_k)$, as in the proof of Lemma 2, we get that f can be written as

$$f = \sum_{i=1}^{s} \alpha_i [z_i, y_{i_1}, \ldots, y_{i_{s-1}}] + \sum_{j_1=2}^{s} \beta_{j_1} [y_{j_1}, y_{j_2}, \ldots, y_{j_s}].$$

where $s < k$.

Suppose that there exists j_1 such that $\beta_{j_1} \neq 0$. By making the evaluation $y_{j_1} = (0, e_{12} + e_{2k-1,2k})$, $y_{j_2} = \ldots = y_{j_s} = (0, E)$ and $z_i = 0$ for all $i = 1, \ldots, s$, we get $\beta_{j_1} = 0$, a contradiction. So $\beta_{j_1} = 0$ for all $j_1 = 2, \ldots, s$.
Now suppose that there exists i such that $\alpha_i \neq 0$. By making the evaluation $z_i = (e_{12} - e_{2k-1,2k}, 0)$, $z_j = 0$ for all $j \neq i$ and $y_{j_1} = \ldots = y_{j_{s-1}} = (E, 0)$ we get $\alpha_i = 0$, a contradiction. Thus $\alpha_i = 0$ for all $i = 1, \ldots, s$. This says that $\mathrm{Id}^*(N_k \oplus U_k) \subseteq I$ and also

$$\gamma_s^*(N_k \oplus U_k) = \begin{cases} 2s - 1 & \text{if } s \leq k-1 \\ 0 & \text{if } s \geq k. \end{cases}.$$

Hence

$$c_n^*(N_k \oplus U_k) = 1 + \sum_{i=1}^{k-1} \binom{n}{i}(2i-1) \approx qn^{k-1}, \text{ for some } q > 0.$$

□

4. On minimal $*$-varieties in $\mathrm{var}^*(M)$

In this section we shall prove that A_k, N_k and U_k generate minimal varieties of polynomial growth. We start with the definition of minimal variety.

Definition 1. *A variety $\mathcal{V}$ of $*$-algebras is minimal of polynomial growth if $c_n^*(\mathcal{V}) \approx qn^k$ for some $k \geq 1$, $q > 0$, and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$ we have that $c_n^*(\mathcal{U}) \approx q'n^t$ with $t < k$.*

We recall that if $A = B + J$ is a finite dimensional $*$-algebra over F , where B is a semisimple $*$-subalgebra and J = J(A) is its Jacobson radical, then J can be decomposed into the direct sum of B-bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11},$$

where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, l, m \in \{0, 1\}$, $J_{ik} J_{lm} \subseteq \delta_{k,l} J_{im}$ where $\delta_{k,l}$ is the Kronecker delta and $J_{11} = BN$ for some nilpotent subalgebra N of A commuting with B. For a proof of this result see [10, Lemma 2].

Remark 2. *Let $A = F + J$ be a $*$-algebra with $J = J_{00} + J_{01} + J_{10} + J_{11}$. If A satisfies the ordinary identity $[x_1, \ldots, x_t]$, for some $t \geq 2$, then $J_{10} = J_{01} = 0$.*

Proof. The proof is obvious since $J_{10} = [J_{10}, \underbrace{F, \ldots, F}_{t-1}]$ and $J_{01} = [J_{01}, \underbrace{F, \ldots, F}_{t-1}]$. □

Theorem 4. *For any $k \geq 2$ and $t > 2$, N_k and U_t generate minimal varieties.*

Proof. We shall prove the statement for N_k. In a similar way it is possible to prove the statement also for U_t.
If $k = 2$, the result follows from [20, Lemma 28].

Now assume that $k \geq 3$. Suppose that the algebra $A \in \mathrm{var}^*(N_k)$ generates a subvariety of $\mathrm{var}^*(N_k)$ and $c_n^*(A) \approx qn^{k-1}$, for some $q > 0$. We shall prove that $A \sim_{T^*} N_k$ and this will complete the proof.
Since $c_n^*(A)$ is polynomially bounded, by Theorem 3 we may assume that

$$A = B_1 \oplus \ldots \oplus B_m,$$

where $B_1, \ldots, B_m$ are finite dimensional $*$-algebras such that $\dim_F \frac{B_i}{J(B_i)} \leq 1$ for all $i = 1, \ldots, m$. This implies that either $B_i \cong F + J(B_i)$ or $B_i = J(B_i)$ is a nilpotent algebra. Since

$$c_n^*(A) \leq c_n^*(B_1) + \ldots + c_n^*(B_m),$$

then there exists B_i such that $c_n^*(B_i) \approx bn^{k-1}$, for some $b > 0$. Hence

$$\mathrm{var}^*(N_k) \supseteq \mathrm{var}^*(A) \supseteq \mathrm{var}^*(F + J(B_i)) \supseteq \mathrm{var}^*(F + J_{11}(B_i))$$

and $c_n^*(F + J(B_i)) \approx bn^{k-1}$, for some $b > 0$. Moreover, by the previous remark, since N_k, and so $F + J(B_i)$, satisfies the ordinary identity $[x_1, \ldots, x_k]$, we get that $J_{01}(B_i) = J_{10}(B_i) = 0$. Hence $F + J(B_i) = (F + J_{11}(B_i)) \oplus J_{00}(B_i)$ and $c_n^*(F + J(B_i)) = c_n^*(F + J_{11}(B_i))$ for n large enough. Hence, in order to prove $A \sim_{T^*} N_k$, it is enough to show that $F + J_{11}(B_i) \sim_{T^*} N_k$. Thus, without loss of generality we may assume that A is a unitary algebra.
Now since $c_n^*(A) \approx qn^{k-1}$ then

$$c_n^*(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^*(A)$$

and, by [21, Lemma 2.2], $\gamma_i^*(A) \neq 0$ for all $0 \leq i \leq k - 1$.

Recall that since $\mathrm{Id}^*(A) \supseteq \mathrm{Id}^*(N_k)$, then $\frac{\Gamma_i^*}{(\Gamma_i^* \cap \mathrm{Id}^*(A)}$ is isomorphic to a quotient module of $\frac{\Gamma_i^*}{(\Gamma_i^* \cap \mathrm{Id}^*(N_k))}$. Hence, if $\psi_i^*(A) = \sum_{|\lambda|+|\mu|=i} m_{\lambda,\mu}\chi_{\lambda,\mu}$ and $\psi_i^*(N_k) = \sum_{|\lambda|+|\mu|=i} m'_{\lambda,\mu}\chi_{\lambda,\mu}$ are the i-th proper $*$-cocharacters of A and N_k respectively , we must have $m_{\lambda,\mu} \leq m'_{\lambda,\mu}$ for all $\lambda \vdash r$, $\mu \vdash i - r$, $r = 0, \ldots, i$.

For any $i = 2, \ldots, k - 2$, let $f_1 = [z_1, y_2, \ldots, y_2]$ and $f_2 = [y_1, y_2, y_1, \ldots, y_1]$ be highest weight vectors corresponding to the partitions $(\lambda, \mu) = ((i - 1), (1))$ and $(\lambda, \mu) = ((i - 1, 1), \emptyset)$, respectively (see [1]). It is clear that f_1 and f_2 are not $*$-identities of N_k. Thus, for $i = 2, \ldots, k - 2$, $\chi_{(i-1),(1)}$ and $\chi_{(i-1,1),\emptyset}$ participate in the i-th proper $*$-cocharacter $\psi_i^*(N_k)$ with non-zero multiplicities.
Hence, for $i = 2, \ldots, k - 2$, since $\gamma_i^*(N_k) = 2i - 1 = \deg \chi_{(i-1),(1)} + \deg \chi_{(i-1,1),\emptyset}$, we have that

$$\psi_i^*(N_k) = \chi_{(i-1),(1)} + \chi_{(i-1,1),\emptyset}.$$

Similarly one obtain $\psi_{k-1}^*(N_k) = \chi_{(k-2),(1)}$.
Thus, since $\gamma_{k-1}^*(A) \neq 0$ we get also that $\psi_{k-1}^*(A) = \chi_{(k-2),(1)}$. Moreover, for $2 \leq i \leq k - 2$ one gets $\psi_i^*(A) = \chi_{(i-1),(1)} + \chi_{(i-1,1),\emptyset}$. In fact, if $\psi_i^*(A) = \chi_{(i-1),(1)}$, for some $2 \leq i \leq k-2$, then the highest weight vector $[y_2, \underbrace{y_1, \ldots, y_1}_{i-1}]$ corresponding to the couple of partitions $(\lambda, \mu) = ((i - 1, 1), \emptyset)$ would be a $*$-identity for A. But this implies that also $[z, \underbrace{y, \ldots, y}_{k-2}]$ is a $*$-identity for A, and so, $\psi_{k-1}^*(A) = 0$, a contradiction. In a similar way one can prove that if $\psi_i^*(A) = \chi_{(i-1,1),\emptyset}$ we would reach a contradiction. So we get that $\psi_i^*(A) = \chi_{(i-1),(1)} + \chi_{(i-1,1),\emptyset}$, for $2 \leq i \leq k - 2$ and $\psi_{k-1}^*(A) = \chi_{(k-2),(1)}$.
Hence

$$c_n^*(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^*(A) = 1 + \sum_{i=1}^{k-2} \binom{n}{i} (2i - 1) + \binom{n}{k-1} (k - 1) = c_n^*(N_k).$$

Thus A and N_k have the same sequence of $*$-codimensions and, since $\mathrm{Id}^*(N_k) \subseteq \mathrm{Id}^*(A)$ we get the equality $\mathrm{Id}^*(N_k) = \mathrm{Id}^*(A)$. $\square$

In order to prove that also A_k generates a minimal variety we need to state some preliminary results.

Lemma 5. *Let $A = F + J \in var^*(A_k)$. Then $J_{11}^- = [J_{11}, J_{11}] = 0$.*

Proof. From $J_{11}^- = \underbrace{F\cdots F}_{k-1} J_{11}^- \underbrace{F\cdots F}_{k-1}$ follows $J_{11}^- = 0$, since $y_1\cdots y_{k-1}zy_k\cdots y_{2k-2}$ is a $*$-identity of A. As a consequence, since $[J_{11}^+, J_{11}^+] \subseteq J_{11}^-$ we get that $[J_{11}, J_{11}] = 0$. □

Lemma 6. *Let $A = F + J \in var^*(A_k)$ with $J_{10} \neq 0$ (hence $J_{01} \neq 0$). If $c_n^*(A) \approx qn^{k-1}$, for some $q > 0$, then $A \sim_{T_*} A_k$.*

Proof. By the previous lemma, $A = F + J_{11} + J_{10} + J_{01} + J_{00}$ with $J_{11}^- = [J_{11}, J_{11}] = 0$. Suppose that $J_{10}(J_{00}^+)^{k-2} = 0$. This says that also $(J_{00}^+)^{k-2}J_{01} = 0$. If $J^m = 0$, we claim that for any $n \geq m$, the multilinear polynomial

$$f = y_{i_1}\cdots y_{i_l}y_1\cdots y_{k-2}zy_{k-1}\cdots y_{2k-4}y_{j_1}\cdots y_{j_t} \in \mathrm{Id}^*(A),$$

where $t + l + 2k - 3 = n$.

In fact, by the multilinearity of f, we can evaluate the variables in a basis of A which is the union of a basis of $J_{11}, J_{10} + J_{01}, J_{00}$ and $1 = 1_F$. Since $J^m = 0$, if all variables are evaluated in J we get a zero value of f. Hence at least one variable must be evaluated in 1. Since $J_{11}^- = 0$ we need to check the evaluation of z in $J_{10} + J_{01}$. It is easily checked that since $J_{10}(J_{00}^+)^{k-2} = (J_{00}^+)^{k-2}J_{01} = 0$ then f vanishes on A.

We have proved that $f = y_{i_1}\cdots y_{i_l}y_1\cdots y_{k-2}zy_{k-1}\cdots y_{2k-4}y_{j_1}\cdots y_{j_t} \in \mathrm{Id}^*(A)$.

In a similar way it is proved that for any $n \geq m$,

$$g = y_{i_1}\cdots y_{i_l}y_1\cdots y_{k-2}[y_l, y_m]y_{k-1}\cdots y_{2k-4}y_{j_1}\cdots y_{j_t} \in \mathrm{Id}^*(A),$$

where $t + l + 2k - 2 = n$.

Let $Q \subseteq \mathrm{Id}^*(A)$ be the T^*-ideal generated by f and g plus the generators of the T^*-ideal $\mathrm{Id}^*(A_k)$. For any $n \geq m$ a set of generators of P_n^* (mod $P_n^* \cap \mathrm{Id}^*(Q)$) is given by

$\{y_{i_1}\cdots y_{i_t}z_ly_{j_1}\cdots y_{j_s} \mid 1 \leq l \leq n,\ t < k-2 \text{ or } s < k-2,\ i_1 < \ldots < i_t,\ j_1 < \ldots < j_s\} \cup$
$\{y_{i_1}\cdots y_{i_t}[y_l, y_m]y_{j_1}\cdots y_{j_s} \mid t < k-2 \text{ or } s < k-2,\ i_1 < \ldots < i_t, l > m < j_1 < \ldots < j_s\} \cup \{y_1y_2\cdots y_n\}$.

Hence

$$c_n^*(A) \leq \sum_{\substack{t<k-2\\ \text{or}\\ n-t<k-1}} \binom{n-1}{t} n + \sum_{\substack{t<k-2\\ \text{or}\\ n-t<k-1}} \binom{n}{t}(n-t-1) \approx qn^{k-2},$$

a contradiction.

Therefore we must have $J_{10}(J_{00}^+)^{k-2} \neq 0$. Let $a \in J_{10}$, $b_1, \ldots, b_{k-2} \in J_{00}^+$ be such that $ab_1\cdots b_{k-2} \neq 0$. Also $b_{k-2}^*\cdots b_1^*a^* \neq 0$, with $b_{k-2}^*, \ldots, b_1^* \in J_{00}^+$ and $a^* \in J_{01}$.

Let $f \in \mathrm{Id}^*(A)$ be a multilinear polynomial of degree $n \geq 2k-1$. By Lemma 1, we can write f, modulo $\mathrm{Id}^*(A_k)$, as

$$\begin{aligned} f = \delta y_1\cdots y_n &+ \sum_{\substack{t<k-1\\ \text{or}\\ s<k-1}} \sum_{l,I,J} \alpha_{l,I,J}\, y_{i_1}\cdots y_{i_t}z_ly_{j_1}\cdots y_{j_s} \\ &+ \sum_{\substack{t<k-1\\ or\\ s<k-2}} \sum_{r,P,Q} \beta_{r,P,Q}\, y_{p_1}\cdots y_{p_t}[y_r, y_m]y_{q_1}\cdots y_{q_s} \\ &+ \sum_{i,R,U} \gamma_{i,R,U}\, y_{r_1}\cdots y_{r_{n-k}}[y_1, y_i]y_{u_1}\cdots y_{u_{k-2}} \end{aligned}$$

where $I = \{i_1, \ldots, i_t\}$, $J = \{j_1, \ldots, j_s\}$, $P = \{p_1, \ldots, p_t\}$, $Q = \{m, q_1, \ldots, q_s\}$

are such that $I \uplus J \uplus \{l\} = P \uplus Q \uplus \{r\} = \{1, \ldots, n\}$, and $i_1 < \cdots < i_t, j_1 < \cdots < j_s$, $p_1 < \ldots < p_t$, $r < m < q_1 < \ldots < q_s$. Also $R = \{r_1, \ldots, r_{n-k}\}$ and $U = \{1, u_1, \ldots, u_{k-2}\}$ are such that $R \uplus U \uplus \{i\} = \{1, \ldots, n\}$ and $r_1 < \ldots < r_{n-k}$, $u_1 < \ldots < u_{k-2}$.

The evaluation $y_1 = \cdots = y_n = 1_F$ and $z_l = 0$, for all $l = 1, \ldots, n$, gives $\delta = 0$. Also, for fixed $s < k-1, l, I, J$ the evaluation $y_{i_m} = 1_F$, $m \leq t$, $z_l = a - a^*$ and $y_{j_p} = b_p$, $p \leq s$, gives $\alpha_{l,I,J} ab_1 \cdots b_s + \alpha_{l,J,I} c = 0$ with $ab_1 \cdots b_s \in J_{10}$ and $c \in J_{01}$ linearly independent. Hence $\alpha_{l,I,J} = \alpha_{l,J,I} = 0$. Similarly, for fixed $t < k-1, l, I, J$ the evaluation $y_{i_{t-m}} = b^*_{m+1}$, $0 \leq m \leq t-1$, $z_l = a - a^*$ and $y_{j_p} = 1_F$, $p \leq s$, gives $\alpha_{l,M,N} = 0$ and $\alpha_{l,N,M} = 0$.

Also, for fixed $i \neq 1$, R, U the evaluation $y_1 = 1_F$, $y_{r_j} = 1_F$, $j \leq n-k$, $y_i = a + a^*$ and $y_{u_l} = b_l$, $l \leq k-2$, gives $\gamma_{i,R,U} = 0$ and $\gamma_{i,U,R} = 0$. Finally, for fixed $s < k-2, r, P, Q$ the evaluation $y_{p_j} = 1_F$, $1 \leq j \leq t$, $y_r = a + a^*$, $y_m = b_1$and $y_{q_l} = b_{l+1}$, $1 \leq l \leq s$, gives $\beta_{r,P,Q} = 0$ and $\beta_{r,Q,P} = 0$. Similarly, for fixed $t < k-1, r, P, Q$ the evaluation $y_{p_{t-j}} = b^*_{j+1}$, $0 \leq j \leq t-1$, $y_r = a + a^*$, $y_m = 1_F$ and $y_{q_l} = 1_F$, $1 \leq l \leq s$, gives $\beta_{r,P,Q} = 0$ and $\beta_{r,Q,P} = 0$.

Therefore $f \in \mathrm{Id}^*(A_k)$. Similarly, if $n < 2k-1$ it is proved that $f \in \mathrm{Id}^*(A_k)$. Hence $\mathrm{Id}^*(A) = \mathrm{Id}^*(A_k)$. $\square$

Now we are in a position to prove that the algebra A_k generates a minimal variety.

Theorem 5. *For any $k \geq 2$, A_k generates a minimal variety.*

Proof. Let $A \in \mathrm{var}^*(A_k)$ be such that $c^*_n(A) \approx qn^{k-1}$, for some $q > 0$. As in the prroof of Theorem 4, we may assume that
$$A = B_1 \oplus \cdots \oplus B_m,$$
where $B_1, \ldots, B_m$ are finite dimensional algebras with involution and either $B_i \cong F + J(B_i)$ or $B_i = J(B_i)$ is a nilpotent algebra. Since
$$c^*_n(A) \leq c^*_n(B_1) + \cdots + c^*_n(B_m),$$
then there exists B_i such that $c^*_n(B_i) \approx bn^{k-1}$, for some $b > 0$. Being $B_i \in \mathrm{var}^*(A_k)$ by the previous lemma $B_i \sim_{T^*} A_k$. Hence,
$$\mathrm{var}^*(A_k) = \mathrm{var}^*(B_i) \subseteq \mathrm{var}^*(A) \subseteq \mathrm{var}^*(A_k)$$
and $\mathrm{var}^*(A) = \mathrm{var}^*(A_k)$ follows. $\square$

5. Classifying the subvarieties of $\mathrm{var}^*(M)$

The main goal of this section is to completely classify the subvarieties of $\mathrm{var}^*(M)$ by giving a list of generating $*$-algebras. We start with the following.

Lemma 7. *Let $A = F + J_{11} \in var^*(M)$ and $c^*_n(A) \approx qn^{k-1}$ for some $q > 0, k \geq 1$. Then:*

- *if $k = 1$, A is a commutative algebra with trivial involution;*
- *if $k > 1$, either $A \sim_{T^*} U_k$ or $A \sim_{T^*} N_k$ or $A \sim_{T^*} N_k \oplus U_k$.*

Proof. If $k \leq 2$, from [20, Lemma 28] it follows that either $A \sim_{T^*} N_2$ or $A \sim_{T^*} U_2 \sim_{T^*} F$. Let now $k \geq 3$. We remark that at least one polynomial among $[y_1, \ldots, y_{k-1}]$ and $[z, y_1, \ldots, y_{k-2}]$ cannot be a $*$-identity for A, since otherwise we would have $\gamma^*_{k-1}(A) = 0$, a contradiction since $c^*_n(A) \approx qn^{k-1}$.
Suppose first that $[y_1, \ldots, y_{k-1}]$ is not a $*$-identity and $[z, y_1, \ldots, y_{k-2}] \equiv 0$ on A. This implies that $\mathrm{Id}^*(U_k) \subseteq \mathrm{Id}^*(A)$ and, since U_k generates a minimal variety and $c^*_n(A) \approx qn^{k-1}$, one gets that $A \sim_{T_*} U_k$.
Now suppose that $[z, y_1, \ldots, y_{k-2}]$ is not a $*$-identity and $[y_1, \ldots, y_{k-1}] \equiv 0$ on A. Then $\mathrm{Id}^*(N_k) \subseteq \mathrm{Id}^*(A)$ and since N_k generates a minimal variety, as before, one gets $A \sim_{T_*} N_k$.
Finally, suppose that neither of the polynomials $[y_1, \ldots, y_{k-1}]$ and $[z, y_1, \ldots, y_{k-2}]$ are identities for A. Since $c^*_n(A) \approx qn^{k-1}$, then $\gamma^*_k(A) = 0$, so every proper polynomial of degree k belongs to $\mathrm{Id}^*(A)$. In particular $[y_1, \ldots, y_k], [z, y_1, \ldots, y_{k-1}] \in \mathrm{Id}^*(A)$ and, so, $\mathrm{Id}^*(N_k \oplus U_k) \subseteq \mathrm{Id}^*(A)$.

Let $f_1 = [z, \underbrace{y, \ldots, y}_{i-1}]$ and $f_2 = [y_2, \underbrace{y_1, \ldots, y_1}_{i-1}]$ be highest weight vectos corresponding to the partitions $(\lambda, \mu) = ((i-1), (1))$ and $(\lambda, \mu) = ((i-1, 1), \emptyset)$, respectively, for $i = 2, \ldots, k-1$ Since f_1 and f_2 are not $*$-identities for $N_k \oplus U_k$, we get that $\chi_{(i-1),(1)}$ and $\chi_{(i-1,1),\emptyset}$ participate in the i-th proper $*$-cocharacter of $N_k \oplus U_k$ with non-zero multiplicities. Hence, since $\gamma^*_i(N_k \oplus U_k) = 2i - 1 = \deg \chi_{(i-1),(1)} + \deg \chi_{(i-1,1),\emptyset}$, we have that

$$\psi^*_i(N_k \oplus U_k) = \chi_{(i-1),(1)} + \chi_{(i-1,1),\emptyset}, \text{ for } i = 2, \ldots, k-1. \tag{5}$$

If $\psi_i^*(A) = \sum_{\lambda\vdash r\ \mu\vdash i-r} m_{\lambda,\mu}\chi_{\lambda,\mu}$ and $\psi_i^*(N_k\oplus U_k) = \sum_{\lambda\vdash r\ \mu\vdash i-r} m'_{\lambda,\mu}\chi_{\lambda,\mu}$, then it must be $m_{\lambda,\mu}\le m'_{\lambda,\mu}$ for all $\lambda\vdash r$, $\mu\vdash i-r$, $r=0,\dots,i$. Moreover, since $[y_1,\dots,y_{k-1}]$ and $[z,y_1,\dots,y_{k-2}]$ are not $*$-identies for A we must have $\psi_i^*(A)=\chi_{(i-1,1),\emptyset}+\chi_{(i-1),(1)}$ for all $i=2,\dots,k-2$. Hence

$$c_n^*(A)=\sum_{i=0}^{k-1}\binom{n}{i}\gamma_i^*(A)=1+\sum_{i=1}^{k-1}(2i-1)=c_n^*(N_k\oplus U_k).$$

Thus A and $U_k\oplus N_k$ have the same $*$-codimension sequence and, since $\mathrm{Id}^*(N_k\oplus U_k)\subseteq\mathrm{Id}^*(A)$, we finally get the equality $\mathrm{Id}^*(N_k\oplus U_k)=\mathrm{Id}^*(A)$ and $A\sim_{T^*}N_k\oplus U_k$. This completes the proof. □

Remark 3. *Let $A=F+J_{11}+J_{10}+J_{01}+J_{00}\in var^*(M)$. Then $J_{10}J_{01}=J_{01}J_{10}=J_{11}^-J_{10}=J_{01}J_{11}^-=0$.*

Proof. We start by proving that $J_{10}J_{01}=J_{01}J_{10}=0$. Let $a\in J_{10}$, $b\in J_{01}$. Since $z_1z_2\equiv 0$ we get that $(a-a^*)(b-b^*)=0$, and so, $ab=a^*b^*=0$. Now let $a\in J_{11}^-$ and $b\in J_{10}$. From $z_1z_2\equiv 0$ it follows $a(b-b^*)=ab=0$. Similarly, if $b\in J_{01}$ we get $ba=0$. □

Lemma 8. *Let $A=F+J_{11}+J_{10}+J_{01}+J_{00}\in var^*(M)$ with $J_{10}\neq 0$ (hence $J_{01}\neq 0$). Then there exist constants $k,u\ge 2$ such that*

1) *if $J_{11}^-=0$, $A\sim_{T^*}A_k\oplus N$, where N is a nilpotent $*$-algebra;*
2) *if $J_{11}^-\neq 0$ either $A\sim_{T^*}N_u\oplus A_k\oplus N$ or $A\sim_{T^*}U_u\oplus A_k\oplus N$ or $A\sim_{T^*}N_u\oplus U_u\oplus A_k\oplus N$, where N is a nilpotent $*$-algebra.*

Proof. Let $j\ge 0$ be the largest integer such that $J_{10}J_{00}^j\neq 0$ and hence $J_{00}^jJ_{01}\neq 0$. Notice that $j=0$ means that $J_{10}J_{00}=0$ and in this case $A=F+J_{11}+J_{10}+J_{01}\oplus J_{00}$.

We shall prove that either $A\sim_{T^*}A_{j+2}\oplus J_{00}$ or $A\sim_{T^*}A_{j+2}\oplus N_u\oplus J_{00}$ or $A\sim_{T^*}A_{j+2}\oplus U_u\oplus J_{00}$ or $A\sim_{T^*}A_{j+2}\oplus N_u\oplus U_u\oplus J_{00}$ for some $u\ge 2$.

Suppose first that $J_{11}^-=0$.

Let $\bar A=A/J_{00}^{j+1}$. Then it is easily checked that

$$y_1\cdots y_{j+1}zy_{j+2}\cdots y_{2j+2},\ y_1\cdots y_jst_3(y_{j+1},y_{j+2},y_{j+3})y_{j+4}\cdots y_{2j+3}\in\mathrm{Id}^*(\bar A)$$

and, so, by Lemma 1, $\mathrm{Id}^*(A_{j+2})\subseteq\mathrm{Id}^*(\bar A)$.

Moreover, as in the proof of Lemma 6, it is possible to check the opposite inclusion. This says that $A_{j+2}\sim_{T_*}\bar A$ and, so, $A_{j+2}\in\mathrm{var}^*(A)$. It follows that $\mathrm{Id}^*(A)\subseteq\mathrm{Id}^*(A_{j+2})\cap\mathrm{Id}^*(J_{00})=\mathrm{Id}^*(A_{j+2}\oplus J_{00})$.

Conversely, let $f\in\mathrm{Id}^*(A_{j+2}\oplus J_{00})$ be a multilinear polynomial of degree n.

Suppose $n\le 2j+2$. Since $f\in\mathrm{Id}^*(A_{j+2})$, then f must be a consequence of $z_1z_2\in\mathrm{Id}^*(A)$. Hence $f\in\mathrm{Id}^*(A)$.

Now let $n>2j+2$.

We can write f as

$$f=\delta y_1\cdots y_n+\sum_{\substack{t<j+1\\ \text{or}\\ s<j+1}}\sum_{l,I,J}\alpha_{l,I,J}y_{i_1}\cdots y_{i_t}z_ly_{j_1}\cdots y_{j_s}$$
$$+\sum_{\substack{t<j+1\\ or\\ s<j}}\sum_{r,P,Q}\beta_{r,P,Q}y_{p_1}\cdots y_{p_t}[y_r,y_m]y_{q_1}\cdots y_{q_s}+\sum_{i,R,U}\gamma_{i,R,U}y_{r_1}\cdots y_{r_{n-j-2}}[y_1,y_i]y_{u_1}\cdots y_{u_j}$$
$$+\sum_{\substack{t\ge j+1\\ \text{and}\\ s\ge j+1}}\sum_{l',I',J'}\delta_{l',I',J'}y_{i'_1}\cdots y_{i'_t}z_{l'}y_{j'_1}\cdots y_{j'_s}+\sum_{\substack{t\ge j+1\\ and\\ s\ge j+1}}\sum_{r',P',Q'}\varepsilon_{r',P',Q'}y_{p'_1}\cdots y_{p'_t}[y_{r'},y_{m'}]y_{q'_1}\cdots y_{q'_s}+g,\tag{6}$$

where $g\in\langle z_1z_2\rangle_{T_*}$ and $I=\{i_1,\dots,i_t\}$, $J=\{j_1,\dots,j_s\}$, $P=\{p_1,\dots,p_t\}$, $Q=\{m,q_1,\dots,q_s\}$ are such that $I\uplus J\uplus\{l\}=P\uplus Q\uplus\{r\}=\{1,\dots,n\}$, and $i_1<\dots<i_t$, $j_1<\dots<j_s$, $p_1<\ldots<p_t$, $r>m<q_1<\ldots<q_s$. Also $R=\{r_1,\dots,r_{n-j-2}\}$ and $U=\{u_1,\dots,u_j\}$ are such that $R\uplus U\uplus\{i,1\}=\{1,\dots,n\}$ and $r_1<\ldots<r_{n-j-2}$, $u_1<\ldots<u_j$. Also $I'=\{i'_1,\dots,i'_t\}$, $J'=\{j'_1,\dots,j'_s\}$, $P'=\{p'_1,\dots,p'_t\}$, $Q'=\{m',q'_1,\dots,q'_s\}$ are such that $I'\uplus J'\uplus\{l'\}=P'\uplus Q'\uplus\{r',m'\}=\{1,\dots,n\}$, and $i'_1<\dots<i'_t,j'_1<\dots<j'_s$, $p'_1<\ldots<p'_t$, $r'>m'<q'_1<\ldots<q'_s$.

Notice that g and

$$\sum_{\substack{t\geq j+1\\ \text{and}\\ s\geq j+1}}\sum_{l',I',J'}\delta_{l',I',J'}y_{i'_1}\cdots y_{i'_t}z_{l'}y_{j'_1}\cdots y_{j'_s}+\sum_{\substack{t\geq j+1\\ and\\ s\geq j+1}}\sum_{r',P',Q'}\varepsilon_{r',P',Q'}y_{p'_1}\cdots y_{p'_t}[y_{r'},y_{m'}]y_{q'_1}\cdots y_{q'_s}$$

are $*$-identities of A_{j+2}. Hence, since $f\in \mathrm{Id}^*(A_{j+2})$ and the monomials appearing in the the first two rows in (6) are linearly independent modulo $\mathrm{Id}^*(A_{j+2})$ (see Lemma 1), then $\alpha_{l,I,J}=\beta_{r,P,Q}=\gamma_{i,R,U}=0$ for every l,I,J,r,P,Q,i,R,U. Hence

$$f=\sum_{\substack{t\geq j+1\\ \text{and}\\ s\geq j+1}}\sum_{l',I',J'}\delta_{l',I',J'}y_{i'_1}\cdots y_{i'_t}z_{l'}y_{j'_1}\cdots y_{j'_s}+\sum_{\substack{t\geq j+1\\ and\\ s\geq j+1}}\sum_{r',P',Q'}\varepsilon_{r',P',Q'}y_{p'_1}\cdots y_{p'_t}[y_{r'},y_{m'}]y_{q'_1}\cdots y_{q'_s}+g.$$

Since $f\in \mathrm{Id}^*(J_{00})$, if we evaluate f into J_{00} we get a zero value. Since $J_{10}J_{00}^{j+1}=0$ and $J_{11}^-=0$ it is immediate to see that every evaluation of f into A gives a zero value. Hence f is a $*$-identity of A and $\mathrm{Id}^*(A_{j+2}\oplus J_{00})\subseteq \mathrm{Id}^*(A)$. So $A\sim_{T^*}A_{j+2}\oplus J_{00}$ follows.

Suppose now that$J_{11}^-\neq 0$.

Let $B=F+J_{10}+J_{10}+J_{00}$. By Remark 3, B is a subalgebra of A and, since $J_{11}(B)=0$, we can apply the first part of the lemma to B and conclude that $B\sim_{T^*}A_{j+2}\oplus J_{00}$.

Now let $D=F+J_{11}$. By Lemma 7, either $D\sim_{T^*}N_r$ or $D\sim_{T^*}U_r$ or $D\sim_{T^*}N_r\oplus U_r$, for some $r\geq 2$.

We shall prove that $A\sim_{T^*}D\oplus B$ and this will complete the proof.

Let $f\in \mathrm{Id}^*(A)$. Since B and D are subalgebras of A, $f\in \mathrm{Id}^*(D)\cap \mathrm{Id}^*(B)=\mathrm{Id}^*(D\oplus B)$.

Conversely, let $f\in \mathrm{Id}^*(D\oplus B)$ be a multilinear polynomial of degree n.

As above we can write f as $f=f_1+f_2+g_1+g_2$

where

$$f_1=\delta y_1\cdots y_n+\sum_{\substack{t<j+1\\ or\\ s<j}}\sum_{r,P,Q}\beta_{r,P,Q}y_{p_1}\cdots y_{p_t}[y_r,y_m]y_{q_1}\cdots y_{q_s}$$

$$+\sum_{i,R,U}\gamma_{i,R,U}y_{r_1}\cdots y_{r_{n-j-2}}[y_1,y_i]y_{u_1}\cdots y_{u_j}+\sum_{\substack{t\geq j+1\\ and\\ s\geq j+1}}\sum_{r',P',Q'}\varepsilon_{r',P',Q'}y_{p'_1}\cdots y_{p'_t}[y_{r'},y_{m'}]y_{q'_1}\cdots y_{q'_s},$$

$$f_2=\sum_{\substack{t<j+1\\ \text{or}\\ s<j+1}}\sum_{l,I,J}\alpha_{l,I,J}y_{i_1}\cdots y_{i_t}z_ly_{j_1}\cdots y_{j_s}+\sum_{\substack{t\geq j+1\\ \text{and}\\ s\geq j+1}}\sum_{l',I',J'}\delta_{l',I',J'}y_{i'_1}\cdots y_{i'_t}z_{l'}y_{j'_1}\cdots y_{j'_s}$$

and $g_1,g_2\in\langle z_1z_2\rangle_{T_*}$ are polynomials in the only y_is and in one z and $n-1$ y_is, respectively.

By the multihomogeneity of the T^*-ideals we may assume that either $f=f_1+g_1$ or $f=f_2+g_2$.

We start by proving that $f=f_1+g_1\in \mathrm{Id}^*(A)$. Notice that $f\in \mathrm{Id}^*(A_{j+2})$, since $f\in \mathrm{Id}^*(B)$ and $B\sim_{T_*}A_{j+2}\oplus J_{00}$. Also

$$\sum_{\substack{t\geq j+1\\ and\\ s\geq j+1}}\sum_{r',P',Q'}\varepsilon_{r',P',Q'}y_{p'_1}\cdots y_{p'_t}[y_{r'},y_{m'}]y_{q'_1}\cdots y_{q'_s},\ g_1\in \mathrm{Id}^*(A_{j+2}).$$

Hence, since the monomials $y_1\cdots y_n$ and those ones appearing in the second and in the third sum of f_1 are linearly independent modulo $\mathrm{Id}^*(A_{j+2})$ we must have $\delta=\beta_{r,P,Q}=\gamma_{i,R,U}=0$ for all r,P,Q,i,R,U.

Hence

$$f=\sum_{\substack{t\geq j+1\\ and\\ s\geq j+1}}\sum_{r',P',Q'}\varepsilon_{r',P',Q'}y_{p'_1}\cdots y_{p'_t}[y_{r'},y_{m'}]y_{q'_1}\cdots y_{q'_s}+g_1.$$

Since $f\in \mathrm{Id}^*(D\oplus B)$, if we evaluate f into B or into D we get a zero value. Moreover, since $J_{10}J_{00}^{j+2}=J_{00}^{j+2}J_{01}=J_{11}^-J_{10}=0$ it follows that every evaluation of f into A gives a zero value. Hence f is a $*$-identity of A and we are done.

Similarly, if $f = f_2 + g_2$ we get that $\alpha_{l,I,J} = 0$ for all l, I, J. Hence

$$f = \sum_{\substack{t \geq j+1 \\ \text{and} \\ s \geq j+1}} \sum_{l',I',J'} \delta_{l',I',J'} y_{i'_1} \cdots y_{i'_t} z_{l'} y_{j'_1} \cdots y_{j'_s} + g_2,$$

and it easily follows that every evaluation of f into A gives a zero value. Hence f is a $*$-identity of A and we are done. $\square$

Now we are in a position to classify all the subvarieties of $\mathrm{var}^*(M)$.

Theorem 6. *If $A \in var^*(M)$ then A is T^*-equivalent to one of the following algebras:*

$$M, N, U_k \oplus N, N_k \oplus N, N_k \oplus U_k \oplus N, A_t \oplus N, N_k \oplus A_t \oplus N, U_k \oplus A_t \oplus N, N_k \oplus U_k \oplus A_t \oplus N,$$

for some $k, t \geq 2$, where N is a nilpotent $*$-algebra.

Proof. If $A \sim_{T^*} M$ there is nothing to prove. Hence we may assume that A generates a proper subvariety of M and so, since M generates a variety of almost polynomial growth, $c_n^*(A)$ is polynomially bounded. As in the proof of Theorem 5, we may assume that

$$A = B_1 \oplus \cdots \oplus B_m,$$

where $B_1, \ldots, B_m$ are finite dimensional algebras with involution such that $\dim B_i/J(B_i) \leq 1$. Now, if $\dim B_i/J(B_i) = 0$, B_i is nilpotent. Suppose that i is such that $\dim B_i/J(B_i) = 1$. Then $B_i = F + J(B_i)$ and let $J(B_i) = J_{11} + J_{10} + J_{01} + J_{00}$.

If $J_{10} = J_{01} = 0$, then by Lemma 7, A is T*-equivalent to one of the following algebras: $U_k \oplus N$, $N_k \oplus N$, $U_k \oplus N_k \oplus N$, where N is a nilpotent $*$-algebra, for some $k \geq 2$. Otherwise, by Lemma 8 either $A \sim_{T^*} A_k \oplus N$ or $A \sim_{T^*} N_u \oplus A_k \oplus N$ or $A \sim_{T^*} U_u \oplus A_k \oplus N$ or $A \sim_{T^*} N_u \oplus U_u \oplus A_k \oplus N$, for some $k, u \geq 2$, where N is a nilpotent $*$-algebra.

Since $A = B_1 \oplus \cdots \oplus B_m$, by putting together these results we get the desired conclusion. $\square$

As a consequence of Theorems 4 and 5 we get that U_k, N_k and A_k generate the only minimal subvarieties of the variety generated by M.

Corollary 1. *A $*$-algebra $A \in var^*(M)$ generates a minimal variety if and only if either $A \sim_{T^*} U_r$ or $A \sim_{T^*} N_k$ or $A \sim_{T^*} A_k$, for some $r > 2$, $k \geq 2$.*

6. Classifying the subvarieties of $\mathrm{var}^*(F \oplus F)$

In this section we classify, up to T^*-equivalence, all the $*$-algebras contained in the variety generated by the commutative algebra $F \oplus F$ endowed with the exchange involution $(a, b)^* = (b, a)$.

Notice that since $F \oplus F$ is commutative, any antiautomorphism of $F \oplus F$ is an automorphism. So $D = F \oplus F$ can be viewed as a superalgebra with grading $(D^{(0)}, D^{(1)})$, where $D^{(0)} = D^+$ and $D^{(1)} = D^-$.

Hence, the classification of the $*$-algebras, up to T^* -equivalence, inside $\mathrm{var}^*(F \oplus F)$ is equivalent to the classification of the superalgebras inside *supervar*$(F \oplus F)$. Such a classification was given in [17, 19].

In what follows we present such results in the language of $*$-algebras for convenience of the reader.

We start by constructing, for any fixed $k \geq 1$, $*$-algebras belonging to the variety generated by $F \oplus F$ whose $*$-codimension sequence grows polynomially as n^k.

For $k \geq 2$, let I_k be the $k \times k$ g matrix and $E_1 = \sum_{i=1}^{k-1} e_{i,i+1}$.

We denote by

$$C_k = C_k(F) = \{\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F\} \subseteq UT_k,$$

the commutative subalgebra of UT_k with involution given by

$$(\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i)^* = \alpha I_k + \sum_{1 \leq i < k} (-1)^i \alpha_i E_1^i.$$

We next state the following result characterizing the $*$-identities and the $*$-codimensions of C_k (see [17]).

Lemma 9. *Let $k \geq 2$. Then*

1) $Id^*(C_k) = \langle [y_1, y_2], [y, z], [z_1, z_2], z_1 \cdots z_k \rangle_{T^*}$.
2) $c_n^*(C_k) = \sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!} n^{k-1}$.

The following result classifies the subvarieties of the variety generated by $F \oplus F$.

Theorem 7. [17, 19] *Let $A \in var^*(F \oplus F)$. Then either $A \sim_{T^*} F \oplus F$ or $A \sim_{T^*} N$ or $A \sim_{T^*} C \oplus N$ or $A \sim_{T^*} C_k \oplus N$, for some $k \geq 2$, where N is a nilpotent $*$-algebra and C is a commutative algebra with trivial involution.*

Notice that the previous theorem allows us to classify all $*$-codimension sequences of the $*$-algebras lying in the variety generated by $F \oplus F$.

Corollary 2. *Let $A \in var^*(F \oplus F)$ be such that $var^*(A) \subsetneq var^*(F \oplus F)$. Then there exists n_0 such that for all $n > n_0$ we must have either $c_n^*(A) = 0$ or $c_n^*(A) = \sum_{j=0}^{k-1} \binom{n}{j}$, for some $k \geq 0$.*

As a consequence of the previous theorem, we can also classify all $*$-algebras generating minimal varieties (see [17]).

Corollary 3. *A $*$-algebra $A \in var^*(F \oplus F)$ generates a minimal variety if and only if $A \sim_{T^*} C_k$, for some $k \geq 2$.*

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