Reproducing pairs of measurable functions

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Abstract

We analyze the notion of reproducing pair of weakly measurable functions, which generalizes that of continuous frame. We show, in particular, that each reproducing pair generates two Hilbert spaces, conjugate dual to each other. Several examples, both discrete and continuous, are presented.

AMS classification numbers: 41A99, 46Bxx, 46Exx

Keywords: Reproducing pairs; continuous frames; upper and lower semi-frames

1 Introduction

Frames and their relatives are most often considered in the discrete case, for instance in signal processing [15]. However, continuous frames have also been studied and offer interesting mathematical problems. They have been introduced originally by Ali, Gazeau and one of us [1, 2] and also, independently, by Kaiser [22]. Since then, several papers dealt with various aspects of the concept, see for instance [16] or [23]. However, there may occur situations where it is impossible to satisfy both frame bounds.

Therefore, several generalizations of frames have been introduced. The concept of semi-frames [6, 7], for example, is concerned with functions that only satisfy one of the two frame bounds. It turns out that a large portion of frame theory can be extended to this larger framework, in particular the notion of duality.

More recently, a new generalization of frames was introduced by Balazs and one of us [24], namely, reproducing pairs. Here one considers a couple of weakly measurable functions (ψ, ϕ), instead of a single mapping, and one studies what amounts to the correlation between the two (a precise definition is given below). This definition also includes the original definition of a continuous frame [1, 2] given the choice $\psi = \phi$. Moreover, it gives rise to a continuous and invertible analysis/synthesis process without the need of any frame bounds. The increase of freedom in choosing the mappings ψ and ϕ, however, leads to the problem of characterizing the range of the analysis operators.

We will show in Section 3 that this problem can be solved by introducing a pair of intrinsically generated Hilbert spaces, conjugate dual to each other. We discuss in detail the properties of these spaces, in particular, we examine when a given function has a reproducing partner. In Section 6, we exhibit several concrete examples of the construction, both in the discrete and in the continuous cases. In particular, we show that the wavelet upper semi-frame described in [6] does not admit a second mapping to form a reproducing pair.

2 Preliminaries

Before proceeding, we list our definitions and conventions. The framework is a (separable) Hilbert space $\mathcal{H}$, with the inner product $\langle\cdot|\cdot\rangle$ linear in the first factor. Given an operator A on $\mathcal{H}$, we denote its domain by $\mathsf{Dom}A$, its range by $\mathsf{Ran}\,A$ and its kernel by $\mathsf{Ker}\,A$. $GL(\mathcal{H})$ denotes the set of all invertible bounded operators on $\mathcal{H}$ with bounded inverse. Throughout the paper, we will consider weakly measurable functions $\psi : X \to \mathcal{H}$, where (X, μ) is a locally compact space with a Radon measure μ. Then the weakly measurable function ψ is a *continuous frame* if there exist constants $0 < \mathsf{m} \leq \mathsf{M} < \infty$ (the frame bounds) such that

$$\mathsf{m}\,\|f\|^2 \leq \int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) \leq \mathsf{M}\,\|f\|^2 \,, \forall\, f \in \mathcal{H}. \tag{2.1}$$

Given the continuous frame ψ, the *analysis* operator $C_\psi : \mathcal{H} \to L^2(X, \mathrm{d}\mu)$ [1] is defined as

$$(C_\psi f)(x) = \langle f|\psi_x\rangle,\ f \in \mathcal{H}, \tag{2.2}$$

and the corresponding *synthesis operator* $C_\psi^* : L^2(X, \mathrm{d}\mu) \to \mathcal{H}$ as (the integral being understood in the weak sense, as usual)

$$C_\psi^*\xi = \int_X \xi(x)\,\psi_x \,\mathrm{d}\mu(x),\ \text{for}\ \xi \in L^2(X, \mathrm{d}\mu). \tag{2.3}$$

We set $S_\psi := C_\psi^* C_\psi$, which is self-adjoint.

Then it follows that

$$\langle S_\psi f|g\rangle = \langle C_\psi^* C_\psi f|g\rangle = \langle C_\psi f|C_\psi g\rangle = \int_X \langle f|\psi_x\rangle\langle\psi_x|g\rangle \,\mathrm{d}\mu(x).$$

Thus, for continuous frames, S_ψ and S_ψ^{-1} are both bounded, that is, $S_\psi \in GL(\mathcal{H})$.

The weakly measurable function ψ is said to be *μ-total* if $\langle\psi_x|g\rangle = 0$, a.e., implies $g = 0$, that is, $\mathsf{Ker}\,C_\phi = \{0\}$.

Now, in practice, there are situations where the notion of frame is too restrictive, in the sense that one cannot satisfy *both* frame bounds simultaneously. Thus there is room for two natural generalizations. Following [6, 7], we will say that a family ψ is an *upper (resp. lower) semi-frame*, if it is μ-total in $\mathcal{H}$ and satisfies the upper (resp. lower) frame inequality. For the sake of completeness, we recall the definitions. A weakly measurable function ψ is an *upper semi-frame* if there exists $\mathsf{M} < \infty$ such that

$$0 < \int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) \leq \mathsf{M}\,\|f\|^2 \,,\ \forall\, f \in \mathcal{H},\ f \neq 0. \tag{2.4}$$

Note that an upper semi-frame is also called a total Bessel mapping [16]. On the other hand, a function ψ is a *lower semi-frame* if there exists a constant $\mathsf{m} > 0$ such that

$$\mathsf{m}\,\|f\|^2 \leq \int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x),\ \ \forall\, f \in \mathcal{H}. \tag{2.5}$$

Note that the lower frame inequality automatically implies that the family is μ-total. Thus, if ψ is an upper semi-frame and not a frame, S_ψ is bounded and S_ψ^{-1} is unbounded, as follows immediately from (2.4).

[1] As usual, we identify a function ξ with its residue class in $L^2(X, \mathrm{d}\mu)$.

In the lower case, however, the definition of S_ψ must be changed, since C_ψ need not be densely defined, so that C_ψ^* may not exist. Instead, following [6, Sec.2] one defines the synthesis operator as

$$D_\psi F = \int_X F(x)\, \psi_x \;\mathrm{d}\mu(x), \quad F \in L^2(X, \mathrm{d}\mu), \tag{2.6}$$

on the domain of all elements F for which the integral in (2.6) converges weakly in $\mathcal{H}$, and then $S_\psi := D_\psi C_\psi$. With this definition, it is shown in [6, Sec.2] that S_ψ is unbounded and S_ψ^{-1} is bounded.

All these objects are studied in detail in our previous papers [6, 7]. In particular, it is shown there that a natural notion of duality exists, namely, two measurable functions ψ, ϕ are dual to each other (the relation is symmetric) if one has

$$\langle f|g\rangle = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \;\mathrm{d}\mu(x),\ \forall\, f, g \in \mathcal{H}.$$

3 Hilbert spaces generated by a reproducing pair

The couple of weakly measurable functions (ψ, ϕ) is called a *reproducing pair* if

(a) The sesquilinear form

$$\Omega_{\psi,\phi}(f, g) = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\, \mathrm{d}\mu(x) \tag{3.1}$$

is well-defined and bounded on $\mathcal{H} \times \mathcal{H}$, that is, $|\Omega_{\psi,\phi}(f, g)| \leq c\, \|f\|\, \|g\|$, for some $c > 0$.

(b) The corresponding bounded operator $S_{\psi,\phi}$ belongs to $GL(\mathcal{H})$.

Under these hypotheses, one has

$$S_{\psi,\phi} f = \int_X \langle f|\psi_x\rangle\phi_x\, \mathrm{d}\mu(x),\ \forall\, f \in \mathcal{H}, \tag{3.2}$$

the integral on the r.h.s. being defined in weak sense.

If $\psi = \phi$, we recover the notion of continuous frame.

Notice that $S_{\psi,\phi}$ is in general neither positive, nor self-adjoint, since $S_{\psi,\phi}^* = S_{\phi,\psi}$. However, if (ψ, ϕ) is a reproducing pair, then $(\psi, S_{\psi,\phi}^{-1}\phi)$ is also a reproducing pair, for which the corresponding resolution operator is the identity, that is, ψ and ϕ are in duality. Therefore, there is no restriction of generality to assume that $S_{\phi,\psi} = I$ [24]. The worst that can happen is to replace some norms by equivalent ones.

In this section we will study normed spaces constructed from weakly measurable functions and show that for reproducing pairs these spaces enjoy natural duality properties.

3.1 Construction and characterization of the spaces $V_\phi(X, \mu)$

Let ϕ be a weakly measurable function and let us denote by $\mathcal{V}_\phi(X, \mu)$ the space of all measurable functions $\xi : X \to \mathbb{C}$ such that the integral $\int_X \xi(x)\langle\phi_x|g\rangle\, \mathrm{d}\mu(x)$ exists for every $g \in \mathcal{H}$ and defines a bounded conjugate linear functional on $\mathcal{H}$, i.e., $\exists\, c > 0$ such that

$$\left|\int_X \xi(x)\langle\phi_x|g\rangle\, \mathrm{d}\mu(x)\right| \leq c\, \|g\|\ ,\ \forall\, g \in \mathcal{H}. \tag{3.3}$$

Example 3.1 If the sesquilinear form $\Omega_{\psi,\phi}$ defined in (3.1) is bounded, in particular if (ψ, ϕ) is a reproducing pair, it is clear that all functions $\xi(x) = \langle f|\psi_x\rangle$ belong to $\mathcal{V}_\phi(X, \mu)$ since, by assumption,

$$\int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \, \mathrm{d}\mu(x)$$

exists and is bounded.

For every $\xi \in \mathcal{V}_\phi(X, \mu)$, there exists a unique vector $h_{\phi,\xi} \in \mathcal{H}$ such that

$$\int_X \xi(x)\langle\phi_x|g\rangle \, \mathrm{d}\mu(x) = \langle h_{\phi,\xi}|g\rangle, \quad \forall g \in \mathcal{H}.$$

By the Riesz lemma, we can define a linear map

$$T_\phi : \xi \in \mathcal{V}_\phi(X, \mu) \mapsto T_\phi\xi \in \mathcal{H} \tag{3.4}$$

in the following weak sense

$$\langle T_\phi\xi|g\rangle = \langle h_{\phi,\xi}|g\rangle = \int_X \xi(x)\langle\phi_x|g\rangle \, \mathrm{d}\mu(x), \ \forall g \in \mathcal{H}. \tag{3.5}$$

The kernel of T_ϕ and the notion of degeneracy will be studied in more detail in Section 5. Accordingly, we define the following vector space

$$V_\phi(X, \mu) = \mathcal{V}_\phi(X, \mu)/\mathsf{Ker}\ T_\phi.$$

If $\xi \in \mathcal{V}_\phi(X, \mu)$, we put, for short, $[\xi]_\phi = \xi + \mathsf{Ker}\ T_\phi$ and define

$$\|[\xi]_\phi\|_\phi := \sup_{\|g\|\leq 1} \left| \int_X \xi(x)\langle\phi_x|g\rangle \, \mathrm{d}\mu(x) \right|. \tag{3.6}$$

It is easy to see that the left hand side does not depend on the particular representative of $[\xi]_\phi$.

The following result is immediate.

Proposition 3.2 *Let ϕ be a weakly measurable function. Then $V_\phi(X, \mu)$ is a normed space with respect to $\|\cdot\|_\phi$ and the map $\widehat{T}_\phi : V_\phi(X, \mu) \to \mathcal{H}$, $\widehat{T}_\phi[\xi]_\phi := T_\phi\xi$ is a well-defined isometry of $V_\phi(X, \mu)$ into $\mathcal{H}$.*

Since $\widehat{T}_\phi : V_\phi(X, \mu) \to \mathcal{H}$ is an isometry, we can define on $V_\phi(X, \mu)$ an inner product by setting

$$\langle[\xi]_\phi|[\eta]_\phi\rangle_{(\phi)} := \langle\widehat{T}_\phi[\xi]_\phi|\widehat{T}_\phi[\eta]_\phi\rangle, \ [\xi]_\phi, [\eta]_\phi \in V_\phi(X, \mu).$$

Using (3.5), we get, more explicitly

$$\begin{aligned}\langle[\xi]_\phi|[\eta]_\phi\rangle_{(\phi)} &= \int_X \xi(x) \left(\int_X \overline{\eta(y)}\langle\phi_x|\phi_y\rangle \, \mathrm{d}\mu(y) \right) \mathrm{d}\mu(x) \\ &= \int_X \overline{\eta(y)} \left(\int_X \xi(x)\langle\phi_x|\phi_y\rangle \, \mathrm{d}\mu(x) \right) \mathrm{d}\mu(y).\end{aligned}$$

Then the norm defined by $\langle\cdot|\cdot\rangle_{(\phi)}$ coincides with the norm $\|\cdot\|_\phi$ defined in (3.6), since one has

$$\|[\xi]_\phi\|_{(\phi)} = \left\|\widehat{T}_\phi[\xi]_\phi\right\| = \|T_\phi\xi\| = \sup_{\|g\|\leq 1} |\langle T_\phi\xi|g\rangle| = \|[\xi]_\phi\|_\phi \, .$$

Thus $V_\phi(X,\mu)$ is an inner product (pre-Hilbert) space.

Let us denote by $V_\phi(X,\mu)^*$ the Hilbert dual space of $V_\phi(X,\mu)$, that is, the set of continuous linear functionals on $V_\phi(X,\mu)$. The norm $\|\cdot\|_{\phi^*}$ of $V_\phi(X,\mu)^*$ is defined, as usual, by

$$\|F\|_{\phi^*} = \sup_{\|[\xi]_\phi\|_\phi \leq 1} |F([\xi]_\phi)|.$$

Now we define a conjugate linear map $\mathsf{C}_\phi : \mathcal{H} \to V_\phi(X,\mu)^*$ by

$$(\mathsf{C}_\phi f)([\xi]_\phi) := \int_X \xi(x)\langle \phi_x|f\rangle \,\mathrm{d}\mu(x),\ f \in \mathcal{H}, \tag{3.7}$$

which will take the role of the analysis operator C_ϕ of Section 2. Notice that C_ϕ is a *linear* map, whereas C_ϕ is conjugate linear, hence the difference in notation. The discrepancy is explained in Remark 3.15 below.

Of course, (3.7) means that $(\mathsf{C}_\phi f)([\xi]_\phi) = \langle T_\phi \xi|f\rangle = \langle \widehat{T}_\phi[\xi]_\phi|f\rangle$, for every $f \in \mathcal{H}$. Thus $\mathsf{C}_\phi = \widehat{T}_\phi^*$, the adjoint map of $\widehat{T}_\phi$. By (3.3) it follows that C_ϕ is continuous . This implies that

$$\mathcal{H} = [\mathsf{Ran}\,\widehat{T}_\phi]^\sim \oplus \mathsf{Ker}\,\mathsf{C}_\phi, \tag{3.8}$$

where the first summand denotes the closure of $\mathsf{Ran}\,\widehat{T}_\phi$. Hence $\mathsf{C}_\phi^* = \widehat{T}_\phi^{**} = \widehat{T}_\phi$, if $V_\phi(X,\mu)$ is complete.

By modifying in an obvious way the definition given in Section 2, we say that ϕ is *μ-total* if $\mathsf{Ker}\,\mathsf{C}_\phi = \{0\}$.

Proposition 3.3 *The following statements are equivalent.*

(i) $V_\phi(X,\mu)[\langle\cdot|\cdot\rangle_{(\phi)}]$ is a Hilbert space.

(ii) $\widehat{T}_\phi$ has closed range.

Proof. (i)⇒(ii): Since $V_\phi(X,\mu)$ is complete and $\widehat{T}_\phi$ is an isometry, $\mathsf{Ran}\,\widehat{T}_\phi$ is also complete. (ii)⇒(i): Let $\widehat{T}_\phi$ have closed range. Then $\widehat{T}_\phi : V_\phi(X,\mu) \to \mathsf{Ran}\,\widehat{T}_\phi$ is isometric with isometric inverse. Hence, $V_\phi(X,\mu) = \widehat{T}_\phi^{-1}(\mathsf{Ran}\,\widehat{T}_\phi)$ is the isometric image of a complete space, and therefore it is complete. □

As a consequence of (3.8) we get

Corollary 3.4 *The following statements hold.*

(i) A weakly measurable function ϕ is μ-total if and only if $\mathsf{Ran}\,\widehat{T}_\phi$ is dense in $\mathcal{H}$.

(ii) If $V_\phi(X,\mu)$ is a Hilbert space, $\mathsf{Ran}\,\widehat{T}_\phi$ is equal to $\mathcal{H}$ if and only if ϕ is μ-total.

Lemma 3.5 *If (ψ,ϕ) is a reproducing pair, then $\mathsf{Ran}\,\widehat{T}_\phi = \mathcal{H}$.*

Proof. Since $S_{\psi,\phi} \in GL(\mathcal{H})$, for every $h \in \mathcal{H}$, there exists a unique $f \in \mathcal{H}$ such that $S_{\psi,\phi} f = h$. But, by (3.2), we get

$$h = \int_X \langle f|\psi_x\rangle \phi_x \,\mathrm{d}\mu(x),$$

so that

$$\langle h|g\rangle = \int_X \langle f|\psi_x\rangle\langle \phi_x|g\rangle \,\mathrm{d}\mu(x),\ \forall\, g \in \mathcal{H},$$

that is, $h = \widehat{T}_\phi[C_\psi f]_\phi$. □

Notice that, if (ψ,ϕ) is a reproducing pair, both functions are necessarily μ-total.

3.2 Duality properties of the spaces $V_\phi(X,\mu)$

If the space $V_\phi(X,\mu)$ is complete, *i.e.,* a Hilbert space, it is certainly isomorphic to its dual, via the Riesz operator. Nevertheless if (ψ,ϕ) is a reproducing pair, the dual of $V_\phi(X,\mu)$ can be identified with $V_\psi(X,\mu)$ as we shall prove below. We emphasize that the duality is taken with respect to the sesquilinear form

$$\langle\!\langle \xi|\eta \rangle\!\rangle_\mu := \int_X \xi(x)\overline{\eta(x)}\,\mathrm{d}\mu(x), \tag{3.9}$$

which coincides with the inner product of $L^2(X,\mu)$ whenever the latter makes sense.

Theorem 3.6 *Let ϕ be a weakly measurable function. If F is a continuous linear functional on $V_\phi(X,\mu)$, then there exists a unique $g \in [\mathcal{M}_\phi]^\sim$, the closure of the range of $\widehat{T}_\phi$, such that*

$$F([\xi]_\phi) = \int_X \xi(x)\langle \phi_x|g\rangle\,\mathrm{d}\mu(x),\ \forall\, \xi \in \mathcal{V}_\phi(X,\mu) \tag{3.10}$$

and $\|F\|_{\phi^*} = \|g\|$, where $\|\cdot\|_{\phi^*}$ denotes the (dual) norm on $V_\phi(X,\mu)^*$. Moreover, every $g \in \mathcal{H}$ defines a continuous functional F on $V_\phi(X,\mu)$ with $\|F\|_{\phi^*} \le \|g\|$, by (3.10)*. In particular, if $g \in \mathsf{Ran}\,\widehat{T}_\phi$, then $\|F\|_{\phi^*} = \|g\|$.*

Proof. Let $F \in V_\phi(X,\mu)^*$. Then, there exists $c>0$ such that

$$|F([\xi]_\phi)| \le c\,\|[\xi]_\phi\|_\phi = c\,\|T_\phi\xi\|\,,\ \forall\, \xi \in \mathcal{V}_\phi(X,\mu).$$

Let $\mathcal{M}_\phi := \{T_\phi\xi : \xi \in \mathcal{V}_\phi(X,\mu)\} = \mathsf{Ran}\,\widehat{T}_\phi$. Then $\mathcal{M}_\phi$ is a vector subspace of $\mathcal{H}$, with closure $[\mathcal{M}_\phi]^\sim$.

Let $\widetilde{F}$ be the functional defined on $\mathcal{M}_\phi$ by

$$\widetilde{F}(T_\phi\xi) := F([\xi]_\phi),\ \xi \in \mathcal{V}_\phi(X,\mu).$$

We notice that $\widetilde{F}$ is well-defined. Indeed, if $T_\phi\xi = T_\phi\xi'$, then $\xi-\xi' \in \mathsf{Ker}\ T_\phi$. Hence, $[\xi]_\phi = [\xi']_\phi$ and $F([\xi]_\phi) = F([\xi']_\phi)$

Hence, $\widetilde{F}$ is a continuous linear functional on $\mathcal{M}_\phi$. Thus there exists a unique $g \in [\mathcal{M}_\phi]^\sim$, the closure of the range of $\widehat{T}_\phi$, such that

$$\widetilde{F}(T_\phi\xi) = \langle \widehat{T}_\phi[\xi]_\phi|g\rangle = \int_X \xi(x)\langle\phi_x|g\rangle\,\mathrm{d}\mu(x)$$

and $\|g\| = \|\widetilde{F}\|$.

In conclusion,

$$F([\xi]_\phi) = \int_X \xi(x)\langle \phi_x|g\rangle\,\mathrm{d}\mu(x),\ \forall\, \xi \in \mathcal{V}_\phi(X,\mu)$$

and $\|F\|_{\phi^*} = \|g\|$.

Moreover, every $g \in \mathcal{H}$ obviously defines a continuous linear functional F by (3.10) as $|F([\xi]_\phi)| \le \|g\|\,\|[\xi]_\phi\|_\phi$. This inequality implies that $\|F\|_{\phi^*} \le \|g\|$. In particular, if $g \in \mathsf{Ran}\,\widehat{T}_\phi$, then there exists $[\xi]_\phi \in \mathcal{V}_\phi(X,\mu)$, $\|[\xi]_\phi\|_\phi = 1$, such that $\widehat{T}_\phi[\xi]_\phi = g\|g\|^{-1}$. Hence $F([\xi]_\phi) = \langle\widehat{T}_\phi[\xi]_\phi|g\rangle = \|g\|$. This concludes the proof. □

Corollary 3.7 *Let ϕ be a μ-total weakly measurable function, then $\mathsf{C}_\phi : \mathcal{H} \to V_\phi(X,\mu)^*$ is an isometric isomorphism.*

Proof. C_ϕ is surjective by Theorem 3.6. As ϕ is μ-total, it follows by Corollary 3.4 that $\mathsf{Ran}\,\widehat{T}_\phi$ is dense in $\mathcal{H}$. Consequently, for $f \in \mathcal{H}$ it follows that

$$\|\mathsf{C}_\phi f\|_{\phi^*} = \sup_{\|[\xi]_\phi\|_\phi = 1} \left| \int_X \xi(x) \langle \phi_x | f \rangle \, \mathrm{d}\mu(x) \right| = \sup_{\|[\xi]_\phi\|_\phi = 1} |\langle \widehat{T}_\phi \xi | f \rangle| = \sup_{\|g\|=1,\ g \in \mathsf{Ran}\,\widehat{T}_\phi} |\langle g | f \rangle| = \|f\|.$$

□

Remark 3.8 It turns out that C_ϕ being an isometric isomorphism is not sufficient to guarantee that $V_\psi(X,\mu)$ is complete. We will see a counterexample in Sec. 6.2.3.

Theorem 3.9 *If (ψ, ϕ) is a reproducing pair, then every bounded linear functional F on $V_\phi(X,\mu)$, i.e., $F \in V_\phi(X,\mu)^*$, can be represented as*

$$F([\xi]_\phi) = \int_X \xi(x) \overline{\eta(x)} \, \mathrm{d}\mu(x), \ \forall\, [\xi]_\phi \in V_\phi(X,\mu), \tag{3.11}$$

with $\eta \in \mathcal{V}_\psi(X,\mu)$. The residue class $[\eta]_\psi \in V_\psi(X,\mu)$ is uniquely determined.

Proof. By Theorem 3.6, we have the representation

$$F(\xi) = \int_X \xi(x) \langle \phi_x | g \rangle \, \mathrm{d}\mu(x).$$

It is easily seen that $\eta(x) = \langle g | \phi_x \rangle \in \mathcal{V}_\psi(X,\mu)$.

It remains to prove uniqueness. Suppose that

$$F(\xi) = \int_X \xi(x) \overline{\eta'(x)} \, \mathrm{d}\mu(x).$$

Then

$$\int_X \xi(x) (\overline{\eta'(x)} - \overline{\eta(x)}) \, \mathrm{d}\mu(x) = 0.$$

Now the function $\xi(x)$ is arbitrary. Hence, taking in particular for $\xi(x)$ the functions $\langle f | \psi_x \rangle \in \mathcal{V}(X,\mu)$, $f \in \mathcal{H}$, we get $[\eta]_\psi = [\eta']_\psi$. □

The lesson of the previous statements is that the map

$$j : F \in V_\phi(X,\mu)^* \mapsto [\eta]_\psi \in V_\psi(X,\mu) \tag{3.12}$$

is well-defined and conjugate linear. On the other hand, $j(F) = j(F')$ implies easily $F = F'$. Therefore $V_\phi(X,\mu)^*$ can be identified with a closed subspace of $\overline{V_\psi}(X,\mu) := \{\overline{[\xi]_\psi} : \xi \in \mathcal{V}_\psi(X,\mu)\}$.

Now we want to prove that the spaces $V_\phi(X,\mu)^*$ and $\overline{V_\psi}(X,\mu)$ can be identified. First, corresponding to $\widehat{T}_\phi$, we introduce the linear operator $\widehat{C}_{\psi,\phi} : \mathcal{H} \to V_\phi(X,\mu)$ by $\widehat{C}_{\psi,\phi} f := [C_\psi f]_\phi$. We note that the construction can distinguish the equivalence classes generated by the analysis operator. Indeed, we have $\widehat{C}_{\psi,\phi} f = \widehat{C}_{\psi,\phi} f'$ if and only if $f = f'$. To see this, let $\widehat{C}_{\psi,\phi} f = \widehat{C}_{\psi,\phi} f'$. Then

$$0 = \int_X \langle f - f' | \psi_x \rangle \langle \phi_x | g \rangle \, \mathrm{d}\mu(x) = \langle S_{\psi,\phi}(f - f') | g \rangle, \ \forall g \in \mathcal{H}. \tag{3.13}$$

Since $S_{\psi,\phi} \in GL(\mathcal{H})$, it follows that $f = f'$.

For proving that the spaces $V_\phi(X,\mu)^*$ and $\overline{V_\psi}(X,\mu)$ can be identified. we will first need two auxiliary lemmas.

Lemma 3.10 *Let (ψ,ϕ) be a reproducing pair. Then* $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}$ *is closed in* $V_\phi(X,\mu)[\|\cdot\|_\phi]$. *In particular, there exist* $\mathsf{m},\mathsf{M}>0$ *such that*

$$\mathsf{m}\,\|f\| \leq \left\|\widehat{C}_{\psi,\phi}f\right\|_\phi \leq \mathsf{M}\,\|f\|\,,\ \forall\, f\in\mathcal{H}. \tag{3.14}$$

Moreover, every $[\eta]_\psi \in V_\psi(X,\mu)$ *defines a bounded linear functional on the closed subspace* $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}[\|\cdot\|_\phi]$.

Proof. Since $S_{\psi,\phi}\in GL(\mathcal{H})$, we have, for $f\in\mathcal{H}$,

$$\begin{aligned}\|f\| &\leq \left\|S_{\psi,\phi}^{-1}\right\|\,\|S_{\psi,\phi}f\| = \left\|S_{\psi,\phi}^{-1}\right\|\,\sup_{\|g\|\leq 1}|\langle S_{\psi,\phi}f|g\rangle| \\ &= \left\|S_{\psi,\phi}^{-1}\right\|\,\sup_{\|g\|\leq 1}\left|\int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\,\mathrm{d}\mu(x)\right| = \left\|S_{\psi,\phi}^{-1}\right\|\,\|[\langle f|\psi(\cdot)\rangle]_\phi\|_\phi = \left\|S_{\psi,\phi}^{-1}\right\|\,\left\|\widehat{C}_{\psi,\phi}f\right\|_\phi.\end{aligned}$$

This relation implies that $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}$ is closed in $V_\phi(X,\mu)[\|\cdot\|_\phi]$. On the other hand,

$$\begin{aligned}\left\|\widehat{C}_{\psi,\phi}f\right\|_\phi &= \sup_{\|g\|\leq 1}\left|\int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\,\mathrm{d}\mu(x)\right| \\ &= \sup_{\|g\|\leq 1}|\langle S_{\psi,\phi}f|g\rangle| = \|S_{\psi,\phi}f\| \leq \|S_{\psi,\phi}\|\,\|f\|\,.\end{aligned}$$

Next, let $\eta\in V_\psi(X,\mu)$. Then, by definition, $\int_X\langle f|\psi_x\rangle\overline{\eta(x)}\,\mathrm{d}\mu(x)$ exists and defines a bounded linear functional on $\mathcal{H}$, i.e.,

$$\left|\int_X \langle f|\psi_x\rangle\overline{\eta(x)}\,\mathrm{d}\mu(x)\right| \leq c\,\|f\|\,,\ \forall\, f\in\mathcal{H}.$$

By the definition of $\|\cdot\|_\psi$, we have, more precisely,

$$\left|\int_X \langle f|\psi_x\rangle\overline{\eta(x)}\,\mathrm{d}\mu(x)\right| \leq \|f\|\,\|\eta\|_\psi\,,\ \forall\, f\in\mathcal{H}.$$

Hence,

$$\left|\int_X \langle f|\psi_x\rangle\overline{\eta(x)}\,\mathrm{d}\mu(x)\right| \leq \left\|S_{\psi,\phi}^{-1}\right\|\,\left\|\widehat{C}_{\psi,\phi}f\right\|_\phi\,\|\eta\|_\psi\,,\ \forall\, f\in\mathcal{H},\eta\in\mathcal{V}_\psi(X,\mu).$$

Thus, by (3.11), $[\eta]_\psi$ defines a bounded linear functional on the space $\mathsf{Ran}\,\widehat{C}_{\psi,\phi} = \mathsf{Ran}\,C_\psi/\mathsf{Ker}\,T_\phi$. □

If (ψ,ϕ) is a reproducing pair and $\left\|\widehat{C}_{\psi,\phi}f\right\|_\phi = \|f\|$, then $S_{\psi,\phi}$ is an isometry, since one has, for every $f\in\mathcal{H}$,

$$\|f\| = \sup_{\|g\|=1}\left|\int_X \langle f|\psi(x)\rangle\langle\phi_x|g\rangle\,\mathrm{d}\mu(x)\right| = \sup_{\|g\|=1}|\langle S_{\psi,\phi}f|g\rangle| = \|S_{\psi,\phi}f\|\,.$$

Lemma 3.11 *Let (ψ,ϕ) be a reproducing pair. Then* $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}$ *is dense in* $V_\phi(X,\mu)$.

Proof. Were it not so, there would be a nonzero $F \in V_\phi(X,\mu)^*$ such that $F(\langle f|\psi(\cdot)\rangle) = 0$ for every $f \in \mathcal{H}$. By Theorem 3.6, there exists $g \in \mathcal{H}\backslash\{0\}$, such that

$$F(\xi) = \int_X \xi(x)\langle \phi_x|g\rangle \, \mathrm{d}\mu(x), \ \forall\, \xi \in V_\phi(X,\mu).$$

Then,

$$F(\langle f|\psi(\cdot)\rangle) = \int_X \langle f|\psi(x)\rangle\langle \phi_x|g\rangle \, \mathrm{d}\mu(x) = 0, \ \forall\, f \in \mathcal{H}.$$

This implies that $\langle S_{\psi,\phi} f|g\rangle = 0$, for every $f \in \mathcal{H}$. This in turn implies that $g = 0$, which is a contradiction. $\square$

Theorem 3.12 *If (ψ,ϕ) is a reproducing pair, the map j defined in* (3.12) *is surjective. Hence, $V_\phi(X,\mu)^* \simeq \overline{V_\psi}(X,\mu)$, where $\simeq$ denotes a bounded isomorphism and the norm $\|\cdot\|_\psi$ is the dual norm of $\|\cdot\|_\phi$. Moreover, $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}[\|\cdot\|_\phi] = V_\phi(X,\mu)[\|\cdot\|_\phi]$ and $\mathsf{Ran}\,\widehat{C}_{\phi,\psi}[\|\cdot\|_\psi] = V_\psi(X,\mu)[\|\cdot\|_\psi]$.*

Proof. By Lemma 3.10, $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}$ is closed in $V_\phi(X,\mu)[\|\cdot\|_\phi]$. By Lemma 3.11, it is also dense. Hence, $\mathsf{Ran}\,\widehat{C}_{\psi,\phi}[\|\cdot\|_\phi]$ and $V_\phi(X,\mu)[\|\cdot\|_\phi]$ coincide. Now, the map j is surjective as every $\eta \in V_\psi$ defines a bounded linear functional on $V_\phi(X,\mu)[\|\cdot\|_\phi]$. $\square$

By Theorems 3.9 and 3.12, it follows that, if (ψ,ϕ) is a reproducing pair, then for every $\eta \in V_\psi(X,\mu)$, there exists $g \in \mathcal{H}$ such that $\eta = \langle\phi(\cdot)|g\rangle$.

In conclusion, we may state

Theorem 3.13 *If (ψ,ϕ) is a reproducing pair, the spaces $V_\phi(X,\mu)$ and $V_\psi(X,\mu)$ are both Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form* (3.9).

Corollary 3.14 *If (ψ,ϕ) is a reproducing pair and $\phi = \psi$, then ψ is a continuous frame and $V_\psi(X,\mu)$ is a closed subspace of $L^2(X,\mu)$.*

Proof. Since the duality takes place with respect to the L^2 inner product, $V_\psi(X,\mu)$ is a subspace of $L^2(X,\mu)$. The equality $\mathsf{Ran}\,\widehat{C}_{\psi,\psi} = V_\psi(X,\mu)$ and the fact that $\widehat{C}_{\psi,\psi}$ is bounded from below with respect to the L^2-norm imply that it is closed. $\square$

Remark 3.15 The operator C_ϕ defined by (2.2) is linear, but the operator C_ϕ given in (3.7) is conjugate linear. However the latter maps $\mathcal{H}$ into $V_\phi(X,\mu)^*$, which is identified with $\overline{V_\psi}(X,\mu)$, thus C_ϕ maps $\mathcal{H}$ *linearly* into $V_\psi(X,\mu)$.

Actually Theorem 3.13 has an inverse. Indeed:

Theorem 3.16 *Let ϕ and ψ be weakly measurable and μ-total. Then, the couple (ψ,ϕ) is a reproducing pair if and only if $V_\phi(X,\mu)$ and $V_\psi(X,\mu)$ are Hilbert spaces, conjugate dual of each other with respect to the sesquilinear form* (3.9).

Proof. The 'if' part is Theorem 3.13. Let now $V_\phi(X,\mu)$ and $V_\psi(X,\mu)$ be Hilbert spaces in conjugate duality. Consider the sesquilinear form

$$\Omega_{\psi,\phi}(f,g) = \int_X \langle f|\psi_x\rangle\langle \phi_x|g\rangle \, \mathrm{d}\mu(x), \ f,g \in \mathcal{H}.$$

By the definition of the norms $\|\cdot\|_\phi, \|\cdot\|_\psi$ and the duality condition, we have, for every $f, g \in \mathcal{H}$, the two inequalities

$$|\Omega_{\psi,\phi}(f,g)| \leq \|[\langle f|\psi(\cdot)\rangle]_\phi\|_\phi \|g\|,$$
$$|\Omega_{\psi,\phi}(f,g)| \leq \|[\langle g|\phi(\cdot)\rangle]_\psi\|_\psi \|f\|.$$

This means the form $\Omega_{\psi,\phi}$ is separately continuous, hence continuous. Therefore there exists a bounded operator $S_{\psi,\phi}$ such that $\Omega_{\psi,\phi}(f,g) = \langle S_{\psi,\phi}f|g\rangle$. First the operator $S_{\psi,\phi}$ is injective. Indeed, by definition of C_ϕ, we have

$$\langle S_{\psi,\phi}f|g\rangle = \langle\!\langle C_\psi f|C_\phi g\rangle\!\rangle_\mu = (\mathsf{C}_\phi g)\big(\widehat{C}_{\psi,\phi}f\big) = \langle \widehat{T}_\phi \widehat{C}_{\psi,\phi}f|g\rangle, \ \forall f,g \in \mathcal{H}.$$

Now $\widehat{T}_\phi$ is isometric and $\widehat{C}_{\psi,\phi}$ is injective, hence $\widehat{T}_\phi \widehat{C}_{\psi,\phi}f = 0$ implies $f = 0$. Next, $S_{\psi,\phi}$ is also surjective, by Corollary 3.4. Hence $S_{\psi,\phi}$ belongs to $GL(\mathcal{H})$. □

Remark 3.17 If the couple (ψ,ϕ) is a reproducing pair, then $V_\phi(X,\mu)$ and $V_\psi(X,\mu)$ are Hilbert spaces, conjugate dual of each other with respect to $\langle\!\langle\cdot|\cdot\rangle\!\rangle_\mu$. Thus, every $[\eta]_\psi \in V_\psi(X,\mu)$ determines a linear functional F_η on $V_\phi(X,\mu)$ by

$$F_\eta([\xi]_\phi) = \int_X \xi(x)\overline{\eta(x)}\, \mathrm{d}\mu(x) = \langle\!\langle \xi|\eta\rangle\!\rangle_\mu.$$

On the other hand (Riesz's lemma) there exists a unique $[\eta']_\phi \in V_\phi(X,\mu)$ such that

$$F_\eta([\xi]_\phi) = \langle [\xi]_\phi|[\eta']_\phi\rangle_{(\phi)} = \int_X \xi(x)\left(\int_X \overline{\eta'(y)}\langle \phi_x|\phi_y\rangle\, \mathrm{d}\mu(y)\right)\mathrm{d}\mu(x).$$

Define $N : [\eta]_\psi \in V_\psi(X,\mu) \to [\eta']_\phi \in V_\phi(X,\mu)$. Then,

$$\langle\!\langle \xi|\eta\rangle\!\rangle_\mu = \langle [\xi]_\phi|N[\eta]_\psi\rangle_{(\phi)}, \quad \forall [\xi]_\phi \in V_\phi(X,\mu), [\eta]_\psi \in V_\psi(X,\mu).$$

In the very same way we can define an operator $M : V_\phi(X,\mu) \to V_\psi(X,\mu)$ such that

$$\langle\!\langle \xi|\eta\rangle\!\rangle_\mu = \langle M[\xi]_\phi|[\eta]_\psi\rangle_{(\psi)}, \quad \forall [\xi]_\phi \in V_\phi(X,\mu), [\eta]_\psi \in V_\psi(X,\mu).$$

Then it is clear that $N^* = M$. Moreover, N is isometric. Hence, $N^* = N^{-1} = M$. From the above equalities we get an explicit form for N^{-1}

$$(N^{-1}[\eta']_\phi)(x) = \int_X \eta'(y)\langle \phi_y|\phi_x\rangle\, \mathrm{d}\mu(y).$$

In addition to Lemma 3.12, there is another characterization of the space $V_\psi(X,\mu)$, in terms of an eigenvalue equation, based on the fact that $\langle S_{\psi,\phi}^{-1}\phi_y|\psi_x\rangle$ is a reproducing kernel [24, Prop.3].

Proposition 3.18 *Let (ψ,ϕ) be a reproducing pair. Let $\xi \in \mathcal{V}_\psi(X,\mu)$ and consider the eigenvalue equation*

$$\int_X \xi(y)\langle S_{\psi,\phi}^{-1}\phi_y|\psi_x\rangle\, \mathrm{d}\mu(y) = \lambda\xi(x). \tag{3.15}$$

Then $\xi \in \mathsf{Ran}\, C_\psi \ \Leftrightarrow\ \lambda = 1$ and $\xi \in \mathsf{Ker}\, T_\phi \ \Leftrightarrow\ \lambda = 0$. Moreover, there are no other eigenvalues.

4 Existence of reproducing partners

Next we present a criterion towards the existence of a specific dual partner to a given measurable function. We remind that the basic sesquilinear form $\langle\!\langle \cdot|\cdot \rangle\!\rangle_\mu$ is given by (3.9).

Theorem 4.1 *Let ϕ be a weakly measurable function and $e = \{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis of $\mathcal{H}$. There exists another measurable function ψ, such that (ψ, ϕ) is a reproducing pair if and only if $\mathsf{Ran}\, \widehat{T}_\phi = \mathcal{H}$ and there exists a family $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{V}_\phi(X, \mu)$ such that*

$$[\xi_n]_\phi = [\widehat{T}_\phi^{-1} e_n]_\phi,\ \forall n \in \mathbb{N}, \quad \textit{and} \quad \sum_{n\in\mathbb{N}} |\xi_n(x)|^2 < \infty,\ \textit{for a.e. } x \in X. \tag{4.1}$$

Proof. If $\mathsf{Ran}\, \widehat{T}_\phi = \mathcal{H}$, then $V_\phi(X, \mu)$ is a Hilbert space, $\mathsf{C}_\phi : \mathcal{H} \to V_\phi^*(X, \mu)$ is an isometric isomorphism and $\mathsf{C}_\phi^* = \widehat{T}_\phi$. Hence, for $f, g \in \mathcal{H}$, one has

$$\begin{aligned} \langle f|g\rangle &= \langle f|\mathsf{C}_\phi^{-1}\mathsf{C}_\phi g\rangle = (\mathsf{C}_\phi g)\big((\mathsf{C}_\phi^{-1})^* f\big) = \langle\!\langle (\mathsf{C}_\phi^{-1})^*\Big(\sum_{n\in\mathbb{N}} \langle f|e_n\rangle e_n\Big) | C_\phi g\rangle\!\rangle_\mu \\ &= \langle\!\langle \sum_{n\in\mathbb{N}} \langle f|e_n\rangle (\mathsf{C}_\phi^{-1})^* e_n | C_\phi g\rangle\!\rangle_\mu = \langle\!\langle \sum_{n\in\mathbb{N}} \langle f|e_n\rangle \widehat{T}_\phi^{-1} e_n | C_\phi g\rangle\!\rangle_\mu \end{aligned} \tag{4.2}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$.

Let (ψ, ϕ) be a reproducing pair. As $S_{\psi,\phi} \in GL(\mathcal{H})$, it immediately follows that $\mathsf{Ran}\, \widehat{T}_\phi = \mathcal{H}$ and thus (4.2) holds. For the sake of simplicity assume that $S_{\psi,\phi} = I$. Using (4.2) we get

$$\langle\!\langle C_\psi f|C_\phi g\rangle\!\rangle_\mu = \langle\!\langle \sum_{n\in\mathbb{N}} \widehat{T}_\phi^{-1} e_n \langle f|e_n\rangle | C_\phi g\rangle\!\rangle_\mu,\ \forall\, f, g \in \mathcal{H},$$

and, consequently,

$$[C_\psi f]_\phi = \Big[\sum_{n\in\mathbb{N}} \langle f|e_n\rangle C_\psi e_n\Big]_\phi = \Big[\sum_{n\in\mathbb{N}} \langle f|e_n\rangle \widehat{T}_\phi^{-1} e_n\Big]_\phi,\ \forall\, f \in \mathcal{H}.$$

In particular, the choice $f = e_n$ implies $[C_\psi e_n]_\phi = [\widehat{T}_\phi^{-1} e_n]_\phi$, $\forall n \in \mathbb{N}$. Moreover, $C_\psi e(x) := \{C_\psi e_n(x)\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ for almost every $x \in X$, since $\|C_\psi e(x)\|_{\ell^2} = \|\psi_x\|$.

Conversely, if $\mathsf{Ran}\, \widehat{T}_\phi = \mathcal{H}$ the following holds weakly by (4.2)

$$f = \int_X C_\phi f(x) \Big(\sum_{n\in\mathbb{N}} \overline{\widehat{T}_\phi^{-1} e_n(x)} e_n\Big)\, \mathrm{d}\mu(x),\ \forall\, f \in \mathcal{H}.$$

By (4.1) we can find $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{V}_\phi(X, \mu)$ such that

$$f = \int_X C_\phi f(x) \Big(\sum_{n\in\mathbb{N}} \overline{\xi_n(x)} e_n\Big)\, \mathrm{d}\mu(x),\ \forall\, f \in \mathcal{H},$$

holds weakly and $\psi_x := \sum_{n\in\mathbb{N}} \overline{\xi_n(x)} e_n$ is a well defined vector in $\mathcal{H}$ for almost every $x \in X$. $\square$

Remark 4.2 If ϕ is in fact a frame, then the reproducing partner ψ given by the proof of Theorem 4.1 is also a frame. To see this, we first observe that if ψ is an upper semi-frame (Bessel mapping), then its reproducing partner ϕ is necessarily a lower semi-frame [6, Lemma 2.5]. The operator $\widehat{T}_\phi^{-1}$ is given by $C_\phi S_\phi^{-1}$. Hence, for some $\gamma > 0$ and for every $f \in \mathcal{H}$,

$$\begin{aligned}\|C_\psi f\|_2^2 &= \int_X \Big| \sum_{n\in\mathbb{N}} \overline{\langle S_\phi^{-1} e_n|\phi_x\rangle} \langle f|e_n\rangle \Big|^2 \,\mathrm{d}\mu(x) \\ &= \int_X \left| \sum_{n\in\mathbb{N}} \langle f|e_n\rangle\langle e_n|(S^{-1})^*\phi_x\rangle \right|^2 \,\mathrm{d}\mu(x) \leq \gamma \left\| S_\phi^{-1} \right\|^2 \|f\|^2.\end{aligned}$$

Observe that, in principle, there may exist a reproducing partner ψ which is not Bessel. Take for example the frame $\phi := \{e_n\}_{n\in\mathbb{N}} \cup \{\frac{1}{n} e_n\}_{n\in\mathbb{N}}$, where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis, and define $\psi := \{e_n\}_{n\in\mathbb{N}} \cup \{n e_n\}_{n\in\mathbb{N}}$.

Given the weakly measurable function ϕ, the fact that (ψ, ϕ) is a reproducing pair does not determine the function ψ uniquely. Indeed we have :

Theorem 4.3 *Let (ψ, ϕ) be a reproducing pair, then (θ, ϕ) is a reproducing pair if and only if $\theta = A\psi + \theta_0$, where $A \in GL(\mathcal{H})$ and $[\langle f|\theta_0(\cdot)\rangle]_\phi = [0]_\phi,\ \forall f \in \mathcal{H}, i.e., \widehat{C}_{\theta_0,\phi} f = 0, \forall\, f \in \mathcal{H}$.*

Proof. If $\theta = A\psi + \theta_0$ as above, then $S_{\theta,\phi} f = \widehat{T}_\phi(\widehat{C}_{A\psi,\phi} + \widehat{C}_{\theta_0,\phi}) f = \widehat{T}_\phi(\widehat{C}_{A\psi,\phi} f) = \widehat{T}_\phi \widehat{C}_{\psi,\phi} A^* f = S_{\psi,\phi} A^* f$, hence $S_{\theta,\phi} = S_{\psi,\phi} A^* \in GL(\mathcal{H})$.

Conversely, assume that (θ, ϕ) is a reproducing pair. By Theorem 3.12, we have $V_\phi(X, \mu) = \mathsf{Ran}\, C_\psi / \mathsf{Ker}\, T_\phi = \mathsf{Ran}\, C_\theta / \mathsf{Ker}\, T_\phi$, i.e., for every $f \in \mathcal{H}$ there exists $g \in \mathcal{H}$ such that $[C_\theta f]_\phi = [\mathsf{C}_\psi g]_\phi$. Then, using successively the definition of $S_{\phi,\theta}$, the relation $[C_\theta f]_\phi = [C_\psi g]_\phi$ and the reproducing kernel (3.15), we obtain

$$\begin{aligned}\langle f|S_{\phi,\theta}(S_{\psi,\phi}^{-1})^*\psi(\cdot)\rangle &= \int_X \langle f|\theta(x)\rangle\langle\phi_x|(S_{\psi,\phi}^{-1})^*\psi(\cdot)\rangle \,\mathrm{d}\mu(x) \\ &= \int_X \langle g|\psi(x)\rangle\langle\phi_x|(S_{\psi,\phi}^{-1})^*\psi(\cdot)\rangle \,\mathrm{d}\mu(x) = \langle g|\psi(\cdot)\rangle = \langle f|\theta(\cdot)\rangle\,,\ \forall\, f \in \mathcal{H}.\end{aligned}$$

This means that, for all $f \in \mathcal{H}$, we have $[C_\theta f]_\phi = [C_{A\psi} f]_\phi$ or, equivalently, $\widehat{C}_{\theta,\phi} = \widehat{C}_{A\psi,\phi}$, where $A := S_{\phi,\theta}(S_{\psi,\phi}^{-1})^* \in GL(\mathcal{H})$. Moreover, $C_\theta f(x) = C_{A\psi} f(x) + F(f, x)$ for a.e. $x \in X$ and every $f \in \mathcal{H}$, where $F(f, \cdot) \in \mathsf{Ker}\, T_\phi$, i.e., $F(f, x) = \langle f|(\theta - A\psi)(x)\rangle =: \langle f|\theta_0(x)\rangle$. $\square$

5 Nondegenerate systems

The measurable function ϕ is said to be *μ-independent* if $\mathsf{Ker}\, T_\phi = \{0\}$, that is, if it satisfies the following condition

$$\int_X \xi(x)\langle\phi_x|g\rangle \,\mathrm{d}\mu(x) = 0,\ \forall\, g \in \mathcal{H},\ \text{implies } \xi(x) = 0 \text{ a.e.}. \tag{5.1}$$

In that case, of course, $\mathcal{V}_\phi(X, \mu) = V_\phi(X, \mu)$. This definition is modeled on that of ω-independence of sequences, introduced in [15, Def.3.1.2]. The function ϕ is called *μ-nondegenerate* if it is both μ-total and μ-independent.

Proposition 5.1 *Let (ψ, ϕ) be a reproducing pair, where ϕ is Bessel, and assume $(\mathsf{Ran}\, C_\psi \cap L^2(X, \mathrm{d}\mu))^\perp \neq \{0\}$. Then ϕ is not μ-independent, hence it is μ-degenerate.*

Proof. Let us assume that ϕ is μ-independent and, without loss of generality, that $S_{\psi,\phi} = I$ (that is, ϕ and ψ are dual of each other). Take $F \in (\mathsf{Ran}\, C_\psi \cap L^2(X, \mathrm{d}\mu))^\perp \backslash \{0\}$. As ϕ is μ-independent, it follows that $D_\phi F \neq 0$ and consequently $F' = C_\psi D_\phi F \neq 0$ since ψ is μ-total. Moreover, $F - F' \neq 0$ since $F \in (\mathsf{Ran}\, C_\psi \cap L^2(X, \mathrm{d}\mu))^\perp$ and $F' \in C_\psi(\mathcal{H})$. Hence we get

$$\int_X (F(x) - F'(x))\langle \phi_x | g\rangle \,\mathrm{d}\mu(x) = \langle D_\phi F - T_\phi C_\psi D_\phi F | g\rangle = 0, \ \forall\, g \in \mathcal{H},$$

since $T_\phi C_\psi = S_{\psi,\phi} = I$, and this contradicts the assumption of μ-independence of ϕ. $\square$

Actually there is more. Assume that ψ is an upper semi-frame (i.e., a Bessel map). Then ϕ is a lower semi-frame [6, Lemma 2.5] (they can both be frames). Then, if (X, μ) is a nonatomic measure space, it follows from [20, Theorem 2] that $\dim(\mathsf{Ran}\, C_\phi \cap L^2(X, \mathrm{d}\mu))^\perp = \infty$.

Intuitively, μ-nondegeneracy occurs only for discrete systems (atomic measure) or continuous systems closely related to discrete ones, called continuous orthonormal bases in [10] and studied in [11, 16]. Incidentally, in the discrete case, similar considerations have been extended to rigged Hilbert spaces in recent papers by Bellomonte and one of us [13, 14].

6 Examples

In this section, we present a few concrete examples of the construction of Section 3. We begin with discrete examples, that is, $X = \mathbb{N}$ with the counting measure.

6.1 Discrete examples

6.1.1 Orthonormal basis

Let $e = \{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis, then $\mathcal{V}_e(\mathbb{N}) = V_e(\mathbb{N}) = \ell^2(\mathbb{N})$. Indeed, for $\xi \in \mathcal{V}_e(\mathbb{N})$, we have

$$\Big|\sum_{n\in\mathbb{N}} \xi_n \langle e_n | g\rangle\Big| = \Big|\sum_{n\in\mathbb{N}} \xi_n \overline{g_n}\Big| \leq c\, \|g\| = c\, \|\{g_n\}_{n\in\mathbb{N}}\|_{\ell^2}\,, \ \forall g \in \mathcal{H},$$

where $g_n := \langle g | e_n\rangle$. As $C_e : \mathcal{H} \to \ell^2(\mathbb{N})$ is bijective, $\xi \in \ell^2(\mathbb{N})^* = \ell^2(\mathbb{N})$. Moreover, since $\mathsf{Ker}\, T_e = \{0\}$ it follows that $\mathcal{V}_e(\mathbb{N}) = V_e(\mathbb{N})$ and $\|\cdot\|_{\ell^2} = \|\cdot\|_e$.

6.1.2 Riesz basis

Now consider a Riesz basis $r = \{r_n\}_{n\in\mathbb{N}}$. Then $r_n = Ae_n$ for some $A \in GL(\mathcal{H})$ [15]. Therefore $\mathcal{V}_r(\mathbb{N}) = V_r(\mathbb{N}) = \ell^2(\mathbb{N})$ as sets, but with equivalent (not necessary equal) norms, since

$$\begin{aligned}
\|\xi\|_r &= \sup_{\|g\|=1} \Big|\sum_{n\in\mathbb{N}} \xi_n \langle r_n | g\rangle\Big| = \sup_{\|g\|=1} \Big|\sum_{n\in\mathbb{N}} \xi_n \langle e_n | A^* g\rangle\Big| \\
&= \sup_{\|g\|=1} \|A^* g\| \,\Big|\sum_{n\in\mathbb{N}} \xi_n \langle e_n | \frac{A^* g}{\|A^* g\|}\rangle\Big| \leq \|A\| \sup_{\|g\|=1} \Big|\sum_{n\in\mathbb{N}} \xi_n \langle e_n | g\rangle\Big| = \|A\|\, \|\xi\|_{\ell^2}\,, \ \forall\, \xi \in \ell^2.
\end{aligned}$$

The lower inequality follows by a similar argument.

6.1.3 Discrete upper and lower-semi frames

Let $\theta = \{\theta_n\}_{n\in\mathbb{N}}$ be a discrete frame, $m = \{m_n\}_{n\in\mathbb{N}} \subset \mathbb{C}\backslash\{0\}$ and define $\phi := \{m_n\theta_n\}_{n\in\mathbb{N}}$. If $\{|m_n|\}_{n\in\mathbb{N}} \in c_0$, then ϕ is an upper semi-frame and if $\{|m_n|^{-1}\}_{n\in\mathbb{N}} \in c_0$, then ϕ is a lower semi-frame. Observe that in both cases ϕ is not a frame.

To see this, let $\{|m_n|\}_{n\in\mathbb{N}} \in c_0$. Then, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|m_n| \leq \varepsilon,\ \forall n \geq N$. Take $f \in \operatorname{span}\{\phi_1, ..., \phi_{N-1}\}^{\perp}$, then

$$\sum_{n\in\mathbb{N}} |\langle f|\phi_n\rangle|^2 = \sum_{n\geq N} |\langle f|\phi_n\rangle|^2 \leq \varepsilon^2 \sum_{n\in\mathbb{N}} |\langle f|\theta_n\rangle|^2 \leq C\varepsilon^2 \|f\|^2 .$$

Hence the lower frame inequality cannot be satisfied. The same argument with inverse inequalities yields the result for $\{|m_n|^{-1}\}_{n\in\mathbb{N}} \in c_0$.

It can easily be seen that $V_\phi(\mathbb{N}) = M_{1/m}(V_\theta(\mathbb{N})) = M_{1/m}(\mathsf{Ran}\, C_\theta)$ as sets, where M_m is the multiplication operator defined by $(M_m\xi)_n = m_n\xi_n$. Moreover, $\|\cdot\|_\phi \asymp \|\cdot\|_{\ell^2_m}$, where $\|\xi\|_{\ell^2_m} := \sum_{n\in\mathbb{N}} |\xi_n m_n|^2$.

Now we will apply Theorem 4.1 to show that there exists ψ such that (ψ, ϕ) is a reproducing pair. We first identify $\widehat{T}_\phi$. Let $\xi \in V_\phi(\mathbb{N})$, then

$$\widehat{T}_\phi\xi = \sum_{n\in\mathbb{N}} \xi_n\phi_n = \sum_{n\in\mathbb{N}} \xi_n m_n\theta_n = T_\theta(M_m\xi).$$

The identification $V_\phi(\mathbb{N}) = M_{1/m}(\mathsf{Ran}\, C_\theta)$ immediately implies that $\mathsf{Ran}\, \widehat{T}_\phi = \mathcal{H}$. In order to check condition (4.1) we observe that the reproducing kernel property yields

$$\widehat{T}_\phi^{-1} f = M_{1/m} C_\theta S_\theta^{-1} f,\ \forall f \in \mathcal{H}.$$

Hence, for every fixed $k \in \mathbb{N}$, we have

$$\sum_{n\in\mathbb{N}} |m_k^{-1}\langle S_\theta^{-1} e_n|\theta_k\rangle|^2 = |m_k|^{-2} \left\|S_\theta^{-1}\theta_k\right\|^2 < \infty.$$

One natural choice of a reproducing partner is $\psi := \{(1/\overline{m_n})\theta_n\}_{n\in\mathbb{N}}$ as $S_{\psi,\phi} = S_\theta \in GL(\mathcal{H})$.

6.1.4 Gabor systems

Let $a, b > 0$ and $g \in L^2(\mathbb{R})$, the Gabor system $G(g, a, b)$ is given by

$$G(g, a, b) := \{T_{an}M_{bm}\varphi\}_{n,m\in\mathbb{Z}},$$

where T_x denotes the translation and M_ω the modulation operator. For an overview on Gabor analysis, see [18].

Reproducing pairs appear to be a promising approach for the study of Gabor systems at critical density ($a \cdot b = 1$) since the well-known Balian-Low theorem (BLT) states that if g is well-localized in both time and frequency, then $G(g, a, 1/a)$ is not a frame.

In [25] the authors show that it is possible to construct a reproducing pair consisting of two Gabor systems where one window beats the obstructions of the BLT. However, Balian-Low like results still exist in this setup.

The Gabor system of integer time-frequency shifts of the Gaussian $\mathcal{G} := G(\varphi, 1, 1)$, where $\varphi(t) := 2^{1/4}e^{-\pi t^2}$, is one of the most studied object in time-frequency analysis. It is complete but not a frame. Moreover, there is no Gabor system with a window in $L^2(\mathbb{R})$ which is dual to $\mathcal{G}$.

However, Bastiaans [12] and Janssen [21] have shown that there is $\gamma \notin L^2(\mathbb{R})$, such that $G(\gamma, 1, 1)$ is dual to $\mathcal{G}$ in a weak distributional sense.

With the help of the novel tools developed in this paper, in particular Theorem 4.1, it is possible to show the existence of an unstructured reproducing partner for $\mathcal{G}$, see [25]. In other words, the coefficients for the Gabor expansion with $\mathcal{G}$ can be calculated using inner products in $L^2(\mathbb{R})$. This solves one of the last open questions for this system.

6.2 Continuous examples

6.2.1 Continuous frames

If ϕ is a continuous frame, Corollary 3.14 implies that $V_\phi(X, \mu) \subseteq L^2(X, \mu)$. Now, since $L^2(X, \mu) = \mathsf{Ran}\, C_\phi \oplus \mathsf{Ker}\, D_\phi$, it follows that $V_\phi(X, \mu)[\|\cdot\|_\phi] \simeq \mathsf{Ran}\, C_\phi[\|\cdot\|_{L^2}]$.

Observe that there may exist $\xi \in \mathcal{V}_\phi(X, \mu)$, such that $\xi \notin L^2(X, \mu)$. If, for example, there exists $\psi : X \to \mathcal{H}$ and $f \in \mathcal{H}$ such that (ψ, ϕ) is a reproducing pair and $\|C_\psi f\|_{L^2} = \infty$, then $\mathsf{Ran}\, C_\psi \subset \mathcal{V}_\phi(X, \mu) \nsubseteq L^2(X, \mu)$. For a concrete example, see [24, Section 4]. Nevertheless, there is always a unique $f \in \mathcal{H}$ such that $\xi = C_\phi f + \xi_0$, where $[\xi]_\phi = [C_\phi f]_\phi$ and $\xi_0 \in \mathsf{Ker}\, T_\phi$, *i.e.*, $\xi_0 \notin L^2(X, \mu)$.

6.2.2 1D continuous wavelets

Let $\phi, \psi \in L^2(\mathbb{R}, \mathrm{d}x)$ and consider the continuous wavelet systems $\phi_{x,a} = T_x D_a \phi$, where, as usual, T_x denotes the translation and D_a the dilation operator. If

$$\int_{\mathbb{R}} |\widehat{\psi}(\omega)\widehat{\phi}(\omega)| \frac{\mathrm{d}\omega}{|\omega|} < \infty, \tag{6.1}$$

then (ψ, ϕ) is a reproducing pair for $L^2(\mathbb{R}, dx)$ with $S_{\psi,\phi} = c_{\psi,\phi} I$ [18, Theorem 10.1], where

$$c_{\psi,\phi} := \int_{\mathbb{R}} \overline{\widehat{\psi}(\omega)}\widehat{\phi}(\omega) \frac{\mathrm{d}\omega}{|\omega|}.$$

Actually this is just another way of expressing the well-known orthogonality relations of wavelet transforms — or, for that matter, of all coherent states associated to square integrable group representations [3, Chaps. 8 and 12]. For $\psi = \phi$, the cross-admissibility condition (6.1) reduces to the classical admissibility condition

$$c_\phi := \int_{\mathbb{R}} |\widehat{\phi}(\omega)|^2 \frac{\mathrm{d}\omega}{|\omega|} < \infty. \tag{6.2}$$

Considering the obvious inequalities

$$|c_{\psi,\phi}| \leq \int_{\mathbb{R}} |\widehat{\psi}(\omega)\widehat{\phi}(\omega)| \frac{\mathrm{d}\omega}{|\omega|} \leq c_\phi^{1/2} c_\psi^{1/2},$$

we see that condition (6.1) is automatically satisfied whenever ϕ and ψ are both admissible. However, it is possible to choose a mother wavelet ϕ that does not satisfy the admissibility condition (6.2) and still obtain a reproducing pair (ψ, ϕ).

Consider for example the Gaussian window $\phi(x) = e^{-\pi x^2}$, then $c_\phi = \infty$ which implies that ϕ is not a continuous wavelet frame. However, if one defines $\psi \in L^2(\mathbb{R}, dx)$ in the Fourier domain via $\widehat{\psi}(\omega) = |\omega|\widehat{\phi}(\omega)$, it follows that $0 < c_{\psi,\phi} = \|\phi\|_2^2 < \infty$. Thus we conclude that (ψ, ϕ) is a reproducing pair. Needless to say, the same considerations apply to D-dimensional continuous wavelets [3]. This example clearly shows the increasing flexibility obtained when replacing continuous frames by reproducing pairs.

6.2.3 A continuous upper semi-frame: affine coherent states

In [6, Section 2.6] the following example of an upper semi-frame is investigated. Define $\mathcal{H}^{(n)} := L^2(\mathbb{R}^+, r^{n-1}\,\mathrm{d}r)$, where $n \in \mathbb{N}$ and the following measure space $(X, \mu) = (\mathbb{R}, \mathrm{d}x)$. Let $\psi \in \mathcal{H}^{(n)}$ and define the affine coherent state

$$\psi_x(r) = e^{-ixr}\psi(r), \quad r \in \mathbb{R}^+.$$

Then ψ is admissible if $\sup_{r\in\mathbb{R}^+} \mathfrak{s}(r) = 1$, where $\mathfrak{s}(r) := 2\pi r^{n-1}|\psi(r)|^2$, and $|\psi(r)| \neq 0$, for a.e. $r \in \mathbb{R}^+$. The frame operator is given by the multiplication operator on $\mathcal{H}^{(n)}$

$$(S_\psi f)(r) = \mathfrak{s}(r) f(r),$$

and, more generally,

$$(S_\psi^m f)(r) = [\mathfrak{s}(r)]^m f(r), \ \forall\, m \in \mathbb{Z}.$$

Hence S_ψ is bounded and S_ψ^{-1} is unbounded.

First we identify $\mathsf{Ker}\, D_\psi$ as the space $\mathcal{K}_+ := \{\eta \in L^2(\mathbb{R}) : \widehat{\eta}(\omega) = 0, \text{ for a.e. } \omega \geq 0\}$. For every $\xi \in L^2(\mathbb{R})$ and $g \in \mathcal{H}^{(n)}$, we have, indeed, the following equality

$$\langle D_\psi \xi | g\rangle = \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}} \xi(x) e^{-ixr}\psi(r)\,\mathrm{d}x \right) \overline{g(r)} r^{n-1}\,\mathrm{d}r = \int_{\mathbb{R}^+} \widehat{\xi}(r)\psi(r)\overline{g(r)} r^{n-1}\,\mathrm{d}r,$$

which easily implies that $\mathsf{Ker}\, D_\psi = \mathcal{K}_+$.

Thus in this case we find that $\mathsf{Ker}\, D_\psi = (\mathsf{Ran}\, C_\phi)^\perp = \mathcal{K}_+ \neq \{0\}$ (it is infinite dimensional), an example of the situation described in Section 5.

The function ψ enjoys the interesting property that we can characterize the space $V_\psi(\mathbb{R}, \mathrm{d}x)$ and its norm. First, we show that $\xi \in V_\psi(\mathbb{R}, \mathrm{d}x)$ implies $\widehat{\xi}\psi \in \mathcal{H}^{(n)}$ and $\|\xi\|_\psi = \left\|\widehat{\xi}\psi\right\|$. Indeed, let $\xi \in V_\psi(\mathbb{R}, \mathrm{d}x)$ and $\psi, g \in \mathcal{H}^{(n)}$. Then we have,

$$\begin{aligned} \langle\!\langle \xi | C_\psi g\rangle\!\rangle_\mu = \langle \widehat{T}_\psi \xi | g\rangle &= \int_{\mathbb{R}^+}\int_{\mathbb{R}} \xi(x) e^{-ixr}\psi(r)\,\mathrm{d}x\ \overline{g(r)} r^{n-1}\,\mathrm{d}r \\ &= \int_{\mathbb{R}^+} \widehat{\xi}(r)\psi(r)\overline{g(r)} r^{n-1}\,\mathrm{d}r = \langle \widehat{\xi}\psi | g\rangle. \end{aligned} \tag{6.3}$$

Hence, $T_\phi\xi = \widehat{\xi}\psi$ which in turn implies that $\widehat{\xi}$ has to be given by an almost everywhere defined function which satisfies $\widehat{\xi}\psi \in \mathcal{H}^{(n)}$. Moreover, (6.3) yields

$$\|\xi\|_\psi = \sup_{\|g\|\leq 1} |\langle\!\langle \xi | C_\psi g\rangle\!\rangle_\mu| = \sup_{\|g\|\leq 1} |\langle \widehat{\xi}\psi | g\rangle| = \left\|\widehat{\xi}\psi\right\|. \tag{6.4}$$

Then again, by the same reasoning, the previous chain of equalities shows that a measurable function ξ is contained in $V_\psi(\mathbb{R}, \mathrm{d}x)$ provided that $\xi \in \mathcal{F}^{-1}(\psi^{-1}\mathcal{H}^{(n)})$.

Proposition 6.1 *Let $\psi \in \mathcal{H}^{(n)}$, then, as sets,*

$$V_\psi(\mathbb{R}, \mathrm{d}x) = \left\{ \xi : X \to \mathbb{C} \textit{ measurable} : \ \xi \in \mathcal{F}^{-1}(\psi^{-1}\mathcal{H}^{(n)}) \right\} / \mathsf{Ker}\, T_\phi$$

and $\|\xi\|_\psi = \left\|\widehat{\xi}\psi\right\|, \forall\, \xi \in V_\psi(\mathbb{R}, \mathrm{d}x)$.

The inverse Fourier transform is taken in the sense of distributions, if needed.

In the quest of a reproducing partner for ψ we will first treat the question if there exists an affine coherent state $\phi_x(r) = e^{-ixr}\phi(r),\ r \in \mathbb{R}^+,\ \phi \in \mathcal{H}^{(n)}$, such that (ψ, ϕ) forms a reproducing pair. Indeed, since ψ is Bessel and not a frame, its dual ϕ is by necessity a lower semi-frame, whereas an affine coherent state must be Bessel, but can never satisfy the lower frame bound. Hence, there is no pair of affine coherent states forming a reproducing pair. This fact can also be proven by an explicit calculation.

Finally, we have here an example of the situation described in Remark 3.8, namely, C_ψ being an isometry by Corollary 3.7, but $\mathsf{Ran}\,\widehat{T}_\psi \neq \mathcal{H}$. We have already seen in (6.3) that $\widehat{T}_\psi \xi = \widehat{\xi}\psi$. If $\mathsf{Ran}\,\widehat{T}_\psi = \mathcal{H}$, an arbitrary element $h \in \mathcal{H}^{(n)} = L^2(\mathbb{R}^+, r^{n-1}\,\mathrm{d}r)$ could be written as $h = \widehat{T}_\psi \xi = \widehat{\xi}\psi$ for some $\xi \in V_\psi(\mathbb{R}, \mathrm{d}x)$. This applies, in particular, to ψ itself, which also belongs to $\mathcal{H}^{(n)}$. This in turn implies that there exists ξ, such that $\widehat{\xi}(r) = 1$ for a.e. $r \geq 0$. But there is no function that satisfies this condition (however the δ-distribution does the job).

This has two major consequences. First, it shows that $V_\psi(\mathbb{R}, \mathrm{d}x)$ is *not* a Hilbert space, since it is not complete. Second, there is *no* reproducing partner for ψ making it a reproducing pair.

6.2.4 Continuous wavelets on the sphere

Next we consider the continuous wavelet transform on the 2-sphere $\mathbb{S}^2$ [3, 4]. For a mother wavelet $\phi \in \mathcal{H} = L^2(\mathbb{S}^2, \mathrm{d}\mu)$, define the family of spherical wavelets

$$\phi_{\varrho,a} := R_\varrho D_a \phi, \text{ where } (\varrho, a) \in X := SO(3) \times \mathbb{R}^+.$$

Here, D_a denotes the stereographic dilation operator and R_ρ the unitary rotation on $\mathbb{S}^2$.

It has been shown in [4, Theorem 3.3] that the operator S_ϕ is diagonal in Fourier space (harmonic analysis on the 2-sphere reduces to expansions in spherical harmonics Y_l^m, $l \in \mathbb{N}_0, m = -l, \ldots, l$), thus it is given by a Fourier multiplier $\widehat{S_\phi f}(l, n) = s_\phi(l)\widehat{f}(l, n)$ with the symbol s_ϕ given by

$$s_\phi(l) := \frac{8\pi^2}{2l+1} \sum_{|m| \leq l} \int_0^\infty \left|\widehat{D_a\phi}(l, m)\right|^2 \frac{\mathrm{d}a}{a^3},\ l \in \mathbb{N}_0.$$

where $\widehat{D_a\phi}(l, m) := \langle Y_l^m | D_a\phi\rangle$ is the Fourier coefficient of $D_a\phi$. If $\mathsf{m} \leq s_\phi(l) < \infty$ for all $l \in \mathbb{N}_0$, it follows that ϕ is a lower semi-frame and S_ϕ is densely defined.

The result of the analysis is twofold. First, the wavelet $\phi \in L^2(\mathbb{S}^2, \mathrm{d}\mu)$ is admissible if and only if there exists a constant $\mathsf{c} > 0$ such that $s_\phi(l) \leq \mathsf{c}$, $\forall\, l \in \mathbb{N}$, equivalently, if the frame operator S_ϕ is bounded. In addition, for any admissible axisymmetric wavelet ϕ, there exists a constant $\mathsf{d} > 0$ such that $\mathsf{d} \leq s_\phi(l) \leq \mathsf{c}$, $\forall\, l \in \mathbb{N}$. Equivalently, S_ϕ and S_ϕ^{-1} are both bounded, *i.e.,* the family of spherical wavelets $\{\phi_{a,\varrho},\ (\varrho, a) \in X = \mathrm{SO}(3) \times \mathbb{R}_+^*\}$ is a continuous frame. One notices, however, that the upper frame bound, which is implied by the constant c, does depend on ϕ, whereas the lower frame bound, which derives from d, does not, it follows from the asymptotic behavior of the function Y_l^m for large l.

However, it turns out [26] that the reconstruction formula converges if $\mathsf{d} \leq s_\phi(l) < \infty$ for all $l \in \{0\} \cup \mathbb{N}$, and this implies that ϕ (which is *not* admissible) is in fact a lower semi-frame and S_ϕ is unbounded, but densely defined.

We will apply Theorem 4.1 to investigate the existence of a reproducing partner for ϕ. First, we show that $\mathsf{Ran}\,\widehat{T}_\phi = \mathcal{H}$. The operator M_ϕ defined by $\widehat{M_\phi f}(l, m) = s_\phi(l)^{-1}\widehat{f}(l, m)$ is bounded and constitutes a right inverse to S_ϕ. Hence, for every $f \in \mathcal{H}$, it holds

$$f = S_\phi M_\phi f = \widehat{T}_\phi[\mathsf{C}_\phi M_\phi f]_\phi \in \mathsf{Ran}\,\widehat{T}_\phi.$$

The spherical harmonics Y_l^n form an orthonormal basis of $L^2(\mathbb{S}^2, \mathrm{d}\mu)$. Choosing $\xi_{l,m}(x,a) := \mathsf{C}_\phi(S_\phi^{-1}Y_l^m)(\rho,a)$ as a representative of $[\widehat{T}_\phi^{-1}Y_l^m]_\phi$ yields for every $(\rho,a) \in \mathbb{R}\times\mathbb{R}^+$:

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{|m|\le l}|\xi_{l,m}(\rho,a)|^2 &= \sum_{l=0}^{\infty}\sum_{|m|\le l}|\mathsf{C}_\phi(S_\phi^{-1}Y_l^m)(\rho,a)|^2 = \sum_{l=0}^{\infty}\sum_{|m|\le l}|\langle S_\phi^{-1}Y_l^m|\phi_{\rho,a}\rangle|^2\\ &= \sum_{l=0}^{\infty}\sum_{|m|\le l}|\widehat{S_\phi^{-1}\phi_{\rho,a}}(l,m)|^2 = \sum_{l=0}^{\infty}\sum_{|m|\le l}|s_\phi(l)^{-1}\widehat{\phi}_{\rho,a}(l,m)|^2\\ &\le \frac{1}{\mathsf{d}}\sum_{l=0}^{\infty}\sum_{|n|\le l}|\widehat{\phi}_{\rho,a}(l,n)|^2 = \frac{1}{\mathsf{d}}\,\|\phi_{\rho,a}\|^2 < \infty.\end{aligned}$$

Thus there exists (at least one) function $\psi : SO(3)\times\mathbb{R}^+ \to L^2(\mathbb{S}^2, \mathrm{d}\mu)$ such that (ψ,ϕ) is a reproducing pair.

Moreover, as for the wavelets on $\mathbb{R}^d$, it is possible to choose another continuous wavelet system $\psi_{\rho,a}$ as reproducing partner if the symbol $s_{\psi,\phi}$, defined by

$$s_{\psi,\phi}(l) := \frac{8\pi^2}{2l+1}\sum_{|m|\le l}\int_0^\infty \widehat{D_a\psi}(l,m)\overline{\widehat{D_a\phi}(l,m)}\frac{da}{a^3}.$$

satisfies $\mathsf{m} \le |s_{\psi,\phi}(l)| \le \mathsf{M}$ for all $l \in \mathbb{N}_0$.

7 Outcome: Reproducing pairs and PIP-spaces

When trying to generalize the well-known notion of frame, both discrete and continuous, a first step is to consider semi-frames, both upper and lower ones. The main result is that the two types are dual of each other. Indeed, if two semi-frames are in duality, either they are both frames, or at least one is a lower semi-frame.

Then the next step is to drop the restriction imposed by the frame bounds on the two measurable functions in duality, and this leads to the notion of reproducing pair. We have seen that the latter is quite rich. It generates a whole mathematical structure. We have given several concrete examples in Section 6. These, and additional ones, should allow one to better specify the best assumptions to be made on the measurable functions or, more precisely, on the nature of the range of the analysis operators C_ψ, C_ϕ . Let (ψ,ϕ) be a reproducing pair. By definition,

$$\langle S_{\psi,\phi}f|g\rangle = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle\,\mathrm{d}\mu(x) = \int_X C_\psi f(x)\,\overline{C_\phi g(x)}\,\mathrm{d}\mu(x) \qquad (7.1)$$

is well defined for all $f,g\in\mathcal{H}$ (here we revert to the *linear* maps C_ψ, C_ϕ defined in (2.2)). The r.h.s. is the L^2 inner product, but generalized, since in general $C_\psi f, C_\phi$ need not belong to $L^2(X, \mathrm{d}\mu)$. Thus clearly the analysis should be made in the context of PIP-spaces [5].

The question is, how to embed $\mathsf{Ran}\,(C_\psi)$ and $\mathsf{Ran}\,(C_\phi)$ into the corresponding assaying subspaces. Next we have to determine how the Hilbert spaces V_ψ and V_ϕ are related to the latter. This is the topic of a further paper [8]. There we will examine successively the cases of a rigged Hilbert space (RHS) and a genuine PIP-space. Then we will particularize the results to a Hilbert scale and to a PIP-space of L^p spaces. The motivation for the last case is the following. If, following [24], we make the innocuous assumption that the map $x \mapsto \psi_x$ is bounded, *i.e.*, $\sup_{x\in X}\|\psi_x\|_{\mathcal{H}} \le c$ for some $c > 0$ (often $\|\psi_x\|_{\mathcal{H}} =$ const., e.g. for wavelets or coherent states), then $(C_\psi f)(x) = \langle f|\psi_x\rangle \in L^\infty(X, \mathrm{d}\mu)$ so that a PIP-space based on the lattice generated by the family $\{L^p(X, \mathrm{d}\mu), 1 \leq p \leq \infty,\}$ may be a good solution.

Another interesting direction consists in considering a whole family $\mathcal{G}$ of μ-total, weakly measurable functions $\phi : X \to \mathcal{H}$, instead of only one. To each $\phi \in \mathcal{G}$ we can associate the pre-Hilbert space $V_\phi(X,\mu)[\|\!|\cdot\|\!|_\phi]$ and take its completion $\widetilde{V_\phi}(X,\mu)[\|\!|\cdot\|\!|_\phi]$. If ϕ has a partner $\psi \in \mathcal{G}$ such that (ψ,ϕ) is a reproducing pair, both spaces $V_\phi(X,\mu) = \widetilde{V_\phi}(X,\mu)[\|\!|\cdot\|\!|_\phi]$ and $V_\psi(X,\mu) = \widetilde{V_\psi}(X,\mu)[\|\!|\cdot\|\!|_\phi]$ are Hilbert spaces, conjugate dual to each other. In the general case, however, the question of completeness of $V_\phi(X,\mu)[\|\!|\cdot\|\!|_\phi]$ is open. Can one find conditions under which it holds? Also once might study the relationship between different pre-Hilbert spaces $V_\phi(X,\mu)$. When is one contained in another one?

Acknowledgement

This work was partly supported by the Austrian Science Fund (FWF) through the START-project FLAME (Frames and Linear Operators for Acoustical Modeling and Parameter Estimation): Y 551-N13 and by the Istituto Nazionale di Alta Matematica (GNAMPA project "Proprietà spettrali di quasi *-algebre di operatori"). JPA acknowledges gratefully the hospitality of the Acoustic Research Institute, Austrian Academy of Science, Vienna, and that of the Dipartimento di Matematica e Informatica, Università di Palermo, whereas MS and CT acknowledge that of the Institut de Recherche en Mathématique et Physique, Université catholique de Louvain.

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