POLYNOMIAL CODIMENSION GROWTH OF ALGEBRAS WITH INVOLUTIONS AND SUPERINVOLUTIONS

ANTONIO IOPPOLO AND DANIELA LA MATTINA

Abstract. Let A be an associative algebra over a field F of characteristic zero endowed with a graded involution or a superinvolution $*$ and let $c_n^*(A)$ be its sequence of $*$-codimensions. In [4, 12] it was proved that if A is finite dimensional such sequence is polynomially bounded if and only if A generates a variety not containing a finite number of $*$-algebras: the group algebra of $\mathbb{Z}_2$ and a 4-dimensional subalgebra of the 4×4 upper triangular matrices with suitable graded involutions or superinvolutions.

In this paper we focus our attention on such algebras since they are the only finite dimensional $*$-algebras, up to T_2^*-equivalence, generating varieties of almost polynomial growth, i.e., varieties of exponential growth such that any proper subvariety has polynomial growth. We classify the subvarieties of such varieties by giving a complete list of generating finite dimensional $*$-algebras. Along the way we classify all minimal varieties of polynomial growth and surprisingly we show that their number is finite for any given growth. Finally we describe the $*$-algebras whose $*$-codimensions are bounded by a linear function.

1. Introduction

Let F be a field of characteristic zero and let $F\langle X\rangle$ be the free associative algebra on a countable set X over F. One of the most interesting and challenging problems in combinatorial PI-theory is that of finding numerical invariants allowing to classify the T-ideals of $F\langle X\rangle$, i.e., the ideals invariant under all endomorphisms of $F\langle X\rangle$. There is a well understood connection between T-ideals of $F\langle X\rangle$ and varieties of F-algebras: every T-ideal is the ideal of polynomial identities satisfied by a given variety of algebras. Therefore it is often convenient to translate a given problem on T-ideals into the language of varieties of algebras. A very useful numerical invariant that can be attached to a T-ideal is given by the sequence of codimensions. Such numerical sequence was introduced by Regev in [26] and measures the rate of growth of the multilinear polynomials lying in a given T-ideal. A celebrated theorem of Regev asserts that if A is an associative PI-algebra, i.e., it satisfies a non-trivial polynomial identity, then its sequence of codimensions $c_n(A)$, $n = 1, 2, \dots,$ is exponentially bounded. Kemer in [14] proved that for a PI-algebra A, $c_n(A)$ is polynomially bounded if and only if the variety of algebras generated by A does not contain either the Grassmann algebra G of an infinite dimensional vector space or the algebra UT_2 of 2×2 upper triangular matrices. Hence $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety grows polynomially.

The varieties of polynomial growth were extensively studied in later years (see for instance [5, 7, 8, 16, 17, 18]) also in the setting of varieties of graded algebras, algebras with involution, graded involution and superinvolution [4, 10, 11, 12, 27].

In this paper we are interested in the study of associative algebras endowed with a graded involution or a superinvolution. In analogy with the ordinary case, one defines the sequence of $*$-codimensions of a $*$-algebra A, i.e., an algebra endowed with a graded involution or a superinvolution $*$. It turns out that if a $*$-algebra satisfies an ordinary identity, then its sequence of $*$-codimensions is exponentially bounded (see [4, 12]). Recently, much interest has been devoted to the study of varieties of $*$-algebras of polynomial growth. More precisely in [4, 12] it was proved that a finite dimensional $*$-algebra has polynomial growth of the $*$-codimensions if and only if the corresponding variety does not contain the following algebras: the group algebra of a group of order 2 and a 4-dimensional subalgebra of UT_4, both algebras with suitable graded involutions or superinvolutions.

2010 *Mathematics Subject Classification.* Primary 16R10, 16R50; Secondary 16W10, 16W50, 16R99.

Key words and phrases. polynomial identity, graded involution, superinvolution, growth.

The authors were partially supported by GNSAGA of INDAM.

Such algebras are the only finite dimensional $*$-algebras, up to T_2^*-equivalence, generating varieties of almost polynomial growth, i.e., varieties of exponential growth such that any proper subvariety has polynomial growth.

We recall that a variety $\mathcal{V}$ is minimal of polynomial growth if $c_n^*(\mathcal{V}) \approx qn^k$ for some $k \geq 1$, $q > 0$, and for any proper subvariety $\mathcal{U} \subsetneqq \mathcal{V}$ we have that $c_n^*(\mathcal{U}) \approx q'n^t$ with $t < k$. In this paper we completely classify all subvarieties and all minimal subvarieties of the varieties of almost polynomial growth generated by the above algebras by giving a complete list of finite dimensional $*$-algebras generating them. Moreover we characterize varieties of polynomial growth generated by finite dimensional $*$-algebras by relating them to the module structure of the multilinear elements in the corresponding relatively free algebras. Finally we describe in detail the $*$-algebras whose $*$-codimensions are bounded by a linear function.

2. PRELIMINARIES AND BASIC RESULTS

Throughout this paper F will denote a field of characteristic zero and $A = A_0 \oplus A_1$ an associative superalgebra (also called $\mathbb{Z}_2$-graded algebra) over F satisfying a non-trivial polynomial identity (PI-algebra). Recall that the elements of A_0 and A_1 are called homogeneous of degree zero (or even elements) and of degree one (or odd elements), respectively.

The free associative algebra $F\langle X\rangle$ on a countable set $X = \{x_1, x_2, \ldots\}$ has a natural structure of superalgebra as follows: write $X = Y \cup Z$, the disjoint union of two sets. If we denote by $\mathcal{F}_0$ the subspace of $F\langle Y \cup Z\rangle$ spanned by all monomials in the variables of X having even degree in the variables of Z and by $\mathcal{F}_1$ the subspace spanned by all monomials of odd degree in Z, then $F\langle Y \cup Z\rangle = \mathcal{F}_0 \oplus \mathcal{F}_1$ is a superalgebra called the free superalgebra on $Y \cup Z$ over F.

We denote by $\mathrm{Id}^{sup}(A) = \{f \in F\langle Y \cup Z\rangle \mid f \equiv 0 \text{ on } A\}$ the set of superpolynomial identities of A, which is a T_2-ideal of the free superalgebra, i.e., an ideal invariant under all graded endomorphisms of $F\langle Y \cup Z\rangle$.

It is well known that in characteristic zero $\mathrm{Id}^{sup}(A)$ is completely determined by its multilinear polynomials and we denote by P_n^{sup} the vector space of all multilinear polynomials of degree n in the variables $y_1, z_1, \ldots, y_n, z_n$. The non-negative integer

$$c_n^{sup}(A) = \dim_F \frac{P_n^{sup}}{P_n^{sup} \cap \mathrm{Id}^{sup}(A)},\ n \geq 1,$$

is called the n-th supercodimension of A.

Now assume that the superalgebra A is endowed with a graded involution, i.e., an involution preserving the grading or with a superinvolution that is a graded linear map $* : A \longrightarrow A$ such that $(a^*)^* = a$ for all $a \in A$ and $(ab)^* = (-1)^{(\deg a)(\deg b)} b^* a^*$, for any homogeneous elements $a, b \in A$. Here $\deg c$ denotes the homogeneous degree of $c \in A_0 \cup A_1$.

Notice that if $A = A_0 \oplus A_1$ is a superalgebra such that $A_1^2 = 0$ ($z_1 z_2 \equiv 0$ on A) then the superinvolutions on A coincide with the graded involutions on A and, in particular, with the involutions on A, if $A_1 = 0$.

In what follows we shall denote by $*$ a graded involution or a superinvolution on A and we shall say that A is a $*$-algebra. In case $A_1^2 = 0$ we shall call $*$ a gs-involution (i.e., a graded involution and also a superinvolution).

Since $\mathrm{char} F = 0$, we can write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where for $i = 0, 1$, $A_i^+ = \{a \in A_i \mid a^* = a\}$ and $A_i^- = \{a \in A_i \mid a^* = -a\}$ denote the sets of symmetric and skew elements of A_i, respectively.

We shall write $F\langle Y \cup Z, *\rangle$ for the free superalgebra with graded involution or superinvolution on the countable set $Y \cup Z$ over F. It is useful to regard $F\langle Y \cup Z, *\rangle$ as generated by even and odd symmetric variables and by even and odd skew variables, i.e,

$$F\langle Y \cup Z, *\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \ldots\rangle,$$

where $y_i^+ = y_i + y_i^*$, $y_i^- = y_i - y_i^*$, $z_i^+ = z_i + z_i^*$ and $z_i^- = z_i - z_i^*$, $i \geq 1$.

We recall that a polynomial

$$f(y_1^+, \ldots, y_m^+, y_1^-, \ldots, y_n^-, z_1^+, \ldots, z_r^+, z_1^-, \ldots, z_s^-) \in F\langle Y \cup Z, *\rangle$$

is a $*$-polynomial identity of A (or simply a $*$-identity), and we write $f \equiv 0$, if

$$f(u_1^+, \ldots, u_m^+, u_1^-, \ldots, u_n^-, v_1^+, \ldots, v_r^+, v_1^-, \ldots, v_s^-) = 0$$

for all $u_1^+, \ldots, u_m^+ \in A_0^+$, $u_1^-, \ldots, u_n^- \in A_0^-$, $v_1^+, \ldots, v_r^+ \in A_1^+$ and $v_1^-, \ldots, v_s^- \in A_1^-$.

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle Y \cup Z, *\rangle \mid f \equiv 0 \text{ on } A\}$ the T_2^*-ideal of $*$-identities of A, i.e., $\mathrm{Id}^*(A)$ is an ideal of $F\langle Y \cup Z, *\rangle$ invariant under all graded endomorphisms of $F\langle Y \cup Z\rangle$ commuting with $*$.

As in the super case, it is easily seen that in characteristic zero, every $*$-identity is equivalent to a system of multilinear $*$-identities. Hence if we denote by $P_n^* = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i = y_i^+ \text{ or } w_i = y_i^- \text{ or } w_i = z_i^+ \text{ or } w_i = z_i^-,\ i = 1, \dots, n\}$ the space of multilinear polynomials of degree n in the variables $y_1^+, y_1^-, z_1^+, z_1^-, \dots, y_n^+, y_n^-, z_n^+, z_n^-$, the study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^* \cap \mathrm{Id}^*(A)$, for all $n \geq 1$. The non-negative integer

$$c_n^*(A) = \dim_F \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)},\ n \geq 1,$$

is called the n-th $*$-codimension of A.

If A is a PI-algebra, then $c_n^*(A), n = 1, 2, \dots,$ is exponentially bounded (see [4], [12]). Here we are interested in $*$-algebras having polynomial growth of their $*$-codimensions.

Given $\mathcal{V}$ a variety of $*$-algebras the growth of $\mathcal{V}$ is defined as the growth of the sequence of $*$-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^*(A)$. Then we say that $\mathcal{V}$ has polynomial growth if $c_n^*(\mathcal{V})$ is polynomially bounded.

In [4, 12] the authors characterized the varieties of polynomial growth by exhibiting a finite list of $*$-algebras to be excluded from the variety.

Next we are going to describe such algebras.

Let A be a commutative algebra endowed with an automorphism φ of order 2. Then we can write $A = A_0^\varphi \oplus A_1^\varphi$ where $A_0^\varphi = \{a \in A \mid \varphi(a) = a\}$ and $A_1^\varphi = \{a \in A \mid \varphi(a) = -a\}$. It is well known that A can be viewed both as a superalgebra (because of duality between $\mathbb{Z}_2$-gradings and automorphisms of order 2) and as an algebra with involution with $A_0 = A^+ = A_0^\varphi$ and $A_1 = A^- = A_1^\varphi$, where A^+ and A^- denote the subspaces of symmetric and skew elements, respectively.

Hence we can regard A as endowed with 2 structures of $*$-algebras:

1) A is endowed with trivial grading and we treat φ as an antiautomorphism (involution);
2) A is endowed with trivial involution and we treat φ as an automorphism.

Notice that the involution φ on A in 1) is a graded involution and also a superinvolution while the trivial involution on A in 2) is only a graded involution. We denote by A and by A^{sup} the algebra A with the first and with the second structure, respectively. By using the decomposition $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, we have that

$$A = A_0^\varphi \oplus A_1^\varphi \oplus 0 \oplus 0$$

and

$$A^{sup} = A_0^\varphi \oplus 0 \oplus A_1^\varphi \oplus 0.$$

By using the same notation for any commutative algebra B with an automorphism of order 2 we have the following.

Remark 2.1. *$B \in var^*(A)$ if and only if $B^{sup} \in var^*(A^{sup})$.*

Proof. The result follows by observing that if $f \in F\langle Y \cup Z, *\rangle$ and f^{sup} is the polynomial obtained from f by exchanging the variables y^-'s with the variables z^+'s then $f \equiv 0$ on A if and only if $f^{sup} \equiv 0$ on A^{sup}. ☐

Now let $D = F \oplus F$ be the commutative algebra endowed with the automorphism φ of order 2 defined by $\varphi(a, b) = (b, a)$, for all $(a, b) \in D$.

As above D denotes the algebra D with trivial grading and with (graded) involution $* = \varphi$ (also superinvolution) called the exchange gs-involution and D^{sup} denotes the algebra D with trivial (graded) involution and with grading determined by φ: $D_0 = F(1, 1)$ and $D_1 = F(1, -1)$. We recall that $\mathrm{Id}^*(D) = \langle [x_1, x_2], z^+, z^-, \rangle_{T_2^*}$ and $\mathrm{Id}^*(D^{sup}) = \langle [x_1, x_2], y^-, z^- \rangle_{T_2^*}$ ([9]).

We also consider the following algebra with involution:

$$M = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}) \oplus Fe_{12} \oplus Fe_{34},$$

a subalgebra of UT_4, endowed with the reflection involution, i.e., the involution obtained by reflecting a matrix along its secondary diagonal: if $a = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \gamma e_{12} + \delta e_{34}$ then

$$a^* = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \delta e_{12} + \gamma e_{34},$$

where the e_{ij}s denote the usual matrix units.

If we regard M as endowed with trivial grading, then the above involution is a graded involution and also a superinvolution. We recall that $\mathrm{Id}^*(M) = \langle y_1^- y_2^-,\ z^+,\ z^- \rangle_{T_2^*}$ ([25]).

Next we consider a non-trivial grading on M: we denote by M^{sup} the algebra M with grading $M_0 = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33})$ and $M_1 = Fe_{12} \oplus Fe_{34}$. Notice that the reflection involution on M^{sup} is a graded involution and also a superinvolution, since $M_1^2 = 0$. Hence M^{sup} can be viewed as a $*$-algebra whose T_2^*-ideal is $\mathrm{Id}^*(M^{sup}) = \langle y^-, z_1 z_2 \rangle_{T_2^*}$ ([12]).

The above algebras characterize the varieties of $*$-algebras of polynomial growth.

Theorem 2.1. [4] *Let A be a finite dimensional algebra with superinvolution over a field F of characteristic zero. Then the sequence $c_n^*(A), n = 1, 2, \dots,$ is polynomially bounded if and only if $M, M^{sup}, D \notin var^*(A)$.*

Theorem 2.2. [12] *Let A be a finite dimensional algebra with graded involution over a field F of characteristic zero. Then the sequence $c_n^*(A), n = 1, 2, \dots,$ is polynomially bounded if and only if $M, M^{sup}, D, D^{sup} \notin var^*(A)$.*

Recall that given two $*$-algebras A and B, A is T_2^*-equivalent to B and we write $A \sim_{T_2^*} B$ in case $\mathrm{Id}^*(A) = \mathrm{Id}^*(B)$.

As a consequence of the above theorems, we have that the algebras M, M^{sup}, D and D^{sup} are the only finite dimensional $*$-algebras, up to T_2^*-equivalence, generating varieties of almost polynomial growth, i.e., varieties of exponential growth such that any proper subvariety has polynomial growth.

Now, we are going to study the structure of a generating finite dimensional $*$-algebra of a variety of polynomial growth. First we recall some definitions. A subalgebra (ideal) A' of a $*$-algebra A is a $*$-subalgebra (ideal) of A if it is a graded subalgebra (ideal) and $A'^* = A'$. The algebra A is a simple $*$-algebra if $A^2 \neq 0$ and A has no non-trivial $*$-ideals.

By the Wedderburn-Malcev theorems ([4], [12]), if B is a finite dimensional $*$-algebra over an algebraically closed field, we can write

$$B = B' + J$$

where B' is a semisimple $*$-subalgebra of B and $J = J(B)$ is its Jacobson radical. Moreover

$$B' = B_1 \oplus \cdots \oplus B_k$$

where $B_1, \dots, B_k$ are simple $*$-algebras and J is a $*$-ideal of B which can be decomposed into the direct sum of graded B'-bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11},$$

where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively and for $i, k, l, m \in \{0, 1\}$, $J_{ik} J_{lm} \subseteq \delta_{kl} J_{im}$ where δ_{kl} is the Kronecker delta.
Notice that J_{00} and J_{11} are stable under $*$ whereas $J_{10}^* = J_{01}$.

Let $A = A_0 \oplus A_1$ be a $*$-algebra. We say that A is endowed with the trivial gs-involution if $A_1 = 0$ and $*$ is the trivial involution. Clearly this says that A is commutative.

By putting together Theorem 8.3 in [12] and the proof of Theorem 5.1 in [4] we get the following.

Theorem 2.3. *Let A be a finite dimensional $*$-algebra over an algebraically closed field F of characteristic zero. Then the sequence $c_n^*(A), n = 1, 2, \dots,$ is polynomially bounded if and only if*

$$A = A_1 \oplus \cdots \oplus A_m + J,$$

where for every $i = 1, \dots, m$, $A_i \cong F$ is endowed with the trivial gs-involution and $A_i J A_k = 0$, for all $1 \leq i, k \leq m$, $i \neq k$.

We remark that if A is any algebra having the above decomposition then $c_n^*(A)$, $n = 1, 2, \dots,$ is polynomially bounded also if the field is not algebraically closed.

Lemma 2.1. *Let $\bar{F}$ be the algebraic closure of the field F and let A be a finite dimensional $*$-algebra over $\bar{F}$ such that $\dim_{\bar{F}} A/J(A) \leq 1$. Then $A \sim_{T_2^*} B$ for some finite dimensional $*$-algebra B over F with $\dim_{\bar{F}} A/J(A) = \dim_F B/J(B)$.*

Proof. Since $\dim_{\bar{F}} A/J(A) \leq 1$, it follows that either $A \cong \bar{F} + J(A)$ or $A = J(A)$ is a nilpotent algebra.

We now take an arbitrary $*$-basis $\{w_1, \dots, w_p\}$ of $J(A)$ over $\bar{F}$ (i.e., consisting of even and odd symmetric and even and odd skew elements) and we let B be the $*$-algebra over F generated by $\mathcal{B} = \{1_{\bar{F}}, w_1, \dots, w_p\}$ or by $\mathcal{B} = \{w_1, \dots, w_p\}$ according as $A \cong \bar{F} + J(A)$ or $A = J(A)$, respectively.

Clearly $\dim_F B/J(B) = \dim_{\bar{F}} A/J(A)$ and as F-algebras, $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(B)$. On the other hand, if f is a multilinear $*$-identity of B then f vanishes on the basis $\mathcal{B}$. But $\mathcal{B}$ is also a basis of A over $\bar{F}$. Hence $\mathrm{Id}^*(B) \subseteq \mathrm{Id}^*(A)$ and $A \sim_{T_2^*} B$. □

Theorem 2.4. *Let A be a finite dimensional $*$-algebra over a field F of characteristic zero. Then $c_n^*(A)$, $n = 1, 2, \dots$, is polynomially bounded if and only if $A \sim_{T_2^*} B$, where $B = B_1 \oplus \cdots \oplus B_m$ with $B_1, \dots, B_m$ finite dimensional $*$-algebras over F and $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$.*

Proof. Suppose first that $A \sim_{T_2^*} B$, where $B = B_1 \oplus \cdots \oplus B_m$ with $B_1, \dots, B_m$ finite dimensional $*$-algebras over F and $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$. Then $c_n^*(A) = c_n^*(B) \leq c_n^*(B_1) + \cdots + c_n^*(B_m)$ and the claim follows since, by the remark after Theorem 2.3, $c_n^*(B_i)$ is polynomially bounded for all $i = 1, \dots, m$.

Conversely, let $c_n^*(A)$ be polynomially bounded. Suppose first that F is algebraically closed. Then, by Theorem 2.3,

$$A = A_1 \oplus \cdots \oplus A_l + J,$$

where for every $i = 1, \dots, l$, $A_i \cong F$ is endowed with the trivial gs-involution and $A_i J A_k = 0$, for all $1 \leq i, k \leq l$, $i \neq k$.

Set $B_1 = A_1 + J, \dots, B_l = A_l + J$. We claim that $A \sim_{T_2^*} B_1 \oplus \cdots \oplus B_l \oplus J$. Clearly $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(B_1 \oplus \cdots \oplus B_l \oplus J)$. Now let $f \in \mathrm{Id}^*(B_1 \oplus \cdots \oplus B_l \oplus J)$ and suppose that f is not a $*$-identity of A. We may clearly assume that f is multilinear. Moreover, by choosing a $*$-basis of A as the union of a basis of $A_1 \oplus \cdots \oplus A_l$ and a basis of J it is enough to evaluate f on this basis. Let $u_1, \dots, u_t$ be elements of this basis such that $f(u_1, \dots, u_t) \neq 0$. Since $f \in \mathrm{Id}^*(J)$ at least one element, say u_k, does not belong to J. Then $u_k \in A_i$, for some i. Recalling that $A_i A_j = A_j A_i = A_i J A_j = A_j J A_i = 0$, for all $j \neq i$, we must have that $u_1, \dots, u_t \in A_i \cup J$. Thus $u_1, \dots, u_t \in A_i + J = B_i$ and this contradicts the fact that f is a $*$-identity of B_i. This proves the claim. Now the proof is completed by noticing that $\dim B_i/J(B_i) = 1$.

In case F is arbitrary, we consider the algebra $\bar{A} = A \otimes_F \bar{F}$, where $\bar{F}$ is the algebraic closure of F and $\bar{A} = A \otimes_F \bar{F}$ is a $*$-algebra with the induced superinvolution or graded involution $(a \otimes \alpha)^* = a^* \otimes \alpha$, for $a \in A, \alpha \in \bar{F}$. Clearly A is T_2^*-equivalent to $\bar{A}$. Moreover the $*$-codimensions of A over F coincide with the $*$-codimensions of $\bar{A}$ over $\bar{F}$. By the hypothesis it follows that the $*$-codimensions of $\bar{A}$ are polynomially bounded. But then by the first part of the proof, $\bar{A} = B_1 \oplus \cdots \oplus B_m$ where $B_1, \dots, B_m$ are finite dimensional $*$-algebras over $\bar{F}$ and $\dim_{\bar{F}} B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$. By the previous lemma there exist finite dimensional $*$-algebras $C_1, \dots, C_m$ over F such that, for all i, $C_i \sim_{T_2^*} B_i$ and $\dim_F C_i/J(C_i) = \dim_{\bar{F}} B_i/J(B_i) \leq 1$. It follows that $\mathrm{Id}^*(A) = \mathrm{Id}^*(\bar{A}) = \mathrm{Id}^*(B_1 \oplus \cdots \oplus B_m) = \mathrm{Id}^*(C_1 \oplus \cdots \oplus C_m)$ and we are done. □

In some cases we have a stronger result.

Theorem 2.5. *Let $A = A_0 \oplus A_1$ be a $*$-algebra such that $A_1 = 0$ or $*$ is the trivial involution. Then $c_n^*(A)$, $n = 1, 2, \dots$, is polynomially bounded if and only if A is T_2^*-equivalent to a finite direct sum of algebras $B_1 \oplus \cdots \oplus B_m$, where $B_1, \dots, B_m$ are finite dimensional $*$-algebras over F and $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$.*

Proof. If $A \sim_{T_2^*} B$, where $B = B_1 \oplus \cdots \oplus B_m$ with $B_1, \dots, B_m$ finite dimensional $*$-algebras over F and $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$, then by the proof of Theorem 2.4, $c_n^*(A)$ is polynomially bounded for all $i = 1, \dots, m$.

Now suppose that $c_n^*(A)$ is polynomially bounded for all $i = 1, \dots, m$. Notice that if $A_1 = 0$ then A is just an ordinary algebra with involution and the result follows by [22, Theorem 3].

Finally if $*$ is the trivial involution, then $\mathrm{var}^*(A) = \mathrm{var}^{sup}(A)$, where var^{sup} denotes a variety of superalgebras and the result follows by [6, Proposition 4]. □

Next we shall give characterizations of the varieties of polynomial growth through the behaviour of their sequences of cocharacters.

Let $n \geq 1$ and write $n = n_1 + \cdots + n_4$ as a sum of non-negative integers. We denote by $P_{n_1,\dots,n_4} \subseteq P_n^*$

the vector space of the multilinear $*$-polynomials in which the first n_1 variables are even symmetric, the next n_2 variables are even skew, the next n_3 variables are odd symmetric and the last n_4 variables are odd skew. The group $S_{n_1} \times \cdots \times S_{n_4}$ acts on the left on the vector space $P_{n_1,\dots,n_4}$ by permuting the variables of the same homogeneous degree which are all even or all odd at the same time. Thus S_{n_1} permutes the variables $y_1^+, \dots, y_{n_1}^+$, S_{n_2} permutes the variables $y_{n_1+1}^-, \dots, y_{n_1+n_2}^-$, and so on. In this way $P_{n_1,\dots,n_4}$ becomes a module over the group $S_{n_1} \times \cdots \times S_{n_4}$. Now $P_{n_1,\dots,n_4} \cap \mathrm{Id}^*(A)$ is invariant under this action and so the vector space

$$P_{n_1,\dots,n_4}(A) = \frac{P_{n_1,\dots,n_4}}{P_{n_1,\dots,n_4} \cap \mathrm{Id}^*(A)}$$

is an $(S_{n_1} \times \cdots \times S_{n_4})$-module with the induced action. We denote by $\chi_{n_1,\dots,n_4}(A)$ its character and it is called the $(n_1,\dots,n_4)$-th cocharacter of A.

If $\lambda = (\lambda_1, \dots, \lambda_r)$ is a partition of n, we write $\lambda \vdash n$. It is well-known that there is a one-to-one correspondence between partitions of n and irreducible S_n-characters. Hence if $\lambda \vdash n$, we denote by χ_λ the corresponding irreducible S_n-character. If $\lambda(1) \vdash n_1, \dots, \lambda(4) \vdash n_4$ are partitions we write $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(4)) \vdash (n_1, \dots, n_4)$ or $\langle\lambda\rangle \vdash n$ and we say that $\langle\lambda\rangle$ is a multipartition of $n = n_1 + \cdots + n_4$.

Since $\operatorname{char} F = 0$, by complete reducibility, $\chi_{n_1,\dots,n_4}(A)$ can be written as a sum of irreducible characters

$$\chi_{n_1,\dots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}, \tag{1}$$

where $m_{\langle\lambda\rangle} \geq 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ in $\chi_{n_1,\dots,n_4}(A)$.

Now if we set $c_{n_1,\dots,n_4}(A) = \dim_F P_{n_1,\dots,n_4}(A)$ it is immediate to see that

$$c_n^*(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1,\dots,n_4} c_{n_1,\dots,n_4}(A), \tag{2}$$

where $\binom{n}{n_1,\dots,n_4} = \frac{n!}{n_1!\cdots n_4!}$ stands for the multinomial coefficient.

Hence the growth of $c_n^*(A)$ is related to the growth of multinomial coefficients and of degrees of irreducible characters.

Theorem 2.6. *Let A be a finite dimensional $*$-algebra over a field F of characteristic zero. Then $c_n^*(A)$, $n = 1, 2, \dots,$ is polynomially bounded if and only if for every $n_1, \dots, n_4$ with $n_1 + \cdots + n_4 = n$ it holds*

$$\chi_{n_1,\dots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_4) \\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)},$$

where q is such that $J(A)^q = 0$ and $\lambda(1)_1$ denotes the length of the first row of the Young diagram corresponding to the partition $\lambda(1)$.

Proof. This result can be proved following word by word the proof given in [15, Theorem 2.2] for graded algebras. ☐

The following theorem collects results about $*$-varieties of polynomial growth.

Theorem 2.7. *For a finite dimensional $*$-algebra A the following conditions are equivalent:*

1) *$c_n^*(A)$ is polynomially bounded;*
2) *$A \sim_{T_2^*} B$, where $B = B_1 \oplus \cdots \oplus B_m$ with $B_1, \dots, B_m$ finite dimensional $*$-algebras over F and $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$;*
3) *for every $n_1, \dots, n_4$ with $n_1 + \cdots + n_4 = n$ it holds*

$$\chi_{n_1,\dots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_4) \\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)},$$

   where q is such that $J(A)^q = 0$;
4) *$M, M^{sup}, D \notin var^*(A)$ in case $*$ is a superinvolution and $M, M^{sup}, D, D^{sup} \notin var^*(A)$ in case $*$ is a graded involution.*

3. Classifying the subvarieties of $\mathrm{var}^*(D)$ and $\mathrm{var}^*(M)$

In this section we classify, up to T_2^*-equivalence, all the $*$-algebras contained in the variety generated by D or M. Here $*$ is a graded involution and also a superinvolution.

As we have remarked before, this is equivalent to the classification of the algebras with involution inside the varieties of algebras with involution generated by D or M. Such a classification was given in [22]. In what follows we present such results in the language of $*$-algebras for convenience of the reader.

Next we construct, for any fixed $k \geq 1$, $*$-algebras belonging to the variety generated by D whose $*$-codimension sequence grows polynomially as n^k.

For $k \geq 2$, let I_k be the $k \times k$ identity matrix and $E_1 = \sum_{i=1}^{k-1} e_{i,i+1}$, where the e_{ij}s denote the usual matrix units.

We denote by

$$C_k = \{\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F\} \subseteq UT_k,$$

a commutative subalgebra of UT_k. We also write C_k to mean the algebra C_k with trivial grading and with gs-involution given by

$$(\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i)^* = \alpha I_k + \sum_{1 \leq i < k} (-1)^i \alpha_i E_1^i.$$

We next state the following result characterizing the $*$-identities and the $*$-codimensions of C_k (see [22]).

Theorem 3.1. *Let $k \geq 2$. Then*

1) $Id^*(C_k) = \langle [x_1, x_2], y_1^- \cdots y_k^-, z^+, z^- \rangle_{T_2^*}$.
2) $c_n^*(C_k) = \sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!} n^{k-1}$.

The following result classifies all the subvarieties of the variety generated by D.

Theorem 3.2. [22] *Let A be a $*$-algebra such that $A \in var^*(D)$. Then either $A \sim_{T_2^*} D$ or $A \sim_{T_2^*} N$ or $A \sim_{T_2^*} C \oplus N$ or $A \sim_{T_2^*} C_k \oplus N$, for some $k \geq 2$, where N is a nilpotent $*$-algebra and C is a commutative algebra with trivial gs-involution.*

Next we exhibit finite dimensional $*$-algebras belonging to the variety generated by M whose $*$-codimension sequence grows polynomially.

For $k \geq 2$, let

$$\begin{aligned}
A_k &= \mathrm{span}_F \left\{e_{11} + e_{2k,2k}, E, \ldots, E^{k-2}, e_{12}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-1,2k}\right\}, \\
N_k &= \mathrm{span}_F \left\{I, E, \ldots, E^{k-2}, e_{12} - e_{2k-1,2k}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-2,2k}\right\}, \\
U_k &= \mathrm{span}_F \left\{I, E, \ldots, E^{k-2}, e_{12} + e_{2k-1,2k}, e_{13}, \ldots, e_{1k}, e_{k+1,2k}, e_{k+2,2k}, \ldots, e_{2k-2,2k}\right\},
\end{aligned}$$

be subalgebras of UT_{2k}. Here I denotes the $2k \times 2k$ identity matrix and $E = \sum_{i=2}^{k-1} e_{i,i+1} + e_{2k-i,2k-i+1}$. We also write A_k, N_k and U_k to mean the above algebras with trivial grading and with reflection gs-involution.

We next state the following results characterizing the $*$-identities and the growth of the $*$-codimensions of the above algebras (see [22] for more details).

Theorem 3.3. *For every $k \geq 2$ we have:*

1) $Id^*(A_k) = \langle y_1^- y_2^-, z^+, z^-, y_1^+ \cdots y_{k-2}^+ St_3(y_{k-1}^+, y_k^+, y_{k+1}^+) y_{k+2}^+ \cdots y_{2k-1}^+, y_1^+ \cdots y_{k-1}^+ y^- y_k^+ \cdots y_{2k-2}^+ \rangle_{T_2^*}$;
2) $c_n^*(A_k) \approx qn^{k-1}$, *for some* $q > 0$.

Theorem 3.4. *The T_2^*-ideal $Id^*(N_k)$ is generated by the polynomials $[x_1, x_2]$, $y_1^- y_2^-$, z^+, z^-, in case $k = 2$ and by $[y_1^+, \ldots, y_{k-1}^+]$, $y_1^- y_2^-$, z^+, z^-, in case $k \geq 3$. Moreover*

$$c_n^*(N_k) = 1 + \sum_{i=1}^{k-2} \binom{n}{i} (2i-1) + \binom{n}{k-1} (k-1) \approx qn^{k-1}, \text{ for some } q > 0.$$

Theorem 3.5. *The T_2^*-ideal $Id^*(U_k)$ is generated by the polynomials $[x_1,x_2]$, y^-, z^+, z^-, in case $k=2$ and by $[y^-,y_1^+,\dots,y_{k-2}^+]$, $y_1^-y_2^-$, z^+, z^-, in case $k\geq 3$. Moreover $c_n^*(U_2)=1$ and*

$$c_n^*(U_k)=1+\sum_{i=1}^{k-2}\binom{n}{i}(2i-1)+\binom{n}{k-1}(k-2)\approx qn^{k-1},\ \textit{for some } q>0,\ \textit{for } k\geq 3.$$

The following result classifies the subvarieties of $\mathrm{var}^*(M)$.

Theorem 3.6. [22, Theorem 6] *If $A\in var^*(M)$ then A is T_2^*-equivalent to one of the following $*$-algebras:*

$$M,\ N,\ N_k\oplus N,\ U_k\oplus N,\ N_k\oplus U_k\oplus N,\ A_t\oplus N,\ N_k\oplus A_t\oplus N,\ U_k\oplus A_t\oplus N,\ N_k\oplus U_k\oplus A_t\oplus N,$$

for some $k,t\geq 2$, where N is a nilpotent $*$-algebra.

As a consequence of the previous theorems, we can also get the classification of all $*$-algebras generating minimal varieties.

Corollary 3.1. *A $*$-algebra $A\in var^*(D)$ generates a minimal variety of polynomial growth if and only if $A\sim_{T_2^*}C_k$, for some $k\geq 2$.*

Corollary 3.2. *A $*$-algebra $A\in var^*(M)$ generates a minimal variety of polynomial growth if and only if either $A\sim_{T_2^*}U_r$ or $A\sim_{T_2^*}N_k$ or $A\sim_{T_2^*}A_k$, for some $r>2$, $k\geq 2$.*

4. Algebras with 1 of polynomial $*$-codimension growth

In this section we classify, up to T_2^*-equivalence, all the $*$-algebras with 1 contained in the variety generated by M^{sup}, where $*$ is, as we have remarked before, a superinvolution and also a graded involution.

We recall the following result characterizing the $(n_1,\dots,n_4)$-th cocharacter of M^{sup}.

Theorem 4.1 ([12])**.** *If $\chi_{n_1,\dots,n_4}(M^{sup})=\sum_{\langle\lambda\rangle\vdash(n_1,\dots,n_4)}m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(4)}$ is the $(n_1,\dots,n_4)$-th cocharacter of M^{sup}, $n_1+\cdots+n_4=n$, then*

$$m_{\langle\lambda\rangle}=\begin{cases}1 & \textit{if } \langle\lambda\rangle=((n),\emptyset,\emptyset,\emptyset)\\ q+1 & \textit{if } \langle\lambda\rangle=((p+q,p),\emptyset,(1),\emptyset)\\ q+1 & \textit{if } \langle\lambda\rangle=((p+q,p),\emptyset,\emptyset,(1))\\ 0 & \textit{otherwise}\end{cases},$$

where $p,q\geq 0$ and $2p+q+1=n$.

We are going to prove that, in case $A\in\mathrm{var}^*(M^{sup})$ generates a variety of polynomial growth, then A satisfies the same $*$-identities as a finite dimensional $*$-algebra.

We start with the following.

Theorem 4.2. *If $A\in var^*(M^{sup})$ then $var^*(A)=var^*(B)$ for some finitely generated $*$-algebra B.*

Proof. Let B be the relatively free algebra of $\mathrm{var}^*(A)$ with 2 even symmetric, 1 odd symmetric and 1 odd skew generators. We shall prove that $\mathrm{var}^*(A)=\mathrm{var}^*(B)$. Clearly $\mathrm{var}^*(B)\subseteq\mathrm{var}^*(A)$.
In order to get the opposite inclusion we need to prove that $\mathrm{Id}^*(B)\subseteq\mathrm{Id}^*(A)$. Let f be a $*$-identity of B. Since $\mathrm{char}F=0$, we may assume that $f=f(y_1^+,\dots,y_{n_1}^+,y_1^-,\dots,y_{n_2}^-,z_1^+,\dots,z_{n_3}^+,z_1^-,\dots,z_{n_4}^-)$ is multilinear. Let L be the $(S_{n_1}\times\cdots\times S_{n_4})$-module generated by f and let $L=L_1\oplus\cdots\oplus L_m$ be its decomposition into irreducible components with L_i generated by f_i as an $(S_{n_1}\times\cdots\times S_{n_4})$-module, $i=1,\dots,m$. If $f_i\equiv 0$ on A for all $i=1,\dots,m$, then also $f\equiv 0$ on A. Hence, without loss of generality, we may assume that L is irreducible.

Let $\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(4)}$ be the irreducible character of L, where $\lambda(i)\vdash n_i$, $i=1,\dots,4$ and let $e_{T_{\lambda(i)}}=\left(\sum_{\tau\in R_{T_{\lambda(i)}}}\tau\right)\left(\sum_{\sigma\in C_{T_{\lambda(i)}}}(sgn\sigma)\sigma\right)$, $i=1,\dots,4$, be the corresponding essential idempotents (see [13, Chapter 2]).

Notice that, if $\lambda(1)_3\neq 0$ or $\lambda(2)\neq\emptyset$ or $\lambda(3)\notin\{\emptyset,(1)\}$ or $\lambda(4)\notin\{\emptyset,(1)\}$ then, from Theorem 4.1 follows that $f\equiv 0$ on A.

Therefore, in order to complete the proof, we may assume that $\lambda(1)_3 = 0$, $\lambda(2) = \emptyset$ and $\lambda(3), \lambda(4) \in \{\emptyset, (1)\}$.

Now we consider $g = \left(\sum_{\tau \in R_{T_{\lambda(1)}}} \tau\right) f$. Since L is irreducible and $g \neq 0$ then $f \equiv 0$ on A if and only if $g \equiv 0$ on A. We shall prove that $g \equiv 0$ on A.

Notice that g is symmetric on at most 2 disjoint subsets Y_1, Y_2 of $\{y_1^+, y_2^+, \ldots\}$. If we identify all variables of Y_1 with y_1^+ and all variables of Y_2 with y_2^+ we obtain a homogeneous polynomial $t = t(y_1^+, y_2^+, z^+, z^-)$ which is still a $*$-identity of B. From the definition of relatively free algebra, it follows that $t \equiv 0$ on A. But the complete linearization of t on all even symmetric variables is equal to $\gamma g(y_1^+, \ldots, y_{n_1}^+, z^+, z^-)$ where $\gamma = \lambda(1)_1!\ \lambda(1)_2! \neq 0$. Hence $g \equiv 0$ on A and so $f \equiv 0$ on A follows. $\square$

In order to characterize the varieties of polynomial growth we need to apply the following result.

Theorem 4.3. [1]. *If A is a finitely generated algebra with superinvolution over an algebraically closed field F of characteristic zero then A satisfies the same $*$-identities as a finite dimensional algebra over F.*

As a consequence of Theorems 4.2 and 4.3 we get the following.

Corollary 4.1. *Let $A \in var^*(M^{sup})$ be a $*$-algebra over an algebraically closed field F of characteristic zero. Then $Id^*(A) = Id^*(B)$ for some finite dimensional $*$-algebra B.*

In order to study $*$-identities of algebras A with 1 we define the proper $*$-polynomials.

We say that a polynomial $f \in P_n^*$ is a proper $*$-polynomial if it is a linear combination of elements of the type

$$y_{i_1}^- \cdots y_{i_s}^- z_{j_1}^+ \cdots z_{j_t}^+ z_{l_1}^- \cdots z_{l_r}^- w_1 \cdots w_m$$

where $w_1, \ldots, w_m$ are left normed (long) Lie commutators in the variables from $Y \cup Z$ (here the symmetric even variables appear only inside the commutators).

We denote by Γ_n^* the subspace of P_n^* of proper $*$-polynomials and $\Gamma_0^* = \mathrm{span}\{1\}$.

The sequence of proper $*$-codimensions is defined as

$$\gamma_n^*(A) = \dim \frac{\Gamma_n^*}{\Gamma_n^* \cap \mathrm{Id}^*(A)},\ n = 0, 1, 2, \ldots.$$

For a unitary $*$-algebra A, the relation between $*$-codimensions and proper $*$-codimensions (see for instance [3]), is given by the following:

$$c_n^*(A) = \sum_{i=0}^{n} \binom{n}{i} \gamma_i^*(A),\ n = 0, 1, 2 \ldots. \tag{3}$$

Given two sets of polynomials $S, S' \subseteq F\langle Y \cup Z, *\rangle$, we say that S' is a consequence of S if $S' \subseteq \langle S \rangle_{T_2^*}$.

Proposition 4.1. *For every $i \geq 1$, Γ_{k+i}^* is a consequence of Γ_k^*.*

Proof. This result can be proved following closely the proof of Lemma 2.2 in [19, 23]. $\square$

As a consequence we have the following.

Corollary 4.2. *Let A be a $*$-algebra with 1. If for some $k \geq 2$, $\gamma_k^*(A) = 0$ then $\gamma_m^*(A) = 0$ for all $m \geq k$.*

Let $n = n_1 + \cdots + n_4 \geq 1$. We denote by $\Gamma_{n_1,\ldots,n_4} \subseteq P_{n_1,\ldots,n_4}$ the subspace of proper $*$-polynomials, which is also an $(S_{n_1} \times \cdots \times S_{n_4})$-submodule of $P_{n_1,\ldots,n_4}$. Since $\Gamma_{n_1,\ldots,n_4} \cap \mathrm{Id}^*(A)$ is invariant under the action of $S_{n_1} \times \cdots \times S_{n_4}$, the vector space

$$\Gamma_{n_1,\ldots,n_4}(A) = \frac{\Gamma_{n_1,\ldots,n_4}}{\Gamma_{n_1,\ldots,n_4} \cap \mathrm{Id}^*(A)}$$

is an $(S_{n_1} \times \cdots \times S_{n_4})$-module with the induced action. We denote by $\psi_{n_1,\ldots,n_4}(A)$ its character and it is called the $(n_1, \ldots, n_4)$-th proper cocharacter of A.

Since $\mathrm{char}\, F = 0$, by complete reducibility $\psi_{n_1,\ldots,n_4}(A)$ can be written as a sum of irreducible characters

$$\psi_{n_1,\ldots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}, \tag{4}$$

where $m_{\langle\lambda\rangle} \geq 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ in $\psi_{n_1,\ldots,n_4}(A)$.

Now if we set $\gamma_{n_1,\ldots,n_4}(A) = \dim_F \Gamma_{n_1,\ldots,n_4}(A)$ it is immediate to see that

$$\gamma_n^*(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1,\ldots,n_4} \gamma_{n_1,\ldots,n_4}(A). \tag{5}$$

Next we consider the algebras N_k and U_k we have defined before endowed with elementary $\mathbb{Z}_2$-gradings. Recall that if $\mathbf{g} = (g_1,\ldots,g_{2k}) \in \mathbb{Z}_2^{2k}$ is an arbitrary $2k$-tuple of elements of $\mathbb{Z}_2$, then $\mathbf{g}$ defines an elementary $\mathbb{Z}_2$-grading on UT_{2k} by setting

$$(UT_{2k})_0 = \mathrm{span}\{e_{ij} \mid g_i + g_j = 0\} \text{ and } (UT_{2k})_1 = \mathrm{span}\{e_{ij} \mid g_i + g_j = 1\}$$

(recall that equalities are taken modulo 2). If A is a graded subalgebra of UT_{2k} the induced grading on A is also called elementary.

Definition 4.1. *For $k \geq 2$, N_k^{sup} is the algebra N_k with elementary $\mathbb{Z}_2$-grading induced by* $\mathbf{g} = (0, \underbrace{1,\ldots,1}_{k-1}, \underbrace{0,\ldots,0}_{k-1}, 1)$ *and with reflection gs-involution.*

The following result characterizes the $*$-identities and the $*$-codimensions of N_k^{sup}.

Theorem 4.4. *Let $k \geq 2$. Then:*

1) *$Id^*(N_k^{sup}) = \langle y^-,\ z_1z_2,\ [z^+, y_1,\ldots,y_{k-2}]\rangle_{T_2^*}$;*
2) *$c_n^*(N_k^{sup}) = 1 + \sum_{i=1}^{k-2} 2i\binom{n}{i} + \binom{n}{k-1}(k-1) \approx qn^{k-1}$, for some $q > 0$.*

Proof. Let $I = \langle y^-,\ z_1z_2,\ [z^+, y_1,\ldots,y_{k-2}]\rangle_{T_2^*}$. It is easy to see that $I \subseteq \mathrm{Id}^*(N_k^{sup})$. Let now f be a $*$-identity of N_k^{sup}. We may assume that f is multilinear and, since N_k^{sup} is an algebra with 1, we may take f proper. After reducing the polynomial f modulo I we obtain that f is the zero polynomial if $\deg f \geq k$, f is a linear combination of commutators

$$[z_i^-, y_{i_1}^+,\ldots,y_{i_{k-2}}^+],\ i_1 < \cdots < i_{k-2}$$

in case $\deg f = k-1$ and f is a linear combination of commutators

$$[z_i^-, y_{i_1}^+,\ldots,y_{i_{s-1}}^+],\ [z_j^+, y_{j_1}^+,\ldots,y_{j_{s-1}}^+],\ i_1 < \cdots < i_{s-1},\ j_1 < \cdots < j_{s-1}$$

in case $\deg f = s < k-1$. Hence, for some $s = 1,\ldots,k-1$,

$$f = \sum_{i=1}^{s} \alpha_i\ [z_i^-, y_{i_1}^+,\ldots,y_{i_{s-1}}^+] + \sum_{j=1}^{s} \beta_j\ [z_j^+, y_{j_1}^+,\ldots,y_{j_{s-1}}^+].$$

Suppose that there exists i such that $\alpha_i \neq 0$ (resp. $\beta_i \neq 0$). By making the evaluation $z_i^- = e_{12} - e_{2k-1,2k}$, $z_l^- = 0$, for all $l \neq i$, $z_j^+ = 0$, for $j = 1,\ldots,s$ (resp. $z_i^+ = e_{13} + e_{2k-2,2k}$, $z_l^+ = 0$, for all $l \neq i$, $z_j^- = 0$, for $j = 1,\ldots,s$) and $y_l = E$ for all $l = i_1,\ldots,i_{s-1}$, we get that $\alpha_i = 0$ (resp. $\beta_i = 0$), a contradiction. Hence $\alpha_i = \beta_i = 0$, for all $i = 1,\ldots,s$. This says that $f \in I$ and, so, $\mathrm{Id}^*(N_k^{sup}) = I$. The argument above also proves that $\gamma_s^*(N_k^{sup}) = s$ for $s = k-1$, $\gamma_s^*(N_k^{sup}) = 2s$ for $s < k-1$ and $\gamma_s^*(N_k^{sup}) = 0$ for $s \geq k$. Then, by (3) we have

$$c_n^*(N_k^{sup}) = 1 + \sum_{i=1}^{k-2} \binom{n}{i} 2i + \binom{n}{k-1}(k-1) \approx qn^{k-1}, \text{ for some } q > 0.$$

□

Definition 4.2. *For $k \geq 2$, U_k^{sup} is the algebra U_k with elementary $\mathbb{Z}_2$-grading induced by* $\mathbf{g} = (0, \underbrace{1,\ldots,1}_{k-1}, \underbrace{0,\ldots,0}_{k-1}, 1)$ *and with reflection gs-involution.*

The following results characterizing the $*$-identities and the $*$-codimensions of U_k^{sup} and $N_k^{sup} \oplus U_k^{sup}$ can be proved in a similar way as the previous theorem.

Theorem 4.5. *Let $k \geq 2$. Then:*

1) $Id^*(U_k^{sup}) = \langle y^-,\ z_1 z_2,\ [z^-, y_1, \dots, y_{k-2}]\rangle_{T_2^*}$*;*
2) $c_n^*(U_k^{sup}) = 1 + \sum_{i=1}^{k-2} \binom{n}{i} 2i + \binom{n}{k-1}(k-1) \approx qn^{k-1}$*, for some* $q > 0$*.*

Theorem 4.6. *If $k \geq 2$ then:*

1) $Id^*(N_k^{sup} \oplus U_k^{sup}) = \langle y^-,\ z_1 z_2,\ [z, y_1, \dots, y_{k-1}]\rangle_{T_2^*}$*;*
2) $c_n^*(N_k^{sup} \oplus U_k^{sup}) = 1 + \sum_{i=1}^{k-1} \binom{n}{i} 2i \approx qn^{k-1}$*, for some* $q > 0$*.*

Notice that $U_t^{sup} \oplus N_k^{sup} \sim_{T_2^*} U_t^{sup}$ if $t > k$ and $U_t^{sup} \oplus N_k^{sup} \sim_{T_2^*} N_k^{sup}$ if $t < k$.

From now until the end of this section we assume that the field F is algebraically closed.

Theorem 4.7. *For any $k \geq 2$, N_k^{sup} generates a minimal variety of polynomial growth.*

Proof. Let $A \in \mathrm{var}^*(N_k^{sup})$ be such that $c_n^*(A) \approx qn^{k-1}$, for some $q > 0$. We shall prove that $A \sim_{T_2^*} N_k^{sup}$. Since $A \in \mathrm{var}^*(M^{sup})$ by Corollary 4.1 A satisfies the same $*$-identities as a finite dimensional $*$-algebra. Hence, since $c_n^*(A)$ is polynomially bounded, by Theorem 2.4 we may assume that

$$A = B_1 \oplus \cdots \oplus B_m,$$

where $B_1, \dots, B_m$ are finite dimensional $*$-algebras such that $\dim B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$. This implies that either $B_i \cong F + J(B_i)$ or $B_i = J(B_i)$ is a nilpotent $*$-algebra. Since $c_n^*(A) \leq c_n^*(B_1) + \cdots + c_n^*(B_m)$, then there exists B_i such that $c_n^*(B_i) \approx bn^{k-1}$, for some $b > 0$. Hence

$$\mathrm{var}^*(N_k^{sup}) \supseteq \mathrm{var}^*(A) \supseteq \mathrm{var}^*(F + J(B_i)) \supseteq \mathrm{var}^*(F + J_{11}(B_i)).$$

Hence, in order to complete the proof it is enough to show that $F + J_{11}(B_i) \sim_{T_2^*} N_k^{sup}$. Thus, without loss of generality, we may assume that A is a unitary $*$-algebra. Since $c_n^*(A) \approx qn^{k-1}$ then $c_n^*(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^*(A)$ and, by Corollary 4.2, $\gamma_i^*(A) \neq 0$ for all $i = 0, \dots, k-1$.

For $n_1 + \cdots + n_4 = n$, let $\psi_{n_1,\dots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ and $\psi_{n_1,\dots,n_4}(N_k^{sup}) = \sum_{\langle\lambda\rangle \vdash n} m'_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ be the $(n_1, \dots, n_4)$-th proper cocharacters of A and N_k^{sup}, respectively. Since $\mathrm{Id}^*(A) \supseteq \mathrm{Id}^*(N_k^{sup})$, we must have $m_{\langle\lambda\rangle} \leq m'_{\langle\lambda\rangle}$ for all $\langle\lambda\rangle \vdash n = n_1 + \ldots + n_4$.

For any $i = 2, \dots, k-1$, let $f_1 = [z_1^+, y_2^+, \dots, y_2^+]$ and $f_2 = [z_1^-, y_2^+, \dots, y_2^+]$ be highest weight vectors corresponding to the multipartitions $\langle\lambda\rangle = ((i-1),\ \emptyset,\ (1),\ \emptyset)$ and $\langle\mu\rangle = ((i-1),\ \emptyset,\ \emptyset,\ (1))$ (see [2, Chapter 12]). It is easily seen that f_1 is not a $*$-identity of N_k^{sup} for $i = 2, \dots, k-2$ and f_2 is not a $*$-identity of N_k^{sup} for $i = 2, \dots, k-1$.

Thus for $i = 2, \dots, k-2$, $\chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_{(1)} \otimes \chi_\emptyset$ participates in the $(i-1, 0, 1, 0)$-th proper cocharacter $\psi_{i-1,0,1,0}(N_k^{sup})$ with non-zero multiplicity. Also, for $i = 2, \dots, k-1$, $\chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)}$ participates in the $(i-1, 0, 0, 1)$-th proper cocharacter $\psi_{i-1,0,0,1}(N_k^{sup})$ with non-zero multiplicity.

Hence, for $i = 2, \dots, k-2$, since

$$\gamma_i^*(N_k^{sup}) = 2i = \binom{i}{i-1,0,1,0} \deg \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_{(1)} \otimes \chi_\emptyset + \binom{i}{i-1,0,0,1} \deg \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)},$$

by (5) we have that, for $n_1 + \cdots + n_4 = i$

$$\psi_{n_1,n_2,n_3,n_4}(N_k^{sup}) = \begin{cases} \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_{(1)} \otimes \chi_\emptyset & \text{if } (n_1,n_2,n_3,n_4) = (i-1,0,1,0) \\ \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)} & \text{if } (n_1,n_2,n_3,n_4) = (i-1,0,0,1) \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, since

$$\gamma_{k-1}^*(N_k^{sup}) = k-1 = \binom{k-1}{k-2,0,0,1} \deg \chi_{(k-2)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)},$$

we get that

$$\psi_{k-2,0,0,1}(N_k^{sup}) = \chi_{(k-2)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)}$$

and

$$\psi_{n_1,n_2,n_3,n_4}(N_k^{sup}) = 0 \text{ if } (n_1,n_2,n_3,n_4) \neq (k-2,0,0,1), n_1+\cdots+n_4 = k-1.$$

Since $\gamma^*_{k-1}(A) \neq 0$ and $m_{\langle\lambda\rangle} \leq m'_{\langle\lambda\rangle}$, for any $\langle\lambda\rangle \vdash n_1+\cdots+n_4$, then we get that $\psi_{k-2,0,0,1}(A) = \chi_{(k-2)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)}$ and $\psi_{n_1,n_2,n_3,n_4}(A) = 0$ if $(n_1,n_2,n_3,n_4) \neq (k-2,0,0,1), n_1+\cdots+n_4 = k-1$.

Moreover for $n_1+\cdots+n_4 = i$, where $i = 2,\dots,k-2$, we claim that $\psi_{n_1,n_2,n_3,n_4}(A) = \psi_{n_1,n_2,n_3,n_4}(N_k^{sup})$.

In fact, if $\psi_{i-1,0,0,1}(A) = 0$, for some $2 \leq i \leq k-2$, then the highest weight vector $[z_1^-, \underbrace{y_2^+,\dots,y_2^+}_{i-1}]$ corresponding to the multipartition $((i-1),\ \emptyset,\ \emptyset,\ (1))$ would be a $*$-identity for A. But this implies that also $[z_1^+, \underbrace{y_2^+,\dots,y_2^+}_{k-1}]$ is a $*$-identity for A, and so $\psi_{k-2,0,0,1}(A) = 0$, a contradiction. In a similar way one can prove that if $\psi_{i-1,0,1,0}(A) = 0$ we would reach a contradiction.

Hence

$$c_n^*(A) = \sum_{i=0}^{k-1}\binom{n}{i}\gamma_i^*(A) = \sum_{i=0}^{k-1}\binom{n}{i}\sum_{n_1+\cdots+n_4=i}\binom{i}{n_1,\dots,n_4}\gamma_{n_1,\dots,n_4}(A) = 1+\sum_{i=1}^{k-2}\binom{n}{i}2i+\binom{n}{k-1}(k-1) = c_n^*(N_k^{sup}).$$

Thus A and N_k^{sup} have the same sequence of $*$-codimensions and, since $\mathrm{Id}^*(N_k^{sup}) \subseteq \mathrm{Id}^*(A)$ we get the equality $\mathrm{Id}^*(N_k^{sup}) = \mathrm{Id}^*(A)$ and the proof is complete. □

In a similar way it is possible to prove the following.

Theorem 4.8. *For any $k \geq 2$, U_k^{sup} generates a minimal variety of polynomial growth.*

In the following result we classify, up to T_2^*-equivalence, all $*$-algebras with 1 inside $\mathrm{var}^*(M^{sup})$.

Theorem 4.9. *Let $A \in var^*(M^{sup})$ be an algebra with 1 such that $c_n^*(A) \approx qn^{k-1}$ for some $q > 0$, $k \geq 1$. Then either $A \sim_{T_2^*} C$ or $A \sim_{T_2^*} U_k^{sup}$ or $A \sim_{T_2^*} N_k^{sup}$ or $A \sim_{T_2^*} N_k^{sup} \oplus U_k^{sup}$, where C is a commutative algebra with trivial gs-involution.*

Proof. If $k = 1$ it is immediate to see that A is a commutative algebra with trivial gs-involution.

Let now $k \geq 2$. Since $c_n^*(A) \approx qn^{k-1}$, by (3) $\gamma^*_{k-1}(A) \neq 0$. Hence at least one polynomial among $[z^+, y_1^+,\dots,y_{k-2}^+]$ and $[z^-, y_1^+,\dots,y_{k-2}^+]$ cannot be a $*$-identity for A, since otherwise we would have $\gamma^*_{k-1}(A) = 0$, a contradiction.

If $[z^-, y_1^+,\dots,y_{k-2}^+]$ is not a $*$-identity and $[z^+, y_1^+,\dots,y_{k-2}^+] \equiv 0$ on A then $\mathrm{Id}^*(N_k^{sup}) \subseteq \mathrm{Id}^*(A)$ and since $c_n^*(A) \approx qn^{k-1}$, by Theorem 4.7, one gets that $A \sim_{T_2^*} N_k^{sup}$. Similarly, if $[z^+, y_1^+,\dots,y_{k-2}^+]$ is not a $*$-identity and $[z^-, y_1^+,\dots,y_{k-2}^+] \equiv 0$ on A one gets that $A \sim_{T_2^*} U_k^{sup}$.

Finally, suppose that neither of the polynomials $[z^+, y_1^+,\dots,y_{k-2}^+]$ and $[z^-, y_1^+,\dots,y_{k-2}^+]$ are $*$-identities for A. Since $c_n^*(A) \approx qn^{k-1}$, then $\gamma_k^*(A) = 0$, and so $\mathrm{Id}^*(N_k^{sup} \oplus U_k^{sup}) \subseteq \mathrm{Id}^*(A)$. As in the proof of Theorem 4.7, for $i = 2,\dots,k-1$, we get that:

$$\psi_{i-1,0,1,0}(A) = \psi_{i-1,0,1,0}(N_k^{sup} \oplus U_k^{sup}) = \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_{(1)} \otimes \chi_\emptyset,$$

$$\psi_{i-1,0,0,1}(A) = \psi_{i-1,0,0,1}(N_k^{sup} \oplus U_k^{sup}) = \chi_{(i-1)} \otimes \chi_\emptyset \otimes \chi_\emptyset \otimes \chi_{(1)}$$

and

$$\psi_{n_1,n_2,n_3,n_4}(A) = \psi_{n_1,n_2,n_3,n_4}(N_k^{sup} \oplus U_k^{sup}) = 0,$$

if $(n_1,n_2,n_3,n_4) \notin \{(i-1,0,0,1),(i-1,0,1,0)\}, n_1+\cdots+n_4 = i$. Hence A and $N_k^{sup} \oplus U_k^{sup}$ have the same sequence of $*$-codimensions:

$$c_n^*(A) = \sum_{i=0}^{k-1}\binom{n}{i}\gamma_i^*(A) = 1 + \sum_{i=1}^{k-1}2i\binom{n}{i} = c_n^*(N_k^{sup} \oplus U_k^{sup})$$

and, since $\mathrm{Id}^*(N_k^{sup} \oplus U_k^{sup}) \subseteq \mathrm{Id}^*(A)$ we finally get the equality $\mathrm{Id}^*(N_k^{sup} \oplus U_k^{sup}) = \mathrm{Id}^*(A)$. □

5. Classifying the subvarieties of var$^*(M^{sup})$

In this section we classify, up to T_2^*-equivalence, all $*$-algebras contained in the variety generated by M^{sup}. We start by constructing $*$-algebras without unit inside var$^*(M^{sup})$.

Definition 5.1. *For $k \geq 2$, A_k^{sup} is the algebra A_k with elementary $\mathbb{Z}_2$-grading induced by* $\mathbf{g} = (0, \underbrace{1, \dots, 1}_{k-1}, \underbrace{0, \dots, 0}_{k-1}, 1)$ *and with reflection gs-involution.*

Next we describe explicitly the $*$-identities of A_k^{sup}.

Theorem 5.1. *Let $k \geq 2$. Then*

1) *$Id^*(A_k^{sup}) = \langle y^-, z_1 z_2, y_1 \cdots y_{k-1} z y_k \cdots y_{2k-2} \rangle_{T_2^*}$;*
2) *$c_n^*(A_k^{sup}) = 1 + 2 \sum\limits_{\substack{t<k-1 \\ or \\ n-t<k}} \binom{n}{t}(n-t) \approx qn^{k-1}$ for some $q > 0$.*

Proof. Write $I = \langle y^-, z_1 z_2, y_1 \cdots y_{k-1} z y_k \cdots y_{2k-2} \rangle_{T_2^*}$. It is easily seen that $I \subseteq \mathrm{Id}^*(A_k^{sup})$. In order to prove the opposite inclusion, first we find a set of generators of P_n^* modulo $P_n^* \cap I$, for every $n \geq 1$.
Any multilinear polynomial of degree n can be written, modulo I, as a linear combination of monomials of the type

$$y_1^+ \cdots y_n^+, \quad y_{i_1}^+ \cdots y_{i_t}^+ z_l^+ y_{j_1}^+ \cdots y_{j_s}^+, \quad y_{r_1}^+ \cdots y_{r_p}^+ z_l^- y_{s_1}^+ \cdots y_{s_q}^+ \tag{6}$$

where $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$, $t < k-1$ or $s < k-1$, $r_1 < \cdots < r_p$, $s_1 < \cdots < s_q$ and $p < k-1$ or $q < k-1$. We next show that the above elements are linearly independent modulo $\mathrm{Id}^*(A_k^{sup})$. Let $f \in \mathrm{Id}^*(A_k^{sup})$ be a linear combination of the above monomials:

$$f = \delta y_1^+ \cdots y_n^+ + \sum_{\substack{t<k-1 \\ or \\ s<k-1}} \sum_{l,I,J} \alpha_{l,I,J}\, y_{i_1}^+ \cdots y_{i_t}^+ z_l^+ y_{j_1}^+ \cdots y_{j_s}^+ + \sum_{\substack{p<k-1 \\ or \\ q<k-1}} \sum_{m,R,S} \beta_{m,R,S}\, y_{r_1}^+ \cdots y_{r_p}^+ z_m^- y_{s_1}^+ \cdots y_{s_q}^+,$$

where $t + s = p + q = n - 1$ and for any fixed t and p, $I = \{i_1, \dots, i_t\}$, $J = \{j_1, \dots, j_s\}$, $R = \{r_1, \dots, r_p\}$ and $S = \{s_1, \dots, s_q\}$.

By making the evaluation $y_1^+ = \cdots = y_n^+ = e_{11} + e_{2k,2k}$, and $z_l^+ = z_l^- = 0$, for all $l = 1, \dots, n$, one gets $\delta(e_{11} + e_{2k,2k}) = 0$ and, so, $\delta = 0$.
For fixed $t < k-1, l, I, J$ the evaluation $z_l^+ = e_{12} + e_{2k-1,2k}$, $z_{l'}^+ = 0$, for all $l' \neq l$, $y_{i_1}^+ = \cdots = y_{i_t}^+ = E$, $y_{j_1}^+ = \cdots = y_{j_s}^+ = e_{11} + e_{2k,2k}$ and $z_m^- = 0$, for $m = 1, \dots, n$, gives

$$\alpha_{l,I,J}\ e_{2k-t-1,2k} + \alpha_{l,J,I}\ e_{1,2+t} = 0.$$

Thus $\alpha_{l,I,J} = \alpha_{l,J,I} = 0$.
Similarly, for fixed $s < k-1, l, I, J$ the evaluation $z_l^+ = e_{12} + e_{2k-1,2k}$, $z_{l'}^+ = 0$, for all $l' \neq l$, $y_{i_1}^+ = \cdots = y_{i_t}^+ = e_{11} + e_{2k,2k}$, $y_{j_1}^+ = \cdots = y_{j_s}^+ = E$ and $z_m^- = 0$, for $m = 1, \dots, n$, gives $\alpha_{l,I,J} = 0$.

In a similar way it is proved that the coefficients $\beta_{m,R,S} = 0$, for all m, R and S.

Therefore the elements in (6) are linearly independent modulo $P_n^* \cap \mathrm{Id}^*(A_k^{sup})$ and, since $P_n^* \cap \mathrm{Id}^*(A_k^{sup}) \supseteq P_n^* \cap I$, they form a basis of P_n^* modulo $P_n^* \cap \mathrm{Id}^*(A_k^{sup})$ and $\mathrm{Id}^*(A_k^{sup}) = I$. By counting, we obtain

$$c_n^*(A_k^{sup}) = 1 + 2 \sum_{\substack{t<k-1 \\ or \\ n-t<k}} \binom{n}{t}(n-t) \approx qn^{k-1}$$

for some $q > 0$. □

Remark 5.1. *Let $A = F + J \in var^*(M^{sup})$. Then*

$$J_{10}J_{01} = J_{01}J_{10} = (J_{11})_1 J_{10} = J_{01}(J_{11})_1 = 0.$$

In particular, if $A \in var^*(A_k^{sup})$ then $(J_{11})_1 = 0$.

Proof. We start by proving that $J_{10}J_{01} = J_{01}J_{10} = 0$. Let $a = a_0 + a_1 \in J_{10}$, $b = b_0 + b_1 \in J_{01}$. Notice that, since $A_0^- = 0$, $a - a^* = a_1 - a_1^*$ and $b - b^* = b_1 - b_1^*$. Then, because of $z_1 z_2 \equiv 0$, $(a - a^*)(b - b^*) = 0$ and, so, $ab = a^* b^* = 0$.
Now let $a \in (J_{11})_1$, $b = b_0 + b_1 \in J_{10}$. Then $a(b - b^*) = 0$ and, so, $ab = 0$.
Finally, if $A \in \mathrm{var}^*(A_k^{sup})$ then A satisfies the $*$-identity $y_1^+ \cdots y_{k-1}^+ z y_k^+ \cdots y_{2k-2}^+ \equiv 0$. Hence, since $(J_{11})_1 = \underbrace{F \cdots F}_{k-1} (J_{11})_1 \underbrace{F \cdots F}_{k-1}$ we get the desired result. ☐

Lemma 5.1. *Let $A = F + J \in var^*(A_k^{sup})$ with $J_{10} \neq 0$ (hence $J_{01} \neq 0$). If $c_n^*(A) \approx qn^{k-1}$, for some $q > 0$, then $A \sim_{T_2^*} A_k^{sup}$.*

Proof. Since $A \in \mathrm{var}^*(A_k^{sup})$, by the previous remark we must have $(J_{11})_1 = J_{01}J_{10} = J_{10}J_{01} = 0$.

Suppose first that $(J_{10})_1((J_{00})_0^+)^{k-2} = 0$ and, so, $((J_{00})_0^+)^{k-2}(J_{01})_1 = 0$. If $J^m = 0$, it can be proved that, for any $n \geq m$, the multilinear polynomial

$$f = y_{i_1} \cdots y_{i_l} y_1 \cdots y_{k-2} z y_{k-1} \cdots y_{2k-4} y_{j_1} \cdots y_{j_t} \in \mathrm{Id}^*(A),$$

where $l + t + 2k - 3 = n$.

Hence, if $Q \subseteq \mathrm{Id}^*(A)$ is the T_2^*-ideal generated by f plus the generators of the T_2^*-ideal $\mathrm{Id}^*(A_k^{sup})$, it is easy to see that for any $n \geq m$, a set of generators of $P_n^* (\bmod\ P_n^* \cap Q)$ is given by the polynomials

$$y_1^+ \cdots y_n^+,\ y_{i_1}^+ \cdots y_{i_t}^+\ z_l^+\ y_{j_1}^+ \cdots y_{j_s}^+,\ y_{i_1}^+ \cdots y_{i_t}^+\ z_l^-\ y_{j_1}^+ \cdots y_{j_s}^+$$

where $t + s = n - 1, t < k - 2$ or $s < k - 2$, $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$. Hence

$$c_n^*(A) \leq 1 + 2 \sum_{\substack{t<k-2 \\ or \\ n-t<k-1}} \binom{n-1}{t} n \approx qn^{k-2},$$

a contradiction.
Therefore we must have $(J_{10})_1((J_{00})_0^+)^{k-2} \neq 0$. In order to complete the proof it is enough to show that $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(A_k^{sup})$. Let $f \in \mathrm{Id}^*(A)$ be a multilinear polynomial. By Theorem 5.1, we can write f, modulo $\mathrm{Id}^*(A_k^{sup})$ as

$$f = \delta y_1^+ \cdots y_n^+ + \sum_{\substack{t<k-1 \\ or \\ s<k-1}} \sum_{l,I,J} \alpha_{l,I,J}\ y_{i_1}^+ \cdots y_{i_t}^+ z_l^+ y_{j_1}^+ \cdots y_{j_s}^+ + \sum_{\substack{p<k-1 \\ or \\ q<k-1}} \sum_{m,R,S} \beta_{m,R,S}\ y_{r_1}^+ \cdots y_{r_p}^+ z_m^- y_{s_1}^+ \cdots y_{s_q}^+,$$

where $I = \{i_1, \ldots, i_t\}$, $J = \{j_1, \ldots, j_s\}$, $R = \{r_1, \ldots, r_p\}$, $S = \{s_1, \ldots, s_q\}$ are such that $I \cup J \cup \{l\} = R \cup S \cup \{m\} = \{1, \ldots, n\}$ and $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$, $r_1 < \cdots < r_p$ and $s_1 < \cdots < s_q$. It is easy to see that f must be the zero polynomial and so, $f \in \mathrm{Id}^*(A_k^{sup})$. This says that $\mathrm{Id}^*(A) = \mathrm{Id}^*(A_k^{sup})$ and the proof is complete. ☐

Now we are in a position to prove the following theorem.

From now until the end of this section we assume that the field F is algebraically closed.

Theorem 5.2. *For any $k \geq 2$, A_k^{sup} generates a minimal variety of polynomial growth.*

Proof. As in the proof of Theorem 4.7 we may assume that $A = B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are finite dimensional $*$-algebras such that either $B_i \cong F + J(B_i)$ or $B_i = J(B_i)$ is a nilpotent algebra and there exists B_i such that $c_n^*(B_i) \approx bn^{k-1}$, for some $b > 0$. Since $k \geq 2$, we must have that $J_{10}(B_i) \neq 0$ (hence $J_{01}(B_i) \neq 0$). If not $B_i \cong (F + J_{11}) \oplus J_{00}$ and $c_n^*(B_i) = c_n^*(F + J_{11})$, for n large enough. But since $C = F + J_{11} \in \mathrm{var}^*(A_k^{sup})$, we get that C is a commutative algebra with trivial gs-involution and, so, $c_n^*(F + J_{11}) = 1$, a contradiction. Therefore, since B_i satisfies the hypotheses of Lemma 5.1, we get that $B_i \sim_{T_2^*} A_k^{sup}$ and $A \sim_{T_2^*} A_k^{sup}$ follows. ☐

Lemma 5.2. *Let $A = F + J \in var^*(M^{sup})$ be a $*$-algebra. If $J_{10} \neq 0$ (hence $J_{01} \neq 0$) then A is T_2^*-equivalent to one of the following $*$-algebras*

$$A_k^{sup} \oplus N,\ N_u^{sup} \oplus A_k^{sup} \oplus N,\ U_u^{sup} \oplus A_k^{sup} \oplus N,\ N_u^{sup} \oplus U_u^{sup} \oplus A_k^{sup} \oplus N,$$

for some $k, u \geq 2$, where N is a nilpotent $*$-algebra.

Proof. Since the proof is very similar to that given in [22, Lemma 8] we shall just give a sketch of it for convenience of the reader.

Let $j \geq 0$ be the largest integer such that $J_{10}J_{00}^j \neq 0$ (hence $J_{00}^jJ_{01} \neq 0$). We shall see that either $A \sim_{T_2^*} A_{j+2}^{sup} \oplus J_{00}$ or $A \sim_{T_2^*} A_{j+2}^{sup} \oplus N_u^{sup} \oplus J_{00}$ or $A \sim_{T_2^*} A_{j+2}^{sup} \oplus U_u^{sup} \oplus J_{00}$ or $A \sim_{T_2^*} A_{j+2}^{sup} \oplus N_u^{sup} \oplus U_u^{sup} \oplus J_{00}$, for some $u \geq 2$.

Suppose first that $(J_{11})_1 = 0$.

It is checked that $A_{j+2}^{sup} \sim_{T_2^*} A/J_{00}^{j+1}$ and so, $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(A_{j+2}^{sup} \oplus J_{00})$. In order to prove the opposite inclusion, it is taken $f \in \mathrm{Id}^*(A_{j+2}^{sup} \oplus J_{00})$ a multilinear polynomial of degree n. If $n \leq 2j+2$, since $f \in \mathrm{Id}^*(A_{j+2}^{sup})$, then f must be a consequence of $\langle y^-, z_1z_2\rangle_{T_2^*} \subseteq \mathrm{Id}^*(A)$. Hence $f \in \mathrm{Id}^*(A)$ and we are done in this case. Now let $n > 2j+2$. It is checked that f can be written modulo $\mathrm{Id}^*(A_{j+2}^{sup})$ as

$$f = \sum_{\substack{t\geq j+1\\ and\\ s\geq j+1}} \sum_{l,I,J} \alpha_{l,I,J}\; y_{i_1}^+ \cdots y_{i_t}^+ z_l^+ y_{j_1}^+ \cdots y_{j_s}^+ + \sum_{\substack{p\geq j+1\\ and\\ q\geq j+1}} \sum_{m,R,S} \beta_{m,R,S}\; y_{r_1}^+ \cdots y_{r_p}^+ z_m^- y_{s_1}^+ \cdots y_{s_q}^+ + g,$$

where $g \in \langle y^-, z_1z_2\rangle_{T_2^*}$ and $I = \{i_1, \dots, i_t\}$, $J = \{j_1, \dots, j_s\}$, $R = \{r_1, \dots, r_p\}$, $S = \{s_1, \dots, s_q\}$ with $i_1 < \cdots < i_t$, $j_1 < \cdots < j_s$, $r_1 < \cdots < r_p$ and $s_1 < \cdots < s_q$. It is easily seen that f is a $*$-identity of A and $\mathrm{Id}^*(A_{j+2}^{sup} \oplus J_{00}) \subseteq \mathrm{Id}^*(A)$. So $A \sim_{T_2^*} A_{j+2}^{sup} \oplus J_{00}$ follows.

Suppose now that $(J_{11})_1 \neq 0$.

Let $B = F + J_{10} + J_{01} + J_{00}$. From Remark 5.1 it follows that B is a subalgebra of A and, since $(J_{11}(B))_1 = 0$ by applying the first part of the lemma to B we conclude that

$$B \sim_{T_2^*} A_{j+2}^{sup} \oplus J_{00}.$$

Now let $L = F + J_{11}$. By Theorem 4.9, either $L \sim_{T_2^*} F$ or $L \sim_{T_2^*} N_r^{sup}$ or $L \sim_{T_2^*} U_r^{sup}$ or $L \sim_{T_2^*} N_r^{sup} \oplus U_r^{sup}$, for some $r \geq 2$.

It is proved that $A \sim_{T_2^*} L \oplus B$ and this complete the proof. □

Now we are in a position to classify all the subvarieties of $\mathrm{var}^*(M^{sup})$.

Theorem 5.3. *If $A \in var^*(M^{sup})$ then A is T_2^*-equivalent to one of the following $*$-algebras: M^{sup}, N, C, $N_k^{sup} \oplus N$, $U_k^{sup} \oplus N$, $N_k^{sup} \oplus U_k^{sup} \oplus N$, $A_t^{sup} \oplus N$, $N_k^{sup} \oplus A_t^{sup} \oplus N$, $U_k^{sup} \oplus A_t^{sup} \oplus N$, $N_k^{sup} \oplus U_k^{sup} \oplus A_t^{sup} \oplus N$ for some $k, t \geq 2$, where N is a nilpotent $*$-algebra and C is a commutative algebra with trivial gs-involution.*

Proof. If $A \sim_{T_2^*} M^{sup}$ there is nothing to prove. Now let A generates a proper subvariety of M^{sup}. Since M^{sup} generates a variety of almost polynomial growth, $\mathrm{var}^*(A)$, has polynomial growth. Hence by Corollary 4.1 and Theorem 2.4 we may assume that $A = B_1 \oplus \cdots \oplus B_m$, where $B_1, \dots, B_m$ are finite dimensional $*$-algebras such that $\dim B_i/J(B_i) \leq 1$. This means that for every i, either B_i is a nilpotent $*$-algebra or B_i has a decomposition of the type $B_i = F + J = F + J_{11} + J_{10} + J_{01} + J_{00}$. Now, by applying Theorem 4.9 and Lemma 5.2 we get the desired conclusion. □

As a consequence of the previous theorem and of Theorems 4.7, 4.8, 5.2 we can also get the classification of all $*$-algebras generating minimal varieties.

Corollary 5.1. *A $*$-algebra $A \in var^*(M^{sup})$ generates a minimal variety of polynomial growth if and only if either $A \sim_{T_2^*} U_k^{sup}$ or $A \sim_{T_2^*} N_k^{sup}$ or $A \sim_{T_2^*} A_k^{sup}$, for some $k \geq 2$.*

6. Classifying the subvarieties of var$^*(D^{sup})$

In this section we classify, up to T_2^*-equivalence, all the algebras with graded involution contained in the variety generated by the algebra D^{sup}, i.e., the algebra $F \oplus F$ with grading $D_0 = F(1,1)$ and $D_1 = F(1,-1)$ and with trivial graded involution.

Since the graded involution is trivial, this is equivalent to the classification of the subvarieties of the supervariety generated by D^{sup}. Such a classification was given in [20, 21]. Also, by Remark 2.1, the classification of the subvarieties of $\mathrm{var}^*(D^{sup})$ can be obtained from the classification of the subvarieties of $\mathrm{var}^*(D)$. In what follows we present these results in the language of algebras with graded involution for convenience of the reader.

According to the notation introduced before, C_k^{sup} is the algebra

$$C_k = \{\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F\}$$

with elementary grading induced by $g = (0, 1, 0, 1, \ldots) \in \mathbb{Z}_2^k$ and trivial involution.

The following result characterize the $*$-identities and the $*$-codimensions of C_k^{sup}.

Theorem 6.1. *Let $k \geq 2$. Then*

1) $Id^*(C_k^{sup}) = \langle [x_1, x_2],\ z_1^+ \cdots z_k^+,\ y^-,\ z^- \rangle_{T_2^*}$;

2) $c_n^*(C_k^{sup}) = \displaystyle\sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!} n^{k-1}$.

The following result classifies all the subvarieties of the variety generated by D^{sup}.

Theorem 6.2. *Let $A \in var^*(D^{sup})$ be an algebra with graded involution. Then either $A \sim_{T_2^*} D^{sup}$ or $A \sim_{T_2^*} N$ or $A \sim_{T_2^*} C \oplus N$ or $A \sim_{T_2^*} C_k^{sup} \oplus N$, for some $k \geq 2$, where N is a nilpotent algebra with graded involution and C is a commutative algebra with trivial gs-involution.*

As a consequence we get the classification of all algebras generating minimal varieties.

Corollary 6.1. *Let $A \in var^*(D^{sup})$ be an algebra with graded involution. Then A generates a minimal variety of polynomial growth if and only if $A \sim_{T_2^*} C_k^{sup}$, for some $k \geq 2$.*

7. Classifying varieties of at most linear growth

In this section we present a classification, up to T_2^*-equivalence, of the finite dimensional $*$-algebras generating varieties of at most linear growth, where $*$ is a graded involution or a superinvolution. Such classification for the $*$-algebras with trivial grading was given in [24].

The following lemma follows from [15, Theorem 5.1].

Lemma 7.1. *Let A be a finite dimensional $*$-algebra such that $c_n^*(A) \leq an$, for some constant a. Then A satisfies the polynomial identities $x_1 x_2 \equiv 0$, with $x_1, x_2 \in X \setminus Y^+$, where $Y^+ = \{y_1^+, y_2^+, \ldots\}$.*

Lemma 7.2. *Let $A = F + J$ be a finite dimensional $*$-algebra such that $c_n^*(A) \leq an$, for some constant a. Then*

$$A \sim_{T_2^*} (F + J_0) \oplus \left(F + J_1^+\right) \oplus \left(F + J_1^-\right).$$

Proof. Since $c_n^*(A) \leq an$, by Lemma 7.1, A satisfies the polynomial identities $x_1 x_2 \equiv 0$. Hence $F + J_0$, $F + J_1^+$ and $F + J_1^-$ are $*$-subalgebras of A and obviously

$$\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*\left((F + J_0) \oplus (F + J_1^+) \oplus (F + J_1^-)\right).$$

Conversely, let $f \in \mathrm{Id}^*\left((F + J_0) \oplus (F + J_1^+) \oplus (F + J_1^-)\right)$ be a multilinear polynomial of degree n. By multihomogeneity of T_2^*-ideals we may assume, modulo $\mathrm{Id}^*(A)$, that either

$$f = \sum_{\sigma \in S_n} \alpha_\sigma y_{\sigma(1)}^+ \cdots y_{\sigma(n)}^+ \quad \text{or} \quad f = \sum_{\substack{i=1,\ldots,n \\ \sigma \in S_n}} \beta_\sigma y_{\sigma(1)}^+ \cdots y_{\sigma(i-1)}^+ x_{\sigma(i)} y_{\sigma(i+1)}^+ \cdots y_{\sigma(n)}^+$$

where $x_i \in X \setminus Y^+$, $i = 1, \ldots, n$.

If f is of the first type, in order to get a non-zero value, we should evaluate f on $F + J_0$. But $f \in \mathrm{Id}^*(F + J_0)$ by the hypothesis, and so we get that $f \equiv 0$ on A. Similarly, if f is of the second type we get that $f \equiv 0$ on A. Hence $\mathrm{Id}^*\left((F + J_0) \oplus (F + J_1^+) \oplus (F + J_1^-)\right) \subseteq \mathrm{Id}^*(A)$ and we are done. □

Since $F + J_0 \in \mathrm{var}^*(M)$, $F + J_1^+$, $F + J_1^- \in \mathrm{var}^*(M^{sup})$ we get the following.

Corollary 7.1. *Let $A = F + J$ be a finite dimensional $*$-algebra such that $c_n^*(A) \leq an$, for some constant a. Then $A \sim_{T_2^*} B_1$ or $A \sim_{T_2^*} B_2$ or $A \sim_{T_2^*} B_1 \oplus B_2$, where $B_1 \in var^*(M)$ and $B_2 \in var^*(M^{sup})$.*

Now we are ready to present the main result of this section.

Theorem 7.1. *Let A be a finite dimensional $*$-algebra such that $c_n^*(A) \leq an$, for some constant a. Then*

$$A \sim_{T_2^*} B_1 \oplus \cdots \oplus B_m \oplus N,$$

where $B_i \in var^*(M)$ or $B_i \in var^*(M^{sup})$, for all $i = 1, \dots, m$ and N is a nilpotent $*$-algebra.

Proof. Since $c_n^*(A) \leq an$, for some constant a, by Theorem 2.4, we may assume that

$$A = A_1 \oplus \cdots \oplus A_m$$

where $A_1, \dots, A_m$ are finite dimensional $*$-algebras with $\dim A_i/J(A_i) \leq 1$, $1 \leq i \leq m$. Notice that this says that either $A_i \cong F + J(A_i)$ or $A_i = J(A_i)$ is a nilpotent $*$-algebra. Since $c_n^*(A_i) \leq c_n^*(A)$ then $c_n^*(A_i) \leq an$, for all $i = 1, \dots, m$. Now the result follows by applying Corollary 7.1 to each non-nilpotent A_i. □

Finally, by putting together Theorem 7.1 and Theorems 3.6, 5.3, we get a finer classification of the $*$-algebras of at most linear codimension growth.

Theorem 7.2. *Let A be a finite dimensional $*$-algebra such that $c_n^*(A) \leq an$, for some constant a. Then*

$$A \sim_{T_2^*} B_1 \oplus \cdots \oplus B_m \oplus N,$$

where N is a nilpotent $*$-algebra and for all $i = 1, \dots, m$, B_i is T_2^*-equivalent to one of the following algebras:

$$N_i,\ C \oplus N_i,\ N_2 \oplus N_i,\ A_2 \oplus N_i,\ N_2 \oplus A_2 \oplus N_i,$$

$$N_2^{sup} \oplus N_i,\ U_2^{sup} \oplus N_i,\ A_2^{sup} \oplus N_i,\ N_2^{sup} \oplus U_2^{sup} \oplus N_i,\ N_2^{sup} \oplus A_2^{sup} \oplus N_i,\ U_2^{sup} \oplus A_2^{sup} \oplus N_i,\ N_2^{sup} \oplus U_2^{sup} \oplus A_2^{sup} \oplus N_i,$$

where C is a commutative $*$-algebra with trivial gs-involution and N_i is a nilpotent $*$-algebra.

References

[1] E. Aljadeff, A. Giambruno and Y. Karasik, *Polynomial identities with involution, superinvolutions and the Grassmann envelope*, preprint.

[2] V. Drensky, *Free algebras and PI-algebras. Graduate course in algebra*, Springer-Verlag Singapore, Singapore, 2000.

[3] V. Drensky and A. Giambruno, *Cocharacters, codimensions and Hilbert series of the polynomial identities for 2×2 matrices with involution*, Canad. J. Math. **46** (1994), no. 4, 718–733.

[4] A. Giambruno, A. Ioppolo and D. La Mattina, *Varieties of algebras with superinvolution of almost polynomial growth*, Algebr. Represent. Theory **19** (2016), no. 3, 599–611.

[5] A. Giambruno and D. La Mattina, *PI-algebras with slow codimension growth*, J. Algebra **284** (2005), no. 1, 371–391.

[6] A. Giambruno, D. La Mattina and P. Misso, *Polynomial identities on superalgebras: classifying linear growth*, J. Pure Appl. Algebra **207** (2006), no. 1, 215–240.

[7] A. Giambruno, D. La Mattina and V. M. Petrogradsky, *Matrix algebras of polynomial codimension growth,* Israel J. Math.**158** (2007), 367–378.

[8] A. Giambruno, D. La Mattina and M. Zaicev, *Classifying the minimal varieties of polynomial growth*, Canad. J. Math. **66** (2014), no. 3, 625–640.

[9] A. Giambruno and S. Mishchenko, *Polynomial growth of the $*$-codimensions and Young diagrams*, Comm. Algebra **29** (2001), no. 1, 277–284.

[10] A. Giambruno and S. Mishchenko, *On star-varieties with almost polynomial growth*, Algebra Colloq. **8** (2001), no. 1, 33–42.

[11] A. Giambruno, S. Mishchenko and M. Zaicev, *Polynomial identities on superalgebras and almost polynomial growth*, Comm. Algebra **29** (2001), no. 9, 3787–3800.

[12] A. Giambruno, R.B. dos Santos and A.C. Vieira, *Identities of $*$-superalgebras and almost polynomial growth*, Linear Multilinear Algebra **64** (2016), no. 3, 484-501.

[13] A. Giambruno and M. Zaicev, *Polynomial identities and asymptotic methods*, AMS, Math. Surv. Monogr. **122** (2005).

[14] A. R. Kemer, *Varieties of finite rank*, Proc. 15-th All the Union Algebraic Conf., Krasnoyarsk, Vol. 2, p. 73, (1979), (in Russian).

[15] P. Koshlukov and D. La Mattina, *Graded algebras with polynomial growth of their codimensions*, J. Algebra **434** (2015), 115–137.

[16] D. La Mattina, *Varieties of almost polynomial growth: classifying their subvarieties,* Manuscripta Math. **123** (2007), no. 2, 185–203.

[17] D. La Mattina, *Varieties of algebras of polynomial growth,* Boll. Unione Mat. Ital. (9) **1** (2008), no. 3, 525–538.

[18] D. La Mattina, *Characterizing varieties of colength ≤ 4*, Comm. Algebra **37** (2009), no. 5, 1793–1807.

[19] D. La Mattina, *Polynomial codimension growth of graded algebras. Groups, rings and group rings*, 189–197, Contemp. Math., **499**, Amer. Math. Soc., Providence, RI, 2009.

[20] D. La Mattina, *Varieties of superalgebras of almost polynomial growth*, J. Algebra **336** (2011), 209–226.

[21] D. La Mattina, *Almost polynomial growth: classifying varieties of graded algebras*, Israel J. Math **207** (2015), no. 1, 53–75.

[22] D. La Mattina and F. Martino, *Polynomial growth and star-varieties*, J. Pure Appl. Algebra **220** (2016), no. 1, 246–262.
[23] D. La Mattina, S. Mauceri and P. Misso, *Polynomial growth and identities of superalgebras and star-algebras*, J. Pure Appl. Algebra **213** (2009), no. 11, 2087–2094.
[24] D. La Mattina and P. Misso, *Algebras with involution with linear codimension growth*, J. Algebra **305** (2006), 270–291.
[25] S. Mishchenko and A. Valenti, *A star-variety with almost polynomial growth*, J. Algebra **223** (2000), no. 1, 66–84.
[26] A. Regev, *Existence of identities in $A \otimes B$*, Israel J. Math. **11** (1972), 131–152.
[27] A. Valenti, *Group graded algebras and almost polynomial growth*, J. Algebra. **334** (2011), 247–254.

Dipartimento di Matematica e Informatica, Università di Palermo, Via Archirafi 34, 90123, Palermo, Italy
E-mail address: `antonio.ioppolo@unipa.it`

Dipartimento di Matematica e Informatica, Università di Palermo, Via Archirafi 34, 90123, Palermo, Italy
E-mail address: `daniela.lamattina@unipa.it`