Least energy solutions to the Dirichlet problem for the equation $-\Delta u = f(x,u)$

Valeria Iiritano[a] and Francesco Tulone[b]

[a]Department of Mathematics and Computer Science, Physical Science and Earth Science, University of Messina, Messina, Italy;

[b]Department of Mathematics and Computer Science, University of Palermo, Palermo, Italy

ABSTRACT

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. We prove a general existence result of least energy solutions and least energy nodal ones for the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P)$$

where f is a Carathéodory function. Our result includes some previous results related to special cases of f. Finally, we propose some open questions concerning the global minima of the restriction on the Nehari manifold of the energy functional associated with (P) when the nonlinearity is of the type $f(x,u) = \lambda|u|^{s-2}u - \mu|u|^{r-2}u$, with $s, r \in (1,2)$ and $\lambda, \mu > 0$.

KEYWORDS

Elliptic problems; weak solution; nodal solution; least energy; variational methods; sublinear nonlinearity; Nehari manifold

AMS SUBJECT CLASSIFICATIONS 35J20; 35J25

1. Introduction

Let Ω be a nonempty bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In this paper, we establish the existence of least energy solutions and least energy nodal ones for the following elliptic problem

$$\begin{cases} -\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \qquad (P)$$

where f is a Carathéodory function with subcritical growth conditions explained below. Our solutions will be always understood in weak sense: by definition, a weak solution of problem (P) is a function $u \in {W_0}^{1,2}(\Omega)$ such that

$$I^{'}(u)(v) = \int_\Omega (\nabla u(x)\nabla v(x) - f(x,u(x))v(x))\mathrm{d}x = 0 \qquad (1)$$

This article is post-print version of https://doi.org/10.1080/17476933.2017.1307346

for all $v \in {W_0}^{1,2}(\Omega)$, where $I^{'}(u) \in ({W_0}^{1,2}(\Omega))'$ is the Gateaux-derivative at u of the energy functional I associated to (P), which is defined as follows

$$I(u) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\Omega} |\nabla u|^2 \mathrm{d}x - \int_{\Omega}\left(\int_0^{u(x)} f(x,t)\mathrm{d}t\right)\mathrm{d}x, \ \ u \in W_0^{1,2}(\Omega). \tag{2}$$

A weak solution which changes sing in Ω is called *nodal* solution. Moreover, a weak solution is said *least energy solution* (resp. *least energy nodal solution*) if it minimizes I on the set S of all weak nonzero solutions (resp. on the set $S_{\pm}$ of all nodal solutions).

When I is of class C^1, the set

$$\mathcal{N} = \left\{u \in W_0^{1,2}(\Omega) \setminus \{0\} : I'(u)(u) = \int_{\Omega} |\nabla u|^2 \mathrm{d}x - \int_{\Omega} f(x,u)u\mathrm{d}x = 0\right\},$$

is the so-called Nehari manifold associated to I. We also introduce the *nodal* Nehari manifold $\mathcal{N}_{\pm}$ defined as

$$\mathcal{N}_{\pm} = \{u \in W_0^{1,2}(\Omega) : \ u_+, u_- \in \mathcal{N}\}, \tag{3}$$

where $u_+(x) = \max\{u(x), 0\}$, $u_-(x) = \max\{-u(x), 0\}$.

In what follows, we denote by $2^* := \frac{2N}{N-2}$, if $N \geq 3$, $2^* = \infty$ if $N = 1,2$, the critical exponent for the Sobolev embedding.

When $f(x,t) = \lambda|t|^{s-2}t$, with $\lambda > 0$ and $s \in (1,2)$, it is well known that the energy functional I admits exactly two global minima u_1, u_2, with u_1 positive in Ω and $u_2 = -u_1$. For $s \in (2, 2^*)$, the functional I restricted to the Nehari manifold is coercive and, being $\mathcal{N}$, in this case, a weakly closed set, it admits a global minimum $u \in \mathcal{N}$. Then, there exists a Lagrange multiplier $\rho \in \mathbb{R}$ such that $I'(u)(v) + \rho J'(u)(v) = 0$, for all $v \in W_0^{1,2}(\Omega)$, where $J(u) = I'(u)(u)$. Testing this equation with $v = u$, it is easy to infer $\rho = 0$. Therefore, since $\mathcal{N} \supset S$, u turns out to be a least energy solution to problem (P).

In [1] (see also [2]) , Grumiau and Parini investigated the existence of least energy nodal solutions for problem (P) in the more general quasilinear case involving the p-Laplacian operator (with $p > 1$) and again for nonlinearities of the type $f(x,t) = \lambda|t|^{s-2}t$, with $s \in (p, p^*)$, where p^* is the critical exponent for the embedding $W_0^{1,2}(\Omega) \subset L^p(\Omega)$. In particular, they proved the existence of a least energy nodal solution by minimizing the restriction of the energy functional I on the *nodal* Nehari manifold. With respect to the least energy solutions case, it is a bit more delicate proving that the global minimum of $I_{|\mathcal{N}_{\pm}}$ is a critical point of I (and so a weak solution) and the proof is based, in this case, on a deformation lemma and on the classical Miranda Fixed Point Theorem.

We want to point out that in the above arguments the Nehari manifold and the nodal Nehari manifold could be replaced by the sets

$$\mathcal{A} = \{u \in W_0^{1,2}(\Omega)\backslash\{0\} : I'(u)(u) \leq 0\}, \quad \mathcal{A}_{\pm} = \{u \in W_0^{1,2}(\Omega) : \ u_+, u_- \in \mathcal{A}\}. \tag{4}$$

Indeed, a global minimum point $u^* \in \mathcal{A}$ of $I_{|\mathcal{A}}$ must belong to $\mathcal{N}$, for otherwise u^* should belong to the open set $\{u \in W_0^{1,2}(\Omega)\backslash\{0\} : I'(u)(u) < 0\} \subset \mathcal{A}$: this means that u^* should be a (nonzero) local minimum of I, and so $u^* \in S \subset N$. In the same way, we can see that a global minimum point $u^* \in \mathcal{A}_{\pm}$ of $I_{|\mathcal{A}_{\pm}}$ must belongs to $\mathcal{N}_{\pm}$.

To the best of our knowledge, for sublinear functions near 0 of the type $\lambda|u|^{s-2}u$, with $1 < s < 2$, the existence of least energy nodal solutions is not yet known. As a corollary of our main results, we will give a positive answer to this question.

Indeed, our main results apply to even more general nonlinearities of the type $f(x,u) = \lambda|u|^{s-2}u - \mu|u|^{r-2}u$ with $\lambda > 0$, $\mu \in \mathbb{R}$, and $r, s \in (1,2)$ with $r < s$. In particular, using some results from [3,4], we will show that there exists $\mu_0 > 0$ such that the problem

$$\begin{cases} -\Delta u = \lambda|u|^{s-2}u - \mu|u|^{r-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \qquad (P_{\lambda,\mu})$$

admits a least energy solution and a least energy nodal solution for all $\mu \in (-\infty, \mu_0]$. An interesting feature of this problem is that the sets $\mathcal{A}$ and $\mathcal{A}_\pm$ introduced in (4) are not weakly closed for $\mu \leq 0$ (in fact we know, by standard results, that for $\mu \leq 0$ problem $(P_{\lambda,\mu})$ admits a sequence $\{u_n\}$ of (sign-changing) weak solutions strongly converging to 0 in $W_0^{1,2}(\Omega)$) but, instead, the sets $\mathcal{A}$ and $\mathcal{A}_\pm$ are even weakly compact for $\mu > 0$. Nevertheless, we do not know if the global minima of $I_{|\mathcal{A}}$ and $I_{|\mathcal{A}_\pm}$ are weak solutions or not. A positive answer to this question would allow to get a variational characterization for both of the least energy solutions and the least energy nodal ones.

2. Basic definitions and notations

Throughout this paper, we consider a Carathéodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ such that, for some constant $C > 0$ and some $q \in (1, 2^*)$, it holds

$$|f(x,t)| \leq C(1 + |t|^{q-1}), \quad \text{for almost all } x \in \Omega \text{ and for all } t \in \mathbb{R}. \tag{5}$$

It should be noted that, without loss of generality, we can always assume $q > 2$.

Moreover, for any $m \geq 1$, with $m \leq 2^*$ if $N \geq 3$, we denote by c_m the best constant for the Sobolev embedding $W_0^{1,2}(\Omega) \hookrightarrow L^m(\Omega)$, i.e.

$$c_m := \sup_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\|u\|_m}{\|u\|}.$$

where $\|u\| := \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right)^{\frac{1}{2}}$ is the standard norm in $W_0^{1,2}(\Omega)$, and $\|u\|_m := \left(\int_\Omega |u|^m \mathrm{d}x \right)^{\frac{1}{m}}$ is the standard norm in $L^m(\Omega)$. Also, we denote by λ_1 the first eigenvalue of the Laplacian in the domain Ω, i.e. $\lambda_1 = c_2^2$. Finally, we will consider the energy functional I defined in (2). By standard results, it is well known that, under condition (5), the functional I is sequentially weakly lower semicontinuous and of class C^1 in $W_0^{1,2}(\Omega)$.

3. Main results

Let $K \subseteq \Omega$ be a set of positive measure. The following conditions will be considered on the nonlinearity f:

(i) $\liminf_{\xi\to 0^+} \frac{\int_0^\xi f(x,t)\mathrm{d}t}{\xi^2} \geq \frac{\lambda_1}{2}$, uniformly in $\Omega\backslash K$, and $\liminf_{\xi\to 0^+} \frac{\int_0^\xi f(x,t)\mathrm{d}t}{\xi^2} > \frac{\lambda_1}{2}$, uniformly in K.

(ii) $\liminf_{\xi\to 0^+} \frac{\int_0^\xi f(x,t)\mathrm{d}t}{\xi^2} > -\infty$, uniformly in $\Omega\backslash K$, and $\limsup_{\xi\to 0^+} \frac{\int_0^\xi f(x,t)\mathrm{d}t}{\xi^2} = +\infty$, uniformly in K.

(iii) there exists $\delta > 0$ such that $f(x,\xi)\xi - 2\int_0^\xi f(x,t)\mathrm{d}t < 0$, for all $\xi \in [-\delta,\delta]\backslash\{0\}$, and for a.e. $x \in \Omega$.

(iv) $\limsup_{t\to 0} \frac{f(x,t)}{t} < \lambda_1$, uniformly in Ω.

(v) $\limsup_{|\xi|\to+\infty} \frac{\int_0^\xi f(x,t)\mathrm{d}t}{\xi^2} < \frac{\lambda_1}{2}$, uniformly in Ω;

(vi) there exist $c > 2$ and $M > 0$ such that $0 \leq c\displaystyle\int_0^\xi f(x,t)\mathrm{d}t \leq \xi f(x,\xi)$, for a.a. $x \in \Omega$ and for all $\xi \in \mathbb{R}\backslash(-M,M)$.

Our main result concerning the existence of the least energy solutions for problem (P) is as follows.

Theorem 1: *Assume that there hold*

(a) *condition* (i) *or* (ii) *and condition* (v)*;*

(b) *condition* (iii) *or* (iv) *and condition* (v) *or* (vi).

Then, if the set S of all nonzero weak solutions is nonempty, there exists $u_0 \in S$ such that $I(u_0) = \inf_{u\in S} I(u)$.

***Proof*:** Assume $S \neq \emptyset$ and put $F(x,\xi) = \int_0^\xi f(x,t)\mathrm{d}t$, for all $(x,t) \in \Omega\times\mathbb{R}$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in S such that $\lim_{n\to+\infty} I(u_n) = \inf_S I$. We first prove that under condition (v) or under condition (vi), the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Indeed, assume that condition (v) holds. Then, we can fix $\rho \in \mathbb{R}$ and $\delta > 0$ such that

$$\xi^{-2}\int_0^\xi f(x,t)\mathrm{d}t \leq \rho < \frac{\lambda_1}{2}, \quad \text{for almost every } x \in \Omega \text{ and for all } \xi \in \mathbb{R}\backslash]-\delta,\delta[.$$

Consequently, taking (5) into account, for some positive constant C_1 and for every $u \in W_0^{1,2}(\Omega)$, one has

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u(x))\mathrm{d}x = \frac{1}{2}\|u\|^2 - \int_{|u(x)|\leq\delta} F(x,u(x))\mathrm{d}x - \int_{|u(x)|>\delta} F(x,u(x))\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \int_{|u(x)|\leq\delta}\left|\int_0^{u(x)} \sup_{|s|\leq\delta}|f(x,s)|\mathrm{d}t\right|\mathrm{d}x - \int_{|u(x)|>\delta}\rho u(x)^2\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - C_1 - \rho\int_\Omega (u(x))^2\mathrm{d}x \geq \frac{1}{2}\|u\|^2 \\ &\quad - C_1 - \rho\frac{1}{\lambda_1}\|u\|^2 = \frac{1}{\lambda_1}\left(\frac{\lambda_1}{2} - \rho\right)\|u\|^2 - C_1. \end{aligned}$$

Therefore,

$$I(u) \to +\infty, \quad \text{as } \|u\| \to +\infty. \tag{6}$$

This clearly implies the boundedness of $\{u_n\}_{n\in\mathbb{N}}$. The same conclusion is achieved if condition (vi) holds. Indeed, if we assume (vi), taking again (5) into account, for some positive constant C_2 and for every $u \in S$, one has

$$\begin{aligned}
0 &= I'(u)(u) \\
&= \|u\|^2 - \int_\Omega f(x,u(x))u(x)\mathrm{d}x = \|u\|^2 - \int_{|u(x)|\le M} f(x,u(x))u(x)\mathrm{d}x \\
&\quad - \int_{|u(x)|>M} f(x,u(x))u(x)\mathrm{d}x \\
&\le \|u\|^2 - C_2 - c\int_{|u(x)|>M} F(x,u(x))\mathrm{d}x.
\end{aligned}$$

Consequently, for some positive constant C_3, one has

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u(x))\mathrm{d}x = \frac{1}{2}\|u\|^2 - \int_{|u(x)|\le M} F(x,u(x))\mathrm{d}x \\
&\quad - \int_{|u(x)|>M} F(x,u(x))\mathrm{d}x \\
&\ge \left(\frac{1}{2} - \frac{1}{c}\right)\|u\|^2 - C_3.
\end{aligned}$$

In particular, $I(u) \to +\infty$ as $\|u\| \to +\infty$, with u laying in S. Therefore, also under condition (vi), the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded.

Thus, if (v) or (vi) holds, by the reflexivity of $W_0^{1,2}(\Omega)$, there exists $u^* \in W_0^{1,2}(\Omega)$ such that, up to a subsequence, $u_n \to u^*$ weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^m(\Omega)$, for each $m \in (1,p^*)$. To finish, we only have to show that $u^* \in S$. To this end, we first note that, being every u_n a weak solution of problem (P), then, for each fixed $v \in W_0^{1,2}(\Omega)$, one has

$$\int_\Omega \nabla u_n(x)\nabla v(x)\mathrm{d}x = \int_\Omega f(x,u_n(x))v(x)\mathrm{d}x, \quad \text{for each } n \in \mathbb{N}. \tag{7}$$

Passing to the limit as $n \to +\infty$, one has

$$\int_\Omega \nabla u^*(x)\nabla v(x)\mathrm{d}x = \int_\Omega f(x,u^*(x))v(x)\mathrm{d}x.$$

Therefore, u^* is a weak solution of problem (P). Let us to show that u^* is nonzero. We have already observed that when condition (v) holds, then the energy functional I is coercive, i.e. the limit (6) holds. By a routine argument, we infer that I has a global minimizer in $W_0^{1,2}(\Omega)$, which is weak solution to problem (P) as well. We now show that if condition (i) or condition (ii) holds, then $\inf_{W_0^{1,2}(\Omega)} I < 0$ which implies $u^* \neq 0$. Suppose that condition (i) holds. Let $\varphi_1 \in C^1(\overline{\Omega})$ be the positive eigenfunction associated to λ_1, normalized with respect to the sup-norm. From (i), there exist $\delta, \rho > 0$ such that

$$F(x,\xi) \ge \left(\frac{\lambda_1}{2} + \rho\right)\xi^2, \quad \text{for all } \xi \in [0,\delta] \text{ and for almost all } x \in K, \tag{8}$$

$$F(x,\xi) \geq \left(\frac{\lambda_1}{2} - \sigma\rho\right)\xi^2, \text{ for all } \xi \in [0,\delta] \text{ and for almost all } x \in \Omega\backslash K, \tag{9}$$

where $\sigma = \frac{1}{2}\left(\int_K \varphi_1(x)^2 \mathrm{d}x\right)\left(1 + \int_{\Omega\backslash K} \varphi_1(x)^2 \mathrm{d}x\right)^{-1}$.

Now, put $\varphi_\xi = \xi \cdot \varphi_1$, for all $\xi > 0$. Then, using the above inequalities, for $\xi \in [0,\delta]$, we obtain

$$\begin{aligned}
I(\varphi_\xi) &= \frac{\xi^2}{2}\|\varphi_1\|^2 - \int_K F(x,\xi\varphi_1(x))\mathrm{d}x - \int_{\Omega\backslash K} F(x,\xi\varphi_1(x))\mathrm{d}x \\
&\leq \frac{\xi^2}{2}\|\varphi_1\|^2 - \xi^2\left(\frac{\lambda_1}{2} + \rho\right)\int_K \varphi_1(x)^2\mathrm{d}x - \xi^2\left(\frac{\lambda_1}{2} - \sigma\rho\right)\int_{\Omega\backslash K} \varphi_1(x)^2\mathrm{d}x \\
&= \frac{\xi^2}{2}\left(\|\varphi_1\|^2 - \lambda_1\int_\Omega \varphi_1(x)^2\mathrm{d}x\right) - \xi^2\rho\left(\int_K \varphi_1(x)^2\mathrm{d}x - \sigma\int_{\Omega\backslash K}\varphi_1(x)^2\mathrm{d}x\right) \\
&\leq -\frac{\xi^2\rho}{2}\int_K \varphi_1(x)^2\mathrm{d}x < 0.
\end{aligned}$$

Therefore, $\inf_{W_0^{1,2}(\Omega)} I < 0$. We obtain the same conclusion when condition (ii) holds. Indeed, from (ii), we can find $\delta, T > 0$ such that

$$F(x,\xi) \geq -T\xi^2, \text{ for all } \xi \in [0,\delta] \text{ and for almost all } x \in \Omega\backslash K.$$

Now, fix compact set $K_0 \subseteq K$ with positive measure and an open set Ω_0 such that $K_0 \subset \Omega_0 \subset \overline{\Omega_0} \subset \Omega$ and $meas(\Omega_0\backslash K_0) < \frac{1}{2T} meas(K_0)$, and let $\varphi \in C^1(\overline{\Omega})$ be a nonnegative function such that $\sup_{x\in\Omega}\varphi(x) = 1$, $\varphi \equiv 1$ in K_0, $\varphi \equiv 0$ in $\Omega\backslash\overline{\Omega_0}$. Again from (ii), choosing δ smaller if necessary, we get

$$F(x,\xi) \geq R\xi^2, \text{ for all } \xi \in [0,\delta] \text{ and for almost all } x \in K,$$

where $R = meas(K_0)^{-1}\|\varphi\|^2 + 1$.

At this point, putting $\varphi_\xi = \xi\varphi$ for all $\xi > 0$, and taking into account the above inequality, we obtain, for $\xi \in [0,\delta]$,

$$\begin{aligned}
I(\xi\varphi) &= \frac{\xi^2}{2}\|\varphi\|^2 - \int_{K_0} F(x,\xi\varphi(x))\mathrm{d}x - \int_{\Omega\backslash K_0} F(x,\xi\varphi(x))\mathrm{d}x \\
&\leq \frac{\xi^2}{2}\|\varphi\|^2 - R\xi^2 meas(K_0) + T\xi^2\int_{\Omega_0\backslash K_0}\varphi(x)^2\mathrm{d}x \\
&\leq -\frac{\xi^2}{2}\|\varphi\|^2 - \xi^2 meas(K_0) + T\xi^2 meas(\Omega_0\backslash K_0) \\
&< -\frac{\xi^2}{2}\|\varphi\|^2 - \frac{\xi^2}{2} meas(K_0) < 0.
\end{aligned}$$

The proof in the case of assumption (a) is complete.

Now, suppose assumption (b) holds. To conclude we again have to show that u^* is nonzero under condition (iii) or (iv). Assume, on the contrary, that $u^* = 0$. One has, for all $n \in \mathbb{N}$,

$$\begin{cases} -\Delta u_n = f(x, u_n), & \text{in } \Omega \\ u_n = 0 & \text{on } \Omega \end{cases}$$

which means that u_n satisfies equation (7). Testing this equation with $v = u_n$, we get

$$\|u_n\|^2 = \int_\Omega f(x, u_n(x))u_n(x)\mathrm{d}x. \tag{10}$$

Now, let $q \in (2, 2^*)$ be as in (5). Since $u_n \to 0$ strongly in $L^q(\Omega)$, then there exists $g \in L^q(\Omega)$ such that $|u_n(x)| \le g(x)$, for almost all $x \in \Omega$. Then, if we put

$$a_n(x) = \frac{f(x, u_n(x))}{1 + |u_n(x)|},$$

taking into account that the function $t \to \frac{1+t^{q-1}}{1+t}$ is increasing in $[1, +\infty)$, we can obtain, in view of (5), the estimate

$$|a_n(x)| \le C\frac{1 + |u_n(x)|^{q-1}}{1 + |u_n(x)|} \le a(x) := C\left(1 + \frac{1 + |g(x)|^{q-1}}{1 + |g(x)|}\right), \quad \text{for almost all } x \in \Omega.$$

Moreover, it is easily seen that $a \in L^{\frac{q}{q-2}}(\Omega)$ and $\frac{q}{q-2} > \frac{N}{2}$, and that $a_n(x) \to 0$ as $n \to +\infty$, almost everywhere in Ω (this follows from the fact that $u^* = 0$ is a solution of problem (P) if and only if $f(x, 0) = 0$). Therefore, by the Dominated Convergence Theorem, it follows

$$\lim_{n\to+\infty} \|a_n\|_{L^{\frac{q}{q-2}}(\Omega)} = 0. \tag{11}$$

Since u_n is a weak solution of the problem

$$\begin{cases} -\Delta u = a_n(x)(1 + |u|) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

by standard regularity results (see [5], Appendix B), we have $u_n \in C^{1,\beta}(\overline{\Omega})$, for some $\beta \in (0, 1)$. By Theorem 8.16 of [6], there exists a constant $C > 0$ independent of u_n such that

$$\|u_n\|_\infty \le C\|a_n\|_{L^{\frac{q}{q-2}}(\Omega)}(1 + \|u_n\|_\infty)$$

Using the limit (11), we infer

$$\lim_{n\to+\infty} \|u_n\|_\infty = 0. \tag{12}$$

Moreover, being I sequentially weakly lower semicontinuous, we have

$$0 = I(u^*) \le \liminf_{n\to+\infty} I(u_n) = \inf_S I. \tag{13}$$

At this point, in view of (12), we can choose $n \in \mathbb{N}$ such that $\|u_n\|_\infty < \delta$. Thanks to (10), (13) and (iii), we get

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \int_\Omega F(x, u_n(x))\mathrm{d}x$$
$$= \int_\Omega \left(\frac{1}{2}f(x, u_n)u_n(x) - F(x, u_n)\right)\mathrm{d}x < 0 = I(u^*) \le \inf_S I,$$

which is in contradiction with $u_n \in S$ (note that the above strict inequality follows from being u_n a non zero function). Consequently, u^* is nonzero.

Finally, let us suppose that assumption (iv) holds. Let $\eta \in (0, \lambda_1)$ be such that

$$\limsup_{t\to 0^+} \frac{f(x,t)}{t} < \lambda_1 - \eta, \quad \text{uniformly in } \Omega.$$

Then, using also (5), we find a positive constant C_1 such that

$$|f(x,t)| \le (\lambda_1 - \eta)|t| + C_1|t|^{q-1}, \quad \text{for almost all } x \in \Omega \text{ and for all } t \in \mathbb{R}. \tag{14}$$

Recall again that, under condition (v) or condition (vi), every minimizing sequence of $I_{|S}$ is bounded. Thus, if $\{u_n\}$ is a sequence in S which minimizes $I_{|S}$, there exists $u^* \in W_0^{1,2}(\Omega)$ such that $u_n \to u^*$ weakly in $W_0^{1,2}(\Omega)$. As noted above, u^* turns out to be a weak solution of problem (P). Therefore, if we prove that u^* is non zero, we get $u^* \in S$ and $I(u^*) = \min_S I$. Using $u^* = 0$ and (14), one has

$$0 = I'(u_n)(u_n) = \|u_n\|^2 - \int_\Omega f(x, u_n(x))u_n(x)\mathrm{d}x \ge \|u_n\|^2$$
$$- (\lambda_1 - \eta)\int_\Omega |u_n|^2\mathrm{d}x - C_1\int_\Omega |u_n|^p\mathrm{d}x$$
$$\ge \|u_n\|^2 - (\lambda_1 - \eta)\frac{1}{\lambda_1}\|u_n\|^2 - C_1\int_\Omega |u_n|^p\mathrm{d}x =$$
$$= \frac{\eta}{\lambda_1}\|u\|^2 - C_1\int_\Omega |u_n|^p\mathrm{d}x \ge \frac{\eta}{\lambda_1}c_q^{-2}\left(\int_\Omega |u_n|^p\mathrm{d}x\right)^{\frac{2}{q}} - C_1\int_\Omega |u_n|^q\mathrm{d}x, \tag{15}$$

for each $n \in \mathbb{N}$. This implies,

$$\int_\Omega |u_n|^p\mathrm{d}x \ge \left(\frac{\eta}{\lambda_1 C_1 c_q^2}\right)^{\frac{q}{q-2}}. \tag{16}$$

Since the functional $u \in W_0^{1,2}(\Omega) \to \int_\Omega |u|^q\mathrm{d}x$ is sequentially weakly continuous, from (16) one has $\int_\Omega |u^*|^q\mathrm{d}x \ge \left(\frac{\eta}{\lambda_1 C_1 c_q^2}\right)^{\frac{q}{q-2}}$. Therefore, u^* is nonzero. The proof of Theorem 1 is now complete. □

Remark 1: From the proof of Theorem 1, it is clear that, in the conditions (i) and (ii), '$\limsup_{\xi\to 0^+}$' and '$\liminf_{\xi\to 0^+}$' can be replaced by '$\limsup_{\xi\to 0^-}$' and '$\liminf_{\xi\to 0^-}$', respectively.

The next result concerns the existence of the least energy nodal solutions for the problem (P). To state it, we need to introduce the following condition:

($\tilde{i}$) there exists $\delta > 0$ such that $f(x,t)t \geq 0$, for almost all $x \in \Omega$ and for all $t \in [-\delta, \delta]$.

Theorem 2: *Assume* (b) *of Theorem 1 and the additional condition* ($\tilde{i}$) *in the case of* (iii) *holds.*

Then, if the set $S_\pm$ of all sign-changing weak solutions is nonempty, there exists $u_0 \in S_\pm$ such that $I(u_0) = \inf_{u\in S_\pm} I(u)$.

***Proof*:** Assume $S_\pm \neq \emptyset$ and let $\{u_n\}$ be a sequence in $S_\pm$ which minimizes $I_{|S_\pm}$. By the proof of Theorem 1, we know that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. So, $\{u_n\}$ weakly converges to some $u_* \in W_0^{1,2}(\Omega)$, with u_* being a solution of problem (P). Our goal is to show that u_* is sign-changing. Suppose, at first, that (iii) and ($\tilde{i}$) hold. Arguing as in the proof of Theorem 1, we infer that u_* is nonzero. Assume that u_* is not sign-changing. Then, for instance, we can suppose u_* nonnegative. Recall that, by standard regularity results, the weak solutions to problem (P) are at least of class $C^{1,\beta}$ in $\overline{\Omega}$, for some $\beta \in (0,1)$. Moreover, condition ($\tilde{i}$) allows to apply the Strong Maximum Principle of [7] and the classical Hopf Lemma. Therefore, u_* is actually strictly positive in Ω, with $\frac{\partial u_*}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the outer unit normal to $\partial\Omega$. In other words, u_* belongs to the interior P of the positive cone of $C^1(\overline{\Omega})$. At this point, observe that $u_n - u_*$ is solution to the problem

$$\begin{cases} -\Delta u = b_n(x)(1+|u|), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{P_n}$$

where

$$b_n(x) := \frac{f(x,u_n) - f(x,u^*)}{1+|u_n(x) - u_*(x)|} \to 0, \quad \text{as } n \to +\infty, \ \text{ for almost all } \ x \in \Omega.$$

Since $u_n - u^* \to 0$ strongly in $L^q(\Omega)$, there exists $g \in L^q(\Omega)$ such that $|u_n(x) - u^*(x)| \leq g(x)$, for all $n \in \mathbb{N}$ and for almost all $x \in \Omega$. Moreover, since $u^* \in C^{1+\beta}(\overline{\Omega})$, one has $M := \sup_{\overline{\Omega}} |u| < +\infty$. Then, using condition (5) and the elementary inequality $|a+b|^{q-1} \leq 2^{q-1}(|a|^{q-1} + |b|^{q-1})$, we have

$$\begin{aligned} |f(x,u_n(x)) - f(x,u^*(x))| &\leq |f(x,u_n(x))| + |f(x,u^*(x))| \leq C(2 + |u_n(x)|^{q-1} + |u^*(x)|^{q-1}) \\ &\leq C(2 + 2^{q-1}|u_n(x) - u^*(x)|^{q-1} + (2^{q-1}+1)|u^*(x)|^{q-1}) \\ &\leq C_1(1 + |u_n(x) - u^*(x)|^{q-1}) \end{aligned}$$

for almost all $x \in \Omega$, for all $t \in \mathbb{R}$ and where we put $C_1 := \max\{2C + (2^{q-1}+1)CM^{q-1}, 2^{q-1}C\}$. Therefore, taking in mind that the function $t \to \frac{1+t^{q-1}}{1+t}$ is nondecreasing in $[1, +\infty)$, one has

$$|b_n(x)| \leq C_1 \frac{1 + |u_n(x) - u^*(x)|^{q-1}}{1 + |u_n(x) - u^*(x)|} \leq C_1 \left(1 + \frac{1 + g(x)^{q-1}}{1 + g(x)}\right) := b(x),$$

for all $n \in \mathbb{N}$ and for almost all $x \in \Omega$, with $b \in L^{\frac{q}{q-2}}(\Omega)$. Then, arguing as in the proof of Theorem 1, we infer

$$\lim_{n\to+\infty} \|u_n - u_*\|_\infty = 0.$$

In particular, one has $\sup_{n\in\mathbb{N}} \|u_n - u_*\|_\infty < +\infty$. Consequently, by Theorem 1 of [8], one has

$$\sup_{n\in\mathbb{N}} \|u_n - u_*\|_{C^{1,\beta}(\overline{\Omega})} < +\infty.$$

for some $\beta \in (0, 1)$. Then, by the Ascoli–Arzelà Theorem, up to a subsequence, one has

$$\lim_{n\to+\infty} \|u_n - u_*\|_{C^1(\overline{\Omega})} = 0.$$

Since u_* belongs to the interior P of the positive cone of $C^1(\overline{\Omega})$, we then get $u_n \in P$, for large $n \in \mathbb{N}$. This is a contradiction with being u_n sign-changing. The case of u_* nonpositive can be treated in an analogous way. There u_* must be sign-changing.

Now, we pass to consider the case of condition (iv) holds. Let $u_n^+ := \max\{u_n, 0\}$ and $u_n^- := \max\{-u_n, 0\}$, for each $n \in \mathbb{N}$. Then, $u_n^+, u_n^- \in W_0^{1,2}(\Omega)$ and, up to a subsequence, $u_n^+ \to u_*^+ := \max\{u_*, 0\}$, $u_n^- \to u_*^- := \max\{-u_*, 0\}$, a.e. in Ω. Testing Equation (7) with u_n^+ and u_n^- and arguing as in (15), we get the inequalities

$$\int_\Omega |u_n^+|^q \mathrm{d}x \geq \left(\frac{\eta}{\lambda_1 C_1 c_q^2}\right)^{\frac{q}{q-2}}, \quad \int_\Omega |u_n^-|^p \mathrm{d}x \geq \left(\frac{\eta}{\lambda_1 C_1 c_q^2}\right)^{\frac{q}{q-2}}, \quad \text{for all } n \in \mathbb{N},$$

Passing to the limit as $n \to +\infty$, we get

$$\int_\Omega |u_*^+|^q \mathrm{d}x \geq \left(\frac{\eta}{\lambda_1 C_1 c_q^2}\right)^{\frac{q}{q-2}}, \quad \int_\Omega |u_*^-|^p \mathrm{d}x \geq \left(\frac{\eta}{\lambda_1 C_1 c_q^2}\right)^{\frac{q}{q-2}}, \quad \text{for all } n \in \mathbb{N},$$

and so $u_*^+ \neq 0$ and $u_*^- \neq 0$. □

4. Some special cases

Our main results apply, in particular, to a nonlinearity of the type $f(x, t) = \lambda|u|^{s-2}u - \mu|u|^{r-2}u$, where $\lambda > 0$, $\mu \in \mathbb{R}$ and $r, s \in (1, 2)$ with $r < s$.

Indeed, for each $\lambda > 0$, this nonlinearity satisfies all the assumptions (i), (ii), (iii) and (v) and $(\bar{i})$ if $\mu \leq 0$, and all the assumptions (iii), (iv) and (v) if $\mu > 0$. It also to be noted that from Theorem 3.13 of [3] and from Theorem 1 of [4] there exists $\mu_0 > 0$ such that $S \neq \emptyset$, $S_\pm \neq \emptyset$ if $\mu \leq \mu_0$. This means that, in view of Theorems 1 and 2, the problem

$$\begin{cases} -\Delta u = \lambda|u|^{s-2}u - \mu|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P_{\lambda,\mu})$$

admits both a least energy solution and a least energy nodal solution, for $\lambda > 0$ and $\mu \leq \mu_0$.

As mentioned in the introduction, when $\mu \leq 0$, there exists a sequence of sign-changing solutions to problem $(P_{\lambda,\mu})$ which strongly converges to the zero function. As a consequence, the sets $\mathcal{A}$ and $\mathcal{A}_{\pm}$ introduced in (4), where in this case $I'(u)(u) = \|u\|^2 - \lambda\|u\|^s_{L^s(\Omega)} + \mu\|u\|^r_{L^s(\Omega)}$, are not weakly closed. On the contrary, when $\mu > 0$ we have the following result:

Theorem 3: *Let $\lambda, \mu > 0$ and $r, s \in (1, 2)$, with $r < s$. Then, the sets*

$$\begin{aligned}\mathcal{A} &= \{u \in W^{1,2}_0(\Omega)\setminus\{0\} : \|u\|^2 - \lambda\|u\|^s_{L^s(\Omega)} + \mu\|u\|^r_{L^s(\Omega)} \leq 0\},\\ \mathcal{A}_{\pm} &= \{u \in W^{1,2}_0(\Omega) : \ u_+, u_- \in \mathcal{A}\}.\end{aligned}$$

are weakly compact.

***Proof*:** Let $\{u_n\}$ be a sequence in $\mathcal{A}$. Then, for each $n \in \mathbb{N}$, one has

$$\|u_n\|^2 \leq \lambda\|u_n\|_{L^s(\Omega)} \leq c^s_s\lambda\|u\|^s_{L^s(\Omega)}.$$

Therefore, being $s < 2$, $\{u_n\}$ is bounded in $W^{1,2}_0(\Omega)$. So, up to a subsequence, $\{u_n\}$ weakly converges to some $u^* \in W^{1,2}_0(\Omega)$. Moreover, passing to the limit as $n \to +\infty$ in the inequality

$$\|u_n\|^2 \leq \lambda\|u_n\|_{L^s(\Omega)} - \mu\|u_n\|_{L^r(\Omega)} \tag{17}$$

we promptly get

$$\|u^*\|^2 \leq \lambda\|u^*\|^s_{L^s(\Omega)} - \mu\|u^*\|^r_{L^r(\Omega)}.$$

To finish, it remains to prove that $u^* \neq 0$. Indeed, if we fix $m \in (2, 2^*)$, we can find a constant $M > 0$ such that $\lambda t^s - \mu t^\mu \leq M t^m$, for all $t \geq 0$. Consequently, using (17), we obtain

$$\|u_n\|^2_{L^m(\Omega)} \leq c^2_m\|u_n\|^2 \leq c^2_m(\lambda\|u_n\|^s_{L^s(\Omega)} - \mu\|u_n\|^r_{L^r(\Omega)}) \leq c^2_m M\|u_n\|^m_{L^m(\Omega)},$$

from which $\|u_n\|_{L^q(\Omega)} \geq \left(\frac{1}{Mc^2_m}\right)^{\frac{1}{q-2}}$. Passing to the limit as $n \to +\infty$ in this last inequality and taking into account the sequential weak continuity of the functional $u \in W^{1,2}_0(\Omega) \to \|u\|^m_{L^m(\Omega)}$, we get $\|u^*\| \geq \left(\frac{1}{Mc^2_m}\right)^{\frac{1}{m-2}}$. Therefore, $u^* \neq 0$.

The weakly compactness of $\mathcal{A}_{\pm}$ follows at once being $\mathcal{A}_{\pm}$ a weakly closed subset of $\mathcal{A}$.

□

Corollary 1: *Let λ, μ, r, s be as in Theorem 3. Then there exist $u_0 \in \mathcal{N}$ and $v_0 \in \mathcal{N}_{\pm}$, where $\mathcal{N}$ and $\mathcal{N}_{\pm}$ are the Nehari manifolds associated to the functional $I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{s}\|u\|^s_s + \frac{\mu}{r}\|u\|^r_r$, $u \in W^{1,2}_0(\Omega)$, such that*

$$I(u_0) = \inf_{\mathcal{N}} I, \quad I(v_0) = \inf_{\mathcal{N}_{\pm}} I.$$

***Proof*:** The sets $\mathcal{A} = \{u \in W^{1,2}_0(\Omega)\setminus\{0\} : \|u\|^2 - \lambda\|u\|^s_s + \mu\|u\|^r_r \leq 0\}$, $\mathcal{A}_{\pm} = \{u \in W^{1,2}_0(\Omega) : \ u_+, u_- \in \mathcal{A}\}$ are clearly nonempty. So, being I sequentially weakly lower semicontinuous on $\mathcal{A}$ *and* $\mathcal{A}_{\pm}$, by the Eberlein–Smulian Theorem and Theorem 1, there exist $u_0 \in \mathcal{A}$ and $v_0 \in \mathcal{A}_{\pm}$, such that

$$I(u_0) = \inf_{\mathcal{A}} I, \quad I(v_0) = \inf_{\mathcal{A}_{\pm}} I.$$

If $\|u_0\|^2 - \lambda\|u_0\|_s^s + \mu\|u_0\|_r^r < 0$, then u_0 should be a local minimum of I. Thus, $I'(u_0)(u_0) = \|u_0\|^2 - \lambda\|u_0\|_s^s + \mu\|u_0\|_r^r = 0$, a contradiction. Therefore, $u_0 \in \mathcal{N}$. The same argument shows that $v_0 \in \mathcal{N}_{\pm}$. □

An open problem Let λ, μ, r, s be as in Theorem 3 and let u_0,v_0 be as in Corollary 1. From Theorem 3.13 of [3] and Theorem 1 of [4], we know that S *and* $S_{\pm}$ are nonempty when the nonlinearity f is of the form $f(x,t) = \lambda|t|^{s-2}t - \mu|t|^{r-2}t$. In the light of Corollary 1, does u_0 and v_0 belongs to S and $S_{\pm}$, respectively?

Acknowledgements

The authors want to express their gratitude to Prof. Giovanni Anello for his invaluable help.

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