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An algebraic representation of Steiner triple systems of order 13 ☆

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ABSTRACT

In this paper we construct an incidence structure isomorphic to a Steiner triple system of order 13 by defining a set $\mathcal{B}$ of twentysix vectors in the 13-dimensional vector space $V = \mathrm{GF}(5)^{13}$, with the property that there exist precisely thirteen 6-subsets of $\mathcal{B}$ whose elements sum up to zero in V, which can also be characterized as the intersections of $\mathcal{B}$ with thirteen linear hyperplanes of V.

A *Steiner triple system* of order v (STS(v), for short) is a $2-(v, 3, 1)$ block design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, that is, a finite set $\mathcal{P}$ with v elements, called *points*, together with a collection $\mathcal{B}$ of subsets of $\mathcal{P}$ with three elements, called *blocks* or *triples*, such that any two distinct points in $\mathcal{P}$ belong to precisely one triple in $\mathcal{B}$ [1–3]. An STS(v) is also called an $S(2, 3, v)$ Steiner system, or a $TS(v, 1)$ triple system. A less used and more obsolete terminology is Steiner triad system.

Moving from a purely combinatorial viewpoint to a geometrical and maybe more inspiring perspective, a Steiner triple system can be thought of as a geometry of points and lines, where each line contains three points and there exists a unique line through any two distinct points.

Steiner triple systems are one of the earliest and most studied classes of combinatorial structures. It is easy to see that $v \equiv 1$ or $3 \pmod 6$ for any STS(v). Conversely, if $v \geq 3$ and $v \equiv 1$ or $3 \pmod 6$, then there exists at least one Steiner triple system of order v [4].

A Steiner triple system is called *geometric* if it is isomorphic to the point-line design of either a projective geometry $\mathrm{PG}(d, 2)$ over GF(2), or an affine geometry $\mathrm{AG}(d, 3)$ over GF(3), for some $d \geq 1$ (an actual visual representation of PG(3, 2) is given in [5]). In the former case, the points of the system can be identified with the nonzero vectors of the vector space $\mathrm{GF}(2)^{d+1}$, and the blocks are the *projective lines* (that is, the punctured two-dimensional subspaces). In the latter case, the point-set can be identified with the vector space $\mathrm{GF}(3)^d$, and the blocks are the *affine lines* (that is, the cosets of the one-dimensional subspaces). In either case, each block has exactly three elements, whose sum in (the additive group of) the underlying vector space is equal to zero.

This leads to the following definition, which attempts to extend the notion of geometric STS to a more general and purely algebraic setting. We say that an STS is *additive* if it can be embedded in a commutative group in such a way that the sum of the points in any block is zero [6, Definition 2.1 and Proposition 2.7]. More precisely, a Steiner triple system $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is additive if there exist a commutative group $(G, +)$ and an injective map $\psi : \mathcal{P} \longrightarrow G$ such that

$$\psi(X) + \psi(Y) + \psi(Z) = 0$$

for any block $\{X, Y, Z\} \in \mathcal{B}$.

The definition of additivity can be extended, more generally, to all block designs [6]. In [6] and [7] it is shown that all symmetric and affine resolvable 2-designs are additive. Some geometric designs, such as the point-flat designs of an affine geometry $\mathrm{AG}(d, q)$ over the Galois field GF(q), as well as of a projective geometry $\mathrm{PG}(d, 2)$ over GF(2), are basic examples of additive 2-designs.

More generally, the so-called 2-(v, k, λ) designs over GF(2), when seen as 2-$(2^v - 1, 2^k - 1, \lambda)$ designs, whose points are the non-zero vectors of $\mathrm{GF}(2)^v$ and whose blocks are the sets of non-zero vectors of suitable k-dimensional subspaces, form a remarkable class of additive 2-designs [8–11], which are, in turn, subdesigns of one of the classes of additive 2-designs considered in [12]. A similar class of additive 2-(p^n, mp, λ) designs is described in [13], where the full automorphism group is found, on the basis of the results given in [14].

Note that the search for additive 2-$(v, 3, \lambda)$ designs must be necessarily restricted to the class of Steiner triple systems (i.e., $\lambda = 1$). Indeed, the third point of a block through two points P and Q in a group $(G, +)$ is necessarily $-(P + Q)$, hence $\lambda = 1$.

Examples of additive STSs can be constructed in a “natural” way as follows. Given a finite commutative group $(G, +)$, take $\mathcal{P} = G$, and take $\mathcal{B}$ to be the family of all (unordered) triples $\{X, Y, Z\}$ of (distinct) elements of $\mathcal{P}$ such that $X + Y + Z = 0$. If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is an STS, then G

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cannot contain an element g of even order $2n$, nor an element g of odd order $2n+1$, with $n \geq 2$, as $\{0, ng, ng\}$ (respectively, $\{g, ng, ng\}$) would be the only triple summing to zero containing 0 (respectively, g) and ng. Hence a necessary condition for $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ to be an STS is that G be an elementary abelian 3-group.

Conversely, it is easy to see that if G is such a group, then $(G, \mathcal{B})$ is a (necessarily additive) Steiner triple system, which, up to isomorphism, is precisely the point-line design of $\mathrm{AG}(d, 3)$, for some $d \geq 1$.

Similarly, if one takes instead $\mathcal{P} = G \backslash \{0\}$ (with the same $\mathcal{B}$ as above), then it is easy to show that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a (necessarily additive) Steiner triple system if and only if G is an elementary abelian 2-group, that is, if and only if $\mathcal{D}$ is (up to isomorphism) the point-line design of $\mathrm{PG}(d, 2)$, for some $d \geq 1$.

More generally, one may let $\mathcal{P}$ be any subset of G and ask whether, in this case, one can obtain new additive STSs, other than the point-line designs of $\mathrm{PG}(d, 2)$ and $\mathrm{AG}(d, 3)$. It turns out, however, that there exist no other additive STSs. Indeed, a Steiner triple system is additive if and only if it is a geometric STS, that is, if and only if it is the point-line design of either $\mathrm{PG}(d, 2)$ or $\mathrm{AG}(d, 3)$, for some $d \geq 1$ [6, Theorem 3.7].

Moreover, an additive STS $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ satisfies the additivity condition in the strongest possible sense, in that, whenever $\mathcal{P}$ is embedded in a commutative group $(G, +)$, in such a way that the sum of the three elements of each block is zero, it is also true that conversely, if $\{x, y, z\}$ is a 3-subset of $\mathcal{P}$ such that $x + y + z = 0$ in G, then $\{x, y, z\}$ is a block in $\mathcal{B}$.

A strong version of additivity (although not as strong as for STSs) is satisfied also by symmetric and affine resolvable designs [6,7], and by the classes of additive designs described in [12] and [13]. For these 2-(v, k, λ) designs, there exists a distinguished embedding in a commutative group $(G, +)$, in such a way that a k-subset of $\mathcal{P}$ is a block if and only if the sum of its elements is zero in G ([6, Theorem 4.1.ii], [7, Theorem 3.1.ii]; see also [6, Remark 4.4]). A block design with this property is said to be *strongly additive*, and an open problem is posed in [7, 3.10] as to whether any additive design is also strongly additive. We recently showed that there exists a 2-(16, 4, 2) quasidouble of the affine plane of order 4 that is additive but not strongly additive [15].

In [6,7] the authors also found several examples of additive 2-(v, k, λ) designs $(\mathcal{P}, \mathcal{B})$ that could be embedded in a finite vector space V, in such a way that the blocks were characterized not only as the zero-sum k-subsets of $\mathcal{P}$, but also as the intersections of $\mathcal{P}$ with suitable hyperplanes of V, that is, in terms of linear equations.

For instance, the (unique) Steiner triple system of order 9 can be represented in the 3-dimensional vector space $\mathrm{GF}(3)^3$ as the set of the nine points of the affine subplane $\mathcal{P}$ of equation $x_1 + x_2 + x_3 = 1$ [6, Remark 3.8(b)], the twelve blocks of the system being precisely the intersections of $\mathcal{P}$ with the twelve linear subplanes of equations $x_i - x_j = 0$ and $x_i + x_j = 0, 1, 2$, for $i, j \in \{1, 2, 3\}, i \neq j$. The (unique) 2-(11, 5, 2) biplane can be represented as a set $\mathcal{P}$ of eleven points in $\mathrm{GF}(3)^5$, in such a way that the blocks can be characterized as the only 5-sets of elements of $\mathcal{P}$ summing up to 0, and can also be described as the intersections of $\mathcal{P}$ with eleven suitable linear hyperplanes [6, Example 4.11]. In this case, $\mathcal{P}$ is not contained in any affine hyperplane of $\mathrm{GF}(3)^5$.

It is well known that there exist, up to isomorphism, only two Steiner triple systems of order 13 (see the tables of blocks in [16], [2, Table II.1.27] and [3, Table 5.7]). In 1897 Zulauf showed that the two STS(13)s known at his time were non-isomorphic [17], and, in 1899, De Pasquale determined that only two isomorphism classes were possible [18]. In any such system there are twentysix blocks, and each point belongs to precisely six blocks.

Since 13 is not of the form $2^d - 1$ or 3^d, a Steiner triple system of order 13 is not additive, hence it cannot be represented as a subset of a commutative group in such a way that the sum of the elements in each block is zero. Nor can an STS(13) be embedded in a finite Desarguesian projective plane [19, 2.1].

But a surprise is in store.

Let V be the 13-dimensional vector space $\mathrm{GF}(5)^{13}$, and let $\mathcal{B}$ be the set of the twentysix vectors $\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_{26}$ in V listed in the following table (1). We claim that there exist precisely thirteen 6-subsets $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_6\}$ of $\mathcal{B}$ with the property that $\mathfrak{p}_1 + \mathfrak{p}_2 + \cdots + \mathfrak{p}_6 = 0$ in V, and that, moreover, these 6-subsets are the intersections of $\mathcal{B}$ with thirteen linear hyperplanes of V.

$$
\begin{array}{lcl}
\mathfrak{b}_1 & = & (0,0,0,1,1,1,1,1,1,1,1,1,1) \\
\mathfrak{b}_2 & = & (0,1,1,0,1,1,0,1,1,1,1,1,1) \\
\mathfrak{b}_3 & = & (0,1,1,1,0,1,1,1,1,0,1,1,1) \\
\mathfrak{b}_4 & = & (0,1,1,1,1,0,1,1,1,1,0,1,1) \\
\mathfrak{b}_5 & = & (0,1,1,1,1,1,1,0,0,1,1,1,1) \\
\mathfrak{b}_6 & = & (0,1,1,1,1,1,1,1,1,1,1,0,0) \\
\mathfrak{b}_7 & = & (1,0,1,0,0,1,1,1,1,1,1,1,1) \\
\mathfrak{b}_8 & = & (1,0,1,1,1,0,0,1,1,1,1,1,1) \\
\mathfrak{b}_9 & = & (1,0,1,1,1,1,1,0,1,0,1,1,1) \\
\mathfrak{b}_{10} & = & (1,0,1,1,1,1,1,1,0,1,1,0,1) \\
\mathfrak{b}_{11} & = & (1,0,1,1,1,1,1,1,1,1,0,1,0) \\
\mathfrak{b}_{12} & = & (1,1,0,0,1,0,1,1,1,1,1,1,1) \\
\mathfrak{b}_{13} & = & (1,1,0,1,0,1,1,1,1,1,1,1,0) \\
\mathfrak{b}_{14} & = & (1,1,0,1,1,1,0,1,1,1,1,0,1) \\
\mathfrak{b}_{15} & = & (1,1,0,1,1,1,1,0,1,1,0,1,1) \\
\mathfrak{b}_{16} & = & (1,1,0,1,1,1,1,1,0,0,1,1,1) \\
\mathfrak{b}_{17} & = & (1,1,1,0,1,1,1,0,1,1,1,0,1) \\
\mathfrak{b}_{18} & = & (1,1,1,0,1,1,1,1,0,1,0,1,1) \\
\mathfrak{b}_{19} & = & (1,1,1,0,1,1,1,1,1,0,1,1,0) \\
\mathfrak{b}_{20} & = & (1,1,1,1,0,0,1,0,1,1,1,1,1) \\
\mathfrak{b}_{21} & = & (1,1,1,1,0,1,0,1,0,1,1,1,1) \\
\mathfrak{b}_{22} & = & (1,1,1,1,0,1,1,1,1,1,0,0,1) \\
\mathfrak{b}_{23} & = & (1,1,1,1,1,0,1,1,0,1,1,1,0) \\
\mathfrak{b}_{24} & = & (1,1,1,1,1,0,1,1,1,0,1,0,1) \\
\mathfrak{b}_{25} & = & (1,1,1,1,1,1,0,0,1,1,1,1,0) \\
\mathfrak{b}_{26} & = & (1,1,1,1,1,1,0,1,1,0,0,1,1)
\end{array}
\tag{1}
$$

For each $1 \leq i \leq 13$, let $\mathcal{H}_i$ be the linear hyperplane in V of equation $x_i = 0$, that is,

$$\mathcal{H}_i = \{(x_1, x_2, \ldots, x_{13}) \in \mathrm{GF}(5)^{13} \mid x_i = 0\},$$

and let us define

$$P_i = \mathcal{B} \cap \mathcal{H}_i.$$

Then, by construction, for all $1 \leq i \leq 13$, P_i is a 6-subset of $\mathcal{B}$ whose elements sum up to zero in V.

Conversely, let $P = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_6\}$ be a 6-subset of $\mathcal{B}$ with the property that $\mathfrak{p}_1 + \mathfrak{p}_2 + \cdots + \mathfrak{p}_6 = 0$ in V. If, for $\mathfrak{b}$ in $\mathcal{B}$, we denote by $\mathfrak{b}(i)$ the ith coefficient of $\mathfrak{b}$, $1 \leq i \leq 13$, then, by construction, $\mathfrak{b}(i) = 0$ for exactly three values of i and $\mathfrak{b}(i) = 1$ for ten values of i, for all $\mathfrak{b}$ in $\mathcal{B}$. Hence the six vectors $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_6$ have an overall number of $13 \times 6 = 78$ coefficients, precisely 18 zeros and 60 ones. Therefore there exists a coordinate i, $1 \leq i \leq 13$, such that $\mathfrak{p}(i) = 0$ for more than one $\mathfrak{p}$ in P. Since $\mathfrak{p}_1 + \mathfrak{p}_2 + \cdots + \mathfrak{p}_6 = 0$ in $\mathrm{GF}(5)^{13}$, that is, $\mathfrak{p}_1(k) + \mathfrak{p}_2(k) + \cdots + \mathfrak{p}_6(k) = 0$ in $\mathrm{GF}(5)$ for all $1 \leq k \leq 13$, it follows that $\mathfrak{p}_j(i) = 0$ for all $j = 1, \ldots, 6$. Since there exist precisely six elements $\mathfrak{b}$ of $\mathcal{B}$ such that $\mathfrak{b}(i) = 0$, we can finally conclude that $P = \mathcal{B} \cap \mathcal{H}_i = P_i$.

Therefore $P_1, P_2, \ldots, P_{13}$ are the only 6-subsets of $\mathcal{B}$ whose elements sum up to zero in V, and they are the intersections of $\mathcal{B}$ with the thirteen linear hyperplanes $x_1 = 0$, $x_2 = 0$, …, $x_{13} = 0$ of V, respectively.

Moreover, for any choice of distinct i, j in $\{1, 2, \ldots, 13\}$, P_i and P_j intersect in precisely one vector $\mathfrak{b}$ in $\mathcal{B}$, and there exists a unique P_k, with k different from i and j, such that $\mathfrak{b}$ also belongs to P_k. Conversely, for any $\mathfrak{b}$ in $\mathcal{B}$, there exist precisely three indices i in $\{1, 2, \ldots, 13\}$ such that $\mathfrak{b} \in P_i$.

This shows, finally, that the incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P} = \{P_1, P_2, \ldots, P_{13}\}$, $\mathcal{B} = \{\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_{26}\}$, and $P \in \mathcal{P}$ is incident with $\mathfrak{b} \in \mathcal{B}$ if and only if $\mathfrak{b} \in P$, is a 2-(13, 3, 1) design, that is, a Steiner triple system of order 13.

This is precisely the STS(13) denoted by n.1 in [2, Table II.1.27] and [3, Table 5.7]. A similar construction can be given for the STS(13) denoted by n.2, by simply redefining the vectors $\mathfrak{b}_2, \mathfrak{b}_6, \mathfrak{b}_{12}, \mathfrak{b}_{14}, \mathfrak{b}_{19}, \mathfrak{b}_{24}$ in the list (1), by switching the coefficients as follows: $\mathfrak{b}_2(7) \leftrightarrow \mathfrak{b}_2(13)$, $\mathfrak{b}_6(7) \leftrightarrow \mathfrak{b}_6(13)$, $\mathfrak{b}_{12}(6) \leftrightarrow \mathfrak{b}_{12}(7)$, $\mathfrak{b}_{14}(6) \leftrightarrow \mathfrak{b}_{14}(7)$, $\mathfrak{b}_{19}(6) \leftrightarrow \mathfrak{b}_{19}(13)$, and $\mathfrak{b}_{24}(6) \leftrightarrow \mathfrak{b}_{24}(13)$.

Finally, note that this kind of algebraic representation cannot be extended to all possible 2-designs. Indeed, the complete 2-$(4, 3, 2)$ design with points $1, 2, 3, 4$ and blocks $\{2, 3, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 4\}$, $\{1, 2, 3\}$ cannot be represented as above, since there exists no commutative group containing a 4-subset S with the property that $a + b + c = 0$ for any 3-subset $\{a, b, c\}$ of S.

It is possible to show, however, that for any 2-(v, b, r, k, λ) design with $k \leq v/2$ and $r-\lambda \geq 2$, there exists a finite commutative group $(G, +)$ and a b-subset $\mathcal{B} = \{\mathfrak{b}_1, \dots, \mathfrak{b}_b\}$ of G such that there exist precisely v r-subsets $P_1, \dots, P_v$ of $\mathcal{B}$ with the property that the sum of the r elements of P_i is zero in G for all $i = 1, \dots, v$. Moreover, the incidence structure $(\mathcal{P}, \mathcal{B})$, where $\mathcal{P} = \{P_1, \dots, P_v\}$ and P_i is incident with $\mathfrak{b}_j$ if and only if $\mathfrak{b}_j \in P_i$, is a 2-(v, b, r, k, λ) design isomorphic to the given design. This general case exceeds the scope of the present article and will be treated in a forthcoming paper.

It is worth mentioning that an alternative algebraic representation was recently given in [20, Proposition 4], where the Steiner triple systems of order 13 were described as sections of $A(13)/A(12)$ in the alternating group $A(13)$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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