From the theory of "congeneric surd equations" to "Segre's bicomplex numbers"

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Available online 4 April 2017

Abstract

We will study the historical emergence of *Tessarines* or *Bicomplex numbers*, from their origin as "imaginary" solutions of irrational equations, to their insertion in the context of study of the algebras of hypercomplex numbers.

Sommario

Analizzeremo l'evoluzione storica dei *Tessarini* o numeri *Bicomplessi*, dalla loro origine come soluzioni "immaginarie" di equazioni irrazionali, al loro inserimento nel contesto dello studio delle algebre di ipercomplessi.
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MSC: 01A55; 08-03; 51-03

Keywords: James Cockle; William Rowan Hamilton; Corrado Segre; Tessarines; Biquaternions; Bicomplex numbers

1. Introduction

From the mid-nineteenth century, particularly in Great Britain, in the wake of researches on geometrical interpretations of complex numbers, studies developed that led to the birth of new systems of hypercomplex numbers and are at the basis of the birth of modern algebra. In particular, the discovery in 1843 of Quaternions by William Rowan Hamilton (1805–1865) revealed to mathematicians the existence of an algebraic system that had all the properties of real and complex numbers except commutativity of multiplication.[1]

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[1] For an in-depth examination of this topic see for example [Conway, 1949], [Hendry, 1984], [Ohrstrom, 1985], [Krishnaswami and Sachdev, 2016].

This article is post-print version of http://dx.doi.org/10.1016/j.hm.2017.03.001

Further, the studies on symbolic algebra[2] by George Peacock (1791–1858) and on logic by Augustus De Morgan (1806–1871) created a context of reflection and analysis on the laws of arithmetic and their meaning.

As a result, mathematicians in Great Britain carried out researches on new systems of hypercomplex numbers, leading to the discovery of Octonions[3] (1843/1845) by John T. Graves (1806–1870) and Arthur Cayley (1821–1895), of the theory of Pluriquaternions[4] (1848) by Thomas Penyngton Kirkman (1806–1895), of Biquaternions[5] (1873) and the Algebras[6] (1878) of William Kingdon Clifford (1845–1879) and to the systematic presentation of the algebras of hypercomplex numbers[7] (1870), the work of Benjamin Peirce (1809–1880).

In this framework, *tessarines* (or bicomplex numbers) were identified. The idea came to James Cockle beginning from the observation made by William G. Horner on the existence of irrational equations, called "*congeneric surd equations*", which admit neither real solutions nor complex solutions. Beginning from the equation $\sqrt{j}+1=0$, Cockle introduced a new imaginary unity j, which satisfies $j^2=1$, and taking inspiration from Hamilton's theory of quaternions wrote the generic tessarine as $w=a+bi+cj+ijd$, where $j^2=1$. He then immediately hypothesized a generalization of it, introducing *octrines* at the end of his work and conjecturing generalization to sixteen unities. In a series of subsequent articles he analyzed the algebraic properties of tessarines, noticing the existence of zero divisors, and compared these properties with those of other systems of hypercomplex numbers known at the time, like *Quaternions*, *Octonions* etc., also in reference to the symbolic algebra introduced by Peacock. In 1892 Corrado Segre, in his complex geometry studies, and in the general context of the study of the algebras of hypercomplex numbers by Karl Weierstrass (1815–1897) in 1884, rediscovered the algebra of bicomplex numbers. Segre presented bicomplex numbers as the analytical representation of the points of bicomplex geometry and recognized that Hamilton himself in the study of biquaternions had introduced the same quantities; but neither Hamilton before, nor Segre after, noticed Cockle's works on the subject. Segre studied the algebra of bicomplex numbers, also determining a decomposition of them, which is a particular case of Peirce's.

In this paper, we will analyze the historical pathway of the emergence of tessarines or bicomplex numbers, from their origin as "imaginary" solutions of irrational equations, to their insertion in the context of study of the algebras of hypercomplex numbers.

2. "Congeneric surd equations"

The theory of irrational equations, referred to as "congenerics", was worked out by William George Horner (1786–1837) in 1836 and came into being in the context of the study of irrational equations, when it was observed that irrational equations exist that have neither real solutions nor complex solutions. W.G. Horner[8] was a British mathematician. His only significant contribution to mathematics lies in the method of solving algebraic equations which still bears his name.[9] The considerations on the subject of irrational

[2] For an analysis of the rise of symbolic algebra in Great Britain see for example [Pycior, 1981, 1982], [Fisch, 1999], [Lambert, 2013], [Katz and Parshall, 2014].

[3] [Cayley, 1845], [Graves, 1845].

[4] [Kirkman, 1848].

[5] [Clifford, 1873].

[6] [Clifford, 1878].

[7] [Peirce, 1881].

[8] For biographical information see http://www-history.mcs.st-and.ac.uk/Biographies/Horner.html and *The Oxford Dictionary of National Biography* http://www.oxforddnb.com/view/article/13804.

[9] [Horner, 1819].

equations that W.G. Horner wrote in a letter to Thomas Stephens Davies (1794?–1851),[10] were published by the latter. In the introduction to the work Davies wrote:

> "*If those mathematicians who have met with a quadratic equation*[11] *whose 'roots' either under a real or imaginary form could not be exhibited will recall to memory the surprise with which they viewed the circumstance, and the attempts which they made to solve the mystery, they will read with no ordinary gratification the following discussion of the general question of which this forms a part. The general theory of such equations, very happily named by Mr. Horner 'Congeneric Equations' is here laid down with great clearness, and, so far as I know, for the first time,– as it is, indeed, nearly the first time the formation of any general and philosophic views respecting them has been attempted* [...]."
>
> [T.S. Davies in Horner, 1836, p. 43]

From the considerations by T.S. Davies it emerges that he felt the need to go deeper into the matter, in that it had not yet received the necessary attention and the analysis by W.G. Horner was finally seen by Davies as clarifying it.

In this connection, in his article, after some considerations on the need to study the case in which irrational equations do not admit any real root, W.G. Horner, in agreement with Davies, emphasized that the properties of irrational equations have not received sufficient attention from writers on the elements of algebra. He remarked that in solving equations involving radicals it is necessary to put the results to the proof before deciding which of them, if any, could be trusted and that the latter alternative, or the failure of every result, was a rare occurrence in exercise books. He noticed that even classical writers spoke of clearing an equation from radicals, in order to obtain its solution, as a matter of course, this would not in any way affect the conditions and the consequence was, that a habit prevails of talking about equations without any regard to this peculiar case, and therefore in language which when applied to it becomes quite incorrect. [Horner, 1836, pp. 43–44].

Then, he analyzed the reasons why it was not correct to apply, without conditions, the "standard" procedure of simplification of radicals [Horner, 1836, p. 46] and proposed that the possible roots of the given irrational equation should be sought among the solutions of a "group of equations" constituted by the given irrational equation, by the associated "congeneric" irrational equation and by the rational equation obtained by simplifying the radicals of the given equation irrational by raising to power or alternatively as a product of the irrational equation and its "congeneric." Hence for Horner the solution to the equation $a - \sqrt{x} = 0$ is obtained from the solution of the "group of equations":

$$\begin{aligned} &a - \sqrt{x} = 0 \quad \text{or } A \\ &a + \sqrt{x} = A_1 \quad \text{or } 0 \\ &a^2 - x = 0, \end{aligned}$$

obtained from the starting irrational equation, considering all the possible variations in sign of the radical [Horner, 1836, p. 49].

Horner, therefore, wondered what the meaning was of finding a "solution" to the rational equation associated with the irrational equation, and if this solution could be called the "root" of the given irrational equation or not: "*But, is the value of x which effects this, to be called a root of the surd formula? No, it is a root of the rational combination only. – Have irrational equations, then, no roots?*" [Horner, 1836, p. 49]

[10] For biographical information see *The Oxford Dictionary of National Biography* http://www.oxforddnb.com/view/article/7269.

[11] He refers to quadratic irrational equations, which he exemplified as follows: "*For instance*, $2x + \sqrt{x^2 - 7} = 5$, *the 'roots' of which are* 4 *and* $\frac{8}{3}$ *determined by the common process; neither of which substituted in the equation reduces it to zero. These are the roots of its congeneric surd equation* $2x - \sqrt{x^2 - 7} = 5$".

and arrived at the conclusion that in the case of "congeneric" irrational equations, instead of looking for the "roots", it was necessary to look for the "solutions" to the group of equations constituted by the irrational equation, by its congeneric and by the associated rational equation: *"None at all. – What have they, then, in the place of roots? An equitable chance, in common with each formula in the congeneric society, of solution by means of the solution of the stock-equation*." [Horner, 1836, p. 49].

The article concludes with some questions posed by the author, regarding how to set problems on this type of irrational equations and on the degree to attribute to the latter:

> "*But if an equation has no root, nor even a certainty of solutions, in what form can it be intelligibly proposed? A note of interrogation posed either for solution or correction. – To what order can surd equations be assigned? To the fractional order* $\frac{m}{n}$, *when n congeneric formulae produce a rational equation of the mth order?* [...] *Are the chances of solution equal for each individual congener?*"
>
> [Horner, 1836, pp. 49–50]

The importance of Horner's article does not lie in the observations made – though these are interesting, seeing that nobody had treated the matter in any depth until that moment – but rather in having inspired J. Cockle's works on "Tessarines."

3. The "Tessarines" and the systems of quadruple Algebra of James Cockle

Between 1848 and 1850 James Cockle (1819–1895),[12] a British mathematician and a Chief Justice for Queensland (Australia), in a series of works introduced and studied a new system of hypercomplex numbers, which he himself called "*Tessarines*." The idea arose from W. Horner's considerations on "*congeneric*" irrational equations. In this connection, Cockle wrote as follows to T.S. Davies:

> "*You are aware that the new algebraical symbol,*[13] *which I have recently discovered, was suggested to my mind by reflecting on the structure of congeneric surd equations. Did I require any excuse for throwing the present investigations into a form of a letter to you, I should find it in the fact that the remarks of Horner on congenerics were put into a similar shape*."
>
> [Cockle, 1848a, p. 435]

Davies, in the introduction to Cockle's work, which he himself proposed for publication, underlined the central role played by Horner's considerations [Davies in Cockle, 1848a, p. 435].

J. Cockle, within the paradigm of the symbolic algebra introduced by George Peacock and stimulated by the discovery of Hamilton's *Quaternions*, introduced a new "symbol" that he called "impossible quantity", as the solution to an "impossible" irrational equation, and beginning from it introduced the *Tessarines*.

Indeed, Cockle having been trained mathematically in the school of Peacock,[14] wrote:

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[12] For biographical information see *The Oxford Dictionary of National Biography* http://www.oxforddnb.com/view/article/5788, and http://www-history.mcs.st-and.ac.uk/Biographies/Cockle.html.

[13] He refers to the symbol j that $1+\sqrt{j}=0$.

[14] In 1830 Peacock wrote the *Treatise on Algebra* [Peacock, 1830] where the proposal is made to put order in the theory of complex numbers and negative ones, using a strictly logical treatment, of an axiomatic kind, which earned him the nickname "Euclid of algebra." In the *Treatise* Peacock distinguishes two types of algebra: arithmetical and symbolical. Arithmetical algebra is an abstract treatment of arithmetic, in which the operation signs denote the usual arithmetical operations and the letters designate natural numbers. Symbolical algebra is, instead, algebra in which the operation symbols indicate the same operations as in arithmetical algebra, but without taking into account the restrictions under which the operations are valid in arithmetical algebra. As Pycior says [Pycior, 1981]: "*Peacock formulated the symbolical approach to algebra in order to resolve the problem of negative and imaginary numbers. In view of Peacock's standing as one of the earliest British advocates of symbolical algebra, the roots of his algebraic*

"[...] *but to express my opinion that Dr. Peacock's Algebra is the standard work on the philosophy of that science, and that his views form a near approximation to those which will ultimately be received as forming the true basis of symbolical algebra. His view appears to form a basis somewhat analogous to the geometrical one which* MONGE *has taken in his principle of "contingent relations*."

[Cockle, 1848b, p. 365]

he identified in algebraic symbolism the paradigm of the discipline and hence identified in the need to introduce "impossible quantities", the result of the laws of algebra:

"[...] *yet an algebraist of the* SCHOOL OF PEACOCK *sees, in the occurrence of those quantities, an inevitable result of the laws imposed upon his symbols, and, consequently, a fit subject for his researches*."

[Cockle, 1848b, p. 365]

He proceeded as follows. Inspired by Horner's works and therefore by reflections on the fact that some irrational equations exist which can be defined "impossible" since they admit neither real nor imaginary solutions, he believed it was necessary to introduce some quantities that would represent such equations. Hence, after designating as impossible equations which have no root whatever, he defined "impossible quantity" a root of an impossible equation.[15] [Cockle, 1848b, pp. 364–365].

The introduction of a "symbol" that describes the "simplest" element of the "impossible quantity", in analogy with imaginary unity, therefore represented a development of algebra originating from the paradigm of symbolic algebra:

"[...] *But the impossible quantities which we are about to consider, are inevitably forced upon our notice in the contemplation of expressions which ordinary algebra presents to us. Hence the admission of a symbol to denote impossible quantity – should such symbol ever be admitted – into works on algebra, would be an extension of and perhaps an innovation upon the previously existing science, and would, in fact, be a legitimate development of it*."

[J. Cockle, 1848b, p. 365]

Hence Cockle introduced the new "symbol" as the solution to the simplest "impossible" irrational equation, in analogy with positive and negative numbers and imaginary unity. He proceeds as follows: just as the positive, negative and imaginary ("*unreal*") units are respectively the roots of the equations $1-x=0$, $1+x=0$, $1+x^2=0$, so the new impossible unit is determined as the root, "*guided by analogy*", of the equation $0=1+\sqrt{x}$. [Cockle, 1848b, p. 366].

Subsequently, after introducing the simplest "impossible quantity" as the solution to the simplest "congeneric" irrational equation, influenced by Hamilton's works on *Quaternions*,[16] he introduced *Tessarines*, as follows:

"*Let*

$$1+i^2=0$$

then i *is the simplest representation of unreal quantity; and, if*

work are, to a limited extent, the roots of British modern algebra." In this sense Cockle is defined an algebraist of the Peacock School, where operations are only subject to symbolical conditions.

15 Cockle's use of the word "impossible" was at variance with its predominant mathematical usage in Great Britain up to that time. During the eighteenth and early nineteenth centuries, the phrases "impossible numbers" and "impossible quantities" were widely employed to denote what today would be called "imaginary numbers." See, for example, [Rice, 2001].

16 The first article by Hamilton on this subject was [Hamilton, 1844].

$$1+\sqrt{j}=0,$$

then j *is the simplest representation of impossible quantity. It is to be borne in mind that, in the latter equation, the radical is to be considered as essentially affected with the sign* $+$.

Let w, x, y, z *be any real quantities, positive, negative, or zero; also let*

$$w+ix+jy+ijz=t$$

then t *I call a tessarine, and* w, x, y, z *its constituents. The latter term I have adopted from the quaternion theory of Sir W.R. Hamilton*."

[Cockle, 1848a, p. 436]

Again inspired by Hamilton's theory, he introduced the "equations" that describe the *Tessarines*, that is to say the relationships that intervene between imaginary unity and the "impossible quantity", as follows:

"*For convenience, I shall denote the product of* i *and* j *by* k*; I shall assume that* j^2 *is equal to unity. I say assume because, although I have reasons* (*not, however, free from objection*), *for making such a supposition, yet I wish to reserve myself full liberty to modify the assumption, and to discuss ab initio the symbol* j^n *and its symbolic relation to* j. *Then the following system of relations will, together with the second equation given in this letter, furnish us with all the conditions requisite for the formation of a theory. That system is*

$$i^2=k^2=-1 \qquad j^2=1$$
$$ij=k, \qquad jk=i, \qquad ki=-j.\text{''}$$

[Cockle, 1848a, p. 437]

In introducing the relationships between the new symbol j and imaginary unity i, Cockle moves away from considerations on "impossible quantities", through which he had introduced it, and in actual fact generalizes the procedure, to the point that the article ends up hypothesizing the possibility of introducing "*Octrines*" and "beyond":

"*By way of conclusion, I will add that, if we had three independent imaginaries* i, j, k, *we should have to deal, not with tessarines, but with what might be called octrines. These last mentioned expressions would be of the form*

$$w+ix+jy+kz+ijp+ikq+jkr+ijks,$$

where w, x, y, z, p, q, r, s *are any real quantities. These quantities might be termed the constituents of the octrine, which could not vanish unless all its constituents were zero. So, if we had four independent imaginaries* [...] *the corresponding expression would consist of sixteen terms, and would have properties analogous to those of the tessarine and the octrine. I must here break off* [...]"

[Cockle, 1848a, pp. 438–439]

From the latter reference, it emerges that the author had generalized the procedure, probably also influenced by the considerations on Octonions by John Graves and Arthur Cayley.[17]

[17] For an in-depth examination of the birth and development of Octonions see [Baez, 2002], [Cerroni and Vaccaro, 2010], [Cerroni, 2010], [Gray and Parshall, 2007].

After introducing the *Tessarines*, Cockle studied their properties: "[...] *The next step is to ascertain the fundamental properties of the new symbol, its origin and nature being duly considered.*" [Cockle, 1849a, p. 38].

He showed that a *Tessarine* is annulled if its constituents are equal to zero [Cockle, 1848a, p. 436], that the product of two *Tessarines* is a *Tessarine* [Cockle, 1848a, p. 437], [Cockle, 1849b, p. 408], and that, as we say today, there are zero divisors. Cockle defined the latter property "anomalous":

> "*But I am about to point out another anomaly*, [...]: *it is that on the supposition that j^2 is equal to the unity*,
>
> $$(1-j)\times(1+j)=1-1=0;$$
>
> *this is to say, the zero may be considered as the product of two impossible factors, neither of which vanishes*. [...]"
>
> [Cockle, 1849a, p. 42]

and he justified it himself, precisely because of the impossible nature of *Tessarines*. It is to be observed that this was the first time that the possibility of the existence of zero divisors was stressed. This is a property that at first, as observed, appeared anomalous to the author, but subsequently was to be approved by him, faced with the fact that *tessarines* satisfy the other laws of ordinary algebra.

Indeed, Cockle continued to explore "the nature" of *Tessarines* comparing them with Hamilton's *Quaternions*, and identifying the properties that from his point of view made them adhere to the laws of "ordinary" algebra. He observed that while in the case of *Quaternions* the characteristic relationships force us to sacrifice the commutative law, in the case of *Tessarines* this is not the case:

> "*My present object is, however, connected with Mr. Boole's observation on the* '*Laws of Quaternions*' [...] *He has there shown that the quaternion relations* [...] *when considered as of universal application conduct us to the conditions*
>
> $$ji=-k, \qquad kj=-i, \qquad ik=-j,$$
>
> *and thus lead to a sacrifice of the commutative character of multiplication. It becomes therefore a subject of inquiry whether the fundamental equations of my Tessarine System entail upon us the necessity of any such sacrifice. To ascertain this, I shall apply the method of Mr. Boole to the Tessarine conditions*
>
> $$i'^2=-j'^2=k'^2=-1$$
> $$i'j'=k', \qquad j'k'=i', \qquad k'i'=-j'.$$
>
> *Let, then, the subject of operation be $j'y$, and we have*
>
> $$\begin{array}{lrcl} & i'j'j'y &=& k'j'y \\ \textit{or} & i'j'^2y &=& k'j'y \\ \textit{but} & j'^2 &=& 1, \\ \textit{therefore} & i'y &=& k'j'y. \end{array}$$
>
> [...] *and, proceeding thus, we see that the commutative character of multiplication is preserved in the Tessarine Theory.*"
>
> [Cockle, 1849g, p. 124]

From this analysis it emerges that Cockle had realised that the commutative property of multiplication in *Tessarines* is a consequence of the characteristic relationships between imaginary unities and therefore that it is the relationships between imaginary unities that determine the "nature" of systems of hypercomplex numbers. Hence Cockle continued in the analysis of possible relationships between imaginary unities and introduced two new algebras, *Coquaternions* and *Cotessarines:*

> "*If α, β, and γ be three imaginaries of which the respective squares are equal either to positive or negative unity, and which are subject to the relation $\alpha\beta = \gamma$, then these imaginaries afford us four and only four, essentially distinct systems of Quadruple Algebra. The first system in that of* QUATERNIONS; *the second, that of* TESSARINES*; for the third system I propose the name of* COQUATERNIONS*; and for the fourth that of* COTESSARINES."
>
> [Cockle, 1849h, p. 197]

He proceeded as follows. For each case he considered the squares of the imaginary unities and the relationship $\alpha\beta = \gamma$ and multiplying both members of the latter by the imaginary unities obtained the characteristic relationships of the four algebras. In this way he recovered the *Quaternions* and the *Tessarines* and identified two new algebras, *Coquaternions* and *Cotessarines*. He examined the case in which two of the three imaginary unities have squares equal to unity, obtaining the system of the *Co-quaternions*, whose characteristic equations can be expressed by $-a^2 = b^2 = c^2 = 1$ where a, b, and c are the imaginaries, in which $ab = c$ and the commutative character of multiplication is lost. [Cockle, 1849h, p. 198].

And finally, considering the case in which the three imaginary unities have a unitary square, he obtained the system of the *Cotessarines*, defined by the characteristic equations $d^2 = e^2 = f^2 = 1$, with d, e, and f imaginaries of this system and $de = f$ [Cockle, 1849h, p. 198]. In the system of Cotessarines the commutative property of multiplication is valid.

Comparing the four algebras Cockle established which of them are a *normal* system,[18] i.e. those that satisfy the laws of ordinary algebra, and which are a *non-normal* system. *Tessarines*, unlike *Quaternions*, are *normal* and for this reason, according to the author, they are the most adequate to extending the ordinary algebra of couples: "[...] *we see that the Tessarine System is normal*. [...] *This Tessarine System appears to me to be the natural extension of ordinary Double Algebra*."[19] [Cockle, 1849h, p. 198].

In a series of works[20] in which he studied the structure of *Tessarines*, Cockle determined the form of their modulus and its properties. Subsequently he compared the modulus of Tessarines with that of Cotessarines, Quaternions and Coquaternions [Cockle, 1849c, pp. 434–435]. Other works in which there are further observations are [Cockle, 1850a], [Cockle, 1850b]. Cockle's last work on these matters dates from 1852 [Cockle, 1852].

From the previous considerations, it emerges that Cockle and the English context in which he was immersed had a "modern conception" (in the sense that it is close to the concept of abstract algebra of structures, which would be accepted in the twentieth century) of the laws of algebra and their nature and consequently introduced "new" systems of hypercomplex numbers, motivating them both intrinsically with the laws of algebra and with their possible applications to geometry.

18 This nomenclature was introduced by Cockle himself.

19 Double Algebra means the algebra of Complex Numbers.

20 [Cockle, 1849d, 1849e, 1849f].

4. W.R. Hamilton's biquaternions

As long ago as 1850, William Rowan Hamilton (1805–1865)[21] had developed an extension of quaternions, defining the algebra of biquaternions. In that year, he made it the object of a communication to a meeting of the British Association for the Advancement of Science in Edinburgh. All that is extant is the following Report:

> "*the author briefly explained the term which he had been obliged to introduce into this new system; showed the simplicity and the reasons for the leading operations in it; and by a few very simple experiments on the rotation of planes round axes inclined to each other, explained the simple interpretation of some of those results which appeared at first to be inconsistent with the principles of the ordinary analysis*."
>
> [Hamilton, 1852]

In 1853 he wrote "Lectures on Quaternions" [Hamilton, 1853], in which he collected all his results on the theory of quaternions and extended them. In the introduction, he noticed that his researches had inspired many contemporaries, showing that he was aware of developments and connections with his theory, in the context of studies in Great Britain. Hamilton[22] affirmed:

> "*My thanks are due, at this last stage, to the friends who have cheered me throughout by their continued sympathy; to the scientific contemporaries* [note: *In these countries, Messrs Boole, Carmichael, Cayley, Cockle, De Morgan, Donkin, Charles and John Graves, Kirkman, O'Brien, Spottiswoode, Young, and perhaps others: some of whose researches or remarks on subjects connected with quaternions* (*such as the triplets, tessarines, octaves, and pluriquaternions*) *have been elsewhere alluded to, but of which I must regret the impossibility of giving here a fuller account*.[...]] *who have at moments turned aside from their own original researches, to notice, and in some instance to extend, results or speculations of mine*; [...]"
>
> [Hamilton, 1853, Preface, p. 64]

And subsequently, after mentioning some studies by contemporaries in § XCIV of the index of the contents, he writes:

> "[...] *subsequent extension* (*in the same year*) *by J.T. Graves, Esq., to a theorem respecting* ***sums of eight squares****, and to a theory of certain* ***octaves****, involving seven distinct imaginaries; allusion to subsequent publications of Professor De Morgan, and other mathematicians of these countries, in the same general field of research, or at least on analogous subjects, such as* ***triplets****,* ***tessarines****, and* ***pluriquaternions****;* [...]"
>
> [Hamilton, 1853, p. liii]

In the paragraphs mentioned, he also observed that it was not possible for him to go deeper into all those results [Hamilton, 1853, p. 539].

Hence it emerges that Hamilton knew and appreciated the studies by his contemporaries and also Cockle's studies on "*Tessarines*." In going deeper into the algebra of quaternions, and in particular the theory of equations with coefficients in quaternions, in his "Lectures on Quaternions" Hamilton again introduced "*biquaternions*, as imaginary solutions of quadratic equations in quaternions:

> "*We see however, that the imaginary solutions of the proposed equation in quaternions still present themselves under the* GENERAL FORM,

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[21] For biographical information see https://www.britannica.com/biography/William-Rowan-Hamilton.

[22] For further information on the work and friendly relations between the group of scholars quoted by Hamilton see [Crilly, 2006].

$$q = q' + \sqrt{-1}q''$$

> *where q' and q'' are real quaternions, while $\sqrt{-1}$ is still the old and the ordinary imaginary of algebra, and is distinguished from all those other roots of negative unity which are peculiar to the present calculus,* [...] *and IIst, by its being, as a factor, commutative with every other. An expression of this general form is called by me* BIQUATERNIONS."
>
> [Hamilton, 1853, pp. 638–639]

He stressed the importance of considering the imaginary solutions of quadratic equations with coefficients in quaternions, for completeness of the theory, in analogy with the imaginary solutions of equations with real coefficients. Specifically, he affirmed:

> "*The theory of such* ***biquaternions*** *is as necessary and important a complement to the theory of* ***single*** *or* ***real quaternions****, as in algebra the theory of* ***couples*** *[...]. It is admitted that the doctrine of* ***algebraic equations*** *would be entirely incomplete, if their* ***imaginary roots*** *[...] were to be neglected, or kept out of view. And in like manner we may already clearly see, from the foregoing remarks and examples, that no theory of* ***equations in quaternions*** *can be considered as complete, which refuses or neglects to tale into account the* ***biquaternions solutions*** *that may exist, of the form above assigned, in any particular or general inquiry*.[...]."
>
> [Hamilton, 1853, p. 639]

It can hence be observed that Hamilton, in introducing biquaternions, followed a line of thought analogous to that of Cockle in the case of Tessarines. That is to say, new "hypercomplex numbers" are introduced, to complete the theory of equations with coefficients in quaternions.

He also gave some numerical examples in support of the theory. He considered the quadratic equation with coefficients in quaternions

$$q^2 = qi + j$$

and determined its six solutions, two of which are quaternions while four are biquaternions. In particular, they are $q = \frac{1}{2}(i-k) \pm \frac{1}{2}(i+j)$ and $q = \frac{1}{2}i(1 \mp \sqrt{-3}) - k$; $q = \frac{1}{2}(i+k) \pm \frac{1}{2}(1-j)\sqrt{-3}$, where $\sqrt{-3}$ is the imaginary unity [Hamilton, 1853, p. 641].

Hamilton, in studying biquaternions, noticed that they admit zero divisors: "*It must, however, be confessed that such calculations as these with biquaternions*, [...] *are sometimes very delicate, and require great caution, from the following circumstance* [...]. *This circumstance is that the product of two biquaternions may vanish, without either factor separately vanishing. To give a very simple example, the product*

$$(k + \sqrt{-1})(k - \sqrt{-1}) = k^2 + 1 = 0$$

while $k + \sqrt{-1}$ and $k - \sqrt{-1}$ must each be considered as different from zero [...]." [Hamilton, 1853, p. 650], and called biquaternions satisfying such properties "Nullific" or "Nullifiers": "*It seems convenient, therefore, to call biquaternions of this class Nullific or to say that they are Nullifiers* [...]." [Hamilton, 1853, p. 672].

In 1866, in *Elements of Quaternions*, a work published posthumously, Hamilton again returned to the subject, studying quaternions that he denominated "planar", that is to say coplanar with one of the imaginary units, for instance i [Hamilton, 1866, p. 113; p. 240]. He observed that they can be represented in the form $x + iy$, with x and y real, therefore constituting a real sub-algebra isomorphic to the field of complex numbers [Hamilton, 1866, p. 244]. Subsequently, he introduced "*complanar biquaternions*", as solutions of equations with coefficients in planar quaternions, and showed that they can be put in the form $x_1 + hy_1 +$

$i(x_2 + hy_2)$, with x_1, y_1, x_2 and y_2 real and h and i imaginary units [Hamilton, 1866, p. 277], that is in the form of those that C. Segre called "*bicomplex*" numbers, which are in fact Cockle's "*Tessarines*." Although he knew Cockle's works, Hamilton did not notice the analogy between "*complanar biquaternions*" and "*Tessarines*." Instead, Segre noticed that his "*bicomplex numbers*" are the "*complanar biquaternions*", as we will see in the next section. Hence, in Hamilton's formulation, bicomplex numbers constitute a commutative sub-algebra of the algebra of biquaternions.

Thus already at the end of 1860 Cockle's tessarines were essentially forgotten. Except for the brief reference by Hamilton, no one had dealt with the algebra of tessarines. One can find a fleeting mention by Alexander MacFarlane in 1904[23] and then nothing more until the second half of the twentieth century.[24] Faced with this disappearance we can make some historiographical reflections. In effect, the general theory of hypercomplex algebras underwent major development in England (and in general in the Anglo-Saxon countries – one thinks, for instance, of B. Peirce) around personalities like Peacock, Hamilton and Cayley. But this interesting development only contributed in a limited way to the general development of algebra of structures.[25] It throve more in the late nineteenth century thanks to German scholars (K. Weierstrass, R. Dedekind, E. Study, G. Frobenius, down to Emmy Noether). The reason for this, in our opinion, lies in the fact that at that time the position of the English school appears excessively formalistic and only slightly linked to the most advanced areas of research of the day, while the German point of view enters fully into key research issues like the study of groups by Lie.[26] From the beginning of the century the new Anglo-American algebraic school (J.M. Wedderburn, L.E. Dickson, and then A.A. Albert) were to come into full contact with the German one, actually originating the algebra of structures. But in-depth examination of these issues goes beyond the scope of this paper.

A few years later, in 1873[27] it was William Kingdon Clifford that introduced two other types of "biquaternions", again in the form $a + b\omega$, with a and b quaternions, but respectively with unity ω such that $\omega^2 = 0$ or $\omega^2 = +1$.[28] Other developments came in 1878, when Clifford, through a reflection on the comparison between Hamilton's quaternions and the vectorial language of Hermann Günther Grassmann, generalized this theory to those that are known as "*Algebras of Clifford*."[29] In this paper we will not deal with all these works, except as regards the development of studies on bicomplex numbers.[30]

5. C. Segre's bicomplex numbers

In the late 1880s in the Italian mathematical sphere, and more precisely in Turin, interest in linear algebra and geometrical use of complex numbers underwent significant development in the hands of two young mathematicians from Turin, Giuseppe Peano (1858–1932) and Corrado Segre (1863–1924)[31], an Italian mathematician who is remembered today as a major contributor to the early development of algebraic geometry. In 1888 Peano published "Geometric Calculation according to the *Ausdehnungslehre* of H. Grassmann, preceded by operations of deductive logic" [Peano, 1888] in which among other things he

23 [MacFarlane, 1904].

24 Even recent historiographic works, such as [van der Waerden, 1985], or [Gray and Parshall, 2007] though fundamentally devoted to the theory of algebras, do not mention Cockle (nor, actually, does Segre).

25 For an in-depth examination of this topic see for example [Parshall, 1983, 1985], [Corry, 1996].

26 For an in-depth examination in this direction see [Hawkins, 2000, 2013].

27 [Clifford, 1873].

28 A comparison was made between Hamilton's "*biquaternions*" and Clifford's by Arthur Buchheim in 1885 [Buchheim, 1885, p. 293].

29 [Clifford, 1878].

30 For an in-depth examination of the history of geometric calculation see for example [Freguglia, 2004], [Petrunic, 2009], [Cogliati, 2015].

31 For biographical information see http://www-history.mcs.st-and.ac.uk/history/Biographies/Segre_Corrado.html.

gave the first axiomatic definition of real vectorial space; already in 1886 Segre [Segre, 1886] had published a note inspired by the *Beiträge* of K.G. Christian von Staudt [von Staudt, 1856–60] and had started a research programme focused on the geometrical interpretation of complex and hypercomplex numbers.[32] In 1892, continuing this line of thought, Segre published the work "*Real representations of complex forms and hyperalgebraic bodies*" [Segre, 1892] in which he inserted the geometrical interpretation of the algebra of bicomplex numbers, returning after forty years to the interrupted thread of Hamilton's thought:

> "[...] *following that principle of the extension of notions, to which Mathematics owes so much progress, and that in particular for the geometry of algebraic varieties had led from real points to complex points. Now a further extension appears appropriate. Complex points are no longer enough. It is useful to introduce some bicomplex points, that is to say some entities that have for images the complex points of representative forms*."[33]
>
> [Segre, 1892, p. 449]

After dealing at length with the hyperalgebraic entities (complex entities that in a real representation are algebraic), Segre introduced bicomplex points as a natural completion of the complex projective straight line. In the last part of the work he introduced and studied, precisely, a new kind of numbers, which are the analytical representation of bicomplex points, bicomplex numbers:

> "*The introduction of imaginary points in geometry corresponds to the introduction of imaginary numbers* (*coordinates*) *in analysis. What will the further generalization be of the concept of number that will correspond to the extension that we have made of the geometric field by introducing bicomplex points*?"[34]
>
> [Segre, 1892, p. 455]

He proceeded as follows. Let us suppose the complex points of a straight line are represented by the real points of the plane σ, and precisely the point that on the straight line has as its coordinate the complex number $x+iy$ (x real y and $i^2=-1$) has as its image in the plane σ the point of coordinates (x, y). Then to obtain the bicomplex points on the same straight line, one must also consider in the plane σ the complex points (x, y), whose coordinate are $x = x_1 + hx_2$ and $y = y_1 + hy_2$, where x_1, x_2, y_1 and y_2 are real and $h^2=-1$, and therefore consider that the bicomplex point of the straight line has the coordinate $x+iy$, that is to say in the plane

$$x_1 + hx_2 + i(y_1 + hy_2) = x_1 + hx_2 + iy_1 + hiy_2.$$

In this way on the straight line the bicomplex point will have as its coordinates *"bicomplex numbers"* of the type $x_1 + hx_2 + iy_1 + hiy_2$, where x_1, x_2, y_1, y_2 are real and i and h *are* two distinct imaginary units for which $h^2 = i^2 = -1$ (but $h \neq \pm i$) and their product is associative and commutative. The algebra of bicomplex numbers is therefore presented as the commutative algebra of the numbers $x+iy$, where x and y are complex numbers (in the imaginary unity, distinguished from i, denoted as h). Each bicomplex

[32] On this aspect of Segre's work see [Segre, 1889–90, 1890], [Brigaglia, 2013], [Zappulla, 2009].

[33] "[...] *seguendo quel principio dell'ampliamento delle nozioni, a cui la Matematica deve tanti progressi, e che in particolare per la geometria delle varietà algebriche aveva portato dai punti reali ai punti complessi. Ora si presenta opportuna un'ulteriore estensione. Non sono più sufficienti i punti complessi. Conviene introdurre dei punti bicomplessi, cioè degli enti che abbiano per immagini i punti complessi delle forme rappresentative*."

[34] "*L'introduzione dei punti imaginari in geometria corrisponde all'introduzione dei numeri imaginari* (*coordinate*) *in analisi. Quale sarà l'ulteriore generalizzazione del concetto di numero, che corrisponderà all'estensione che abbiamo fatta del campo geometrico introducendo i punti bicomplessi*?"

number is therefore expressed as a linear combination of the units 1, i, h *and* $k = ih$ with $i^2 = h^2 = -1$ and $k^2 = 1$. [Segre, 1892, p. 456].

As mentioned in the previous section, it was Segre himself who realized that bicomplex numbers are equivalent to Hamilton's "*planar biquaternions*": "[...] *but from the need to solve algebraic equations in which the unknowns are quaternions, being led then to consider alongside the quaternions, ordinary or real, whose 4 coefficients* (*scalar*) *are real, the imaginary quaternions or biquaternions in which those coefficients are imaginary, one obtains for the biquaternions of a given plane the representation* (*Elements, no. 257*)[35] $x + iy = x_1 + hx_2 + i(y_1 + hy_2)$, *where h is an imaginary number such that* $h^2 = -1$. *Apart from the distinction between the meanings of versor and number that Hamilton attributes respectively to the two symbols i and h, his biquaternions of a given plane are the same thing as our bicomplex numbers*."[36] [Segre, 1892, p. 457].

Actually, until 1886 Segre [Segre, 1886], was almost certainly not aware of the work done by the English school in the first half of the century. Indeed, in his work on complex numbers prior to that of 1892 he mentions neither Hamilton nor the German school. It would seem, precisely, that after the publication of the above-mentioned paper in 1886, he became aware of the need to extend his knowledge of the algebraic aspects of hypercomplex numbers, carefully analyzing the works of Hamilton, Karl Weierstrass and Richard Dedekind.

Therefore, Segre, like Hamilton, did not realize that *bicomplex numbers* coincide with the *Tessarines* introduced by Cockle in 1848. Segre, moreover, also makes reference [Segre, 1892] to Hermann Hankel (1839–1873), who in 1867[37] had hinted at the subject, referring to Hamilton's biquaternions, and to Rudolf Lipschitz (1832–1903), who in 1886[38] dealt more at length with the subject, referring to bicomplex numbers.

Segre, in particular, inserted the algebra of bicomplex numbers into the general framework of the study of hypercomplex algebras, which can be said to have been given systematic form with the publication, a few years earlier, in 1884, of the letter of Karl Weierstrass to Hermann Schwarz,[39] followed by the works of Schwarz himself[40] and above all of Richard Dedekind.[41] In the eighteen-eighties the subject became of particular interest in German mathematics and, above all thanks to Eduard Study, became an integral part of researches related to the groups of Lie.[42]

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35 This means section 257 of Hamilton's Elements of Quaternions.

36 "[...] *ma dai bisogni della risoluzione delle equazioni algebriche in cui le incognite sono quaternioni, essendo poi condotto a considerare accanto ai quaternioni, ordinari o reali, i cui 4 coefficienti* (*scalari*) *son reali, i quaternioni immaginari o biquaternioni in cui quei coefficienti sono immaginari, ottiene per i biquaternioni di un dato piano la rappresentazione* (*Elements, n. 257*) $x + iy = x_1 + hx_2 + i(y_1 + hy_2)$, *ove h è un numero imaginario tale che* $h^2 = -1$. *A parte la distinzione fra i significati di versore e di numero che Hamilton attribuisce risp. ai due simboli i ed h, i suoi biquaternioni di un dato piano son la stessa cosa che i nostri numeri bicomplessi*."

37 [Hankel, 1867]. Hankel just mentions the structure of bicomplex numbers, but undertakes an extensive and thorough examination of Hamilton's works. Among British authors he knows and cites Kirkman, Graves and above all Peacock, from whom he derives the principle of permanence of formal rules. On Hankel's work there are no lengthy studies. It is possible that it was precisely Hankel's text that attracted Segre's attention to Hamilton's work. See M. Crowe, Biography in *Dictionary of Scientific Biography*, 1970–1990, http://www.encyclopedia.com/doc/1G2-2830901844.html.

38 [Lipschitz, 1886, pp. 125–131]. Lipschitz's treatise, cited by Segre, in addition to the brief reference to bicomplex numbers has, in Segre's eyes, the advantage of setting these new algebras in the more general framework of the theories of Weierstrass and Dedekind. See B. Schoenberg, Biography in *Dictionary of Scientific Biography*, 1970–1990, http://www.encyclopedia.com/doc/1G2-2830902635.html.

39 [Weierstrass, 1884].

40 [Schwarz, 1884].

41 [Dedekind, 1885]. For the history of the theory of algebras in the eighteen-eighties see [Lützen, 2001].

42 For an in-depth examination in this direction see note 23.

Segre, however, operates in a much more clearly geometrical way. As mentioned, his real starting point is Staudt and, if we like, Jean Victor Poncelet.[43] As we have seen, Segre starts from the concept of "bicomplex point" and only later, through a coordinatisation process, reaches the algebraic structure of bicomplex numbers. The procedure is opposite to that of Hamilton, for example, who finds geometrical applications for a structure already completely defined from the algebraic point of view.

Segre's study on this algebra is ample and exhaustive. The first point highlighted is, naturally, the presence of zero divisors, which Segre called "*Nullifics*"[44] taking up the term used by Hamilton [Segre, 1892 p. 458]. He determined that the zero divisors in bicomplex numbers constitute two ideals (in the modern sense), which he called "*infinite sets of nullifics*",[45] respectively generated by $h+i(I_1)$ and by $-h+i(I_2)$. Specifically he wrote: "*Those of the 1st set are those bicomplex numbers that would be annulled if in place of the symbol i we had $-h$: they are the products of any bicomplex numbers* [...] *for the number $h+i$* [...]. *Those of the 2nd set are those bicomplex numbers that would be annulled for $i=h$; that is to say, the products of any bicomplex numbers* [...] *for the number $-h+I$* [...]."[46] [Segre, 1892, p. 459]. He also observed that the product of two non-zero bicomplex numbers will be equal to zero if and only if they respectively belong to the *"two sets of nullifics*",[47] that is to say to the two ideals.

Segre determined the geometrical interpretation of the *"Nullifics"* in the projective geometry of bicomplex numbers. Specifically, he defined among the bicomplex varieties the two sets of *proto-strings*,[48] ("*infinite strings*"[49]) on a (complex) straight line and showed that a *proto-string*[50] will have as the coordinates of the points the bicomplex numbers of a coset module I_1 or I_2 depending on the type of proto-string involved: "*the points of the objective straight line that are on the same proto-string of the 1st set are those that have for coordinates different bicomplex numbers for nullifics of the 2nd set.*"[51] [Segre, 1892, p. 459].

Subsequently, he proceeded to determine the structure of bicomplex algebra. Specifically he determined two decompositions of it. He observed, in particular, that given a bicomplex number $x+iy$, setting $Z=x+hy$, $Z'=x-hy$ and $g=\frac{1-hi}{2}$, $g'=\frac{1+hi}{2}$, one obtains $x+iy=Zg+Z'g'$, with $g\in I_1$ and $g'\in I_2$. He therefore established that a bicomplex number can be decomposed into the sum of two zero divisors, each of the two possible types: "*every bicomplex number $x+iy$ can be decomposed in a clearly determined way into the sum of two nullifics, one (Zg) of the 1st set, the other $(Z'g')$ of the 2nd*"[52] [Segre, 1892, p. 459].

Having established this first result, Segre was able to deduce another, more significant, decomposition. He proceeded as follows; given the bicomplex number $x+iy=x_1+hx_2+iy_1+hiy_2$, setting $X=x_1-y_2$, $Y=x_2+y_1$, $X'=x_1+y_2$, $Y'=-x_2+y_1$, he obtained $x+iy=Xg+Yk+X'g'+Y'k'$, with X, Y, X' and Y' evidently real and $k=hg=\frac{h+i}{2}$, $k'=-hg'=\frac{-h+i}{2}$. From this he deduced, given $g^2=g$, $k^2=-g$,

43 [Poncelet, 1822].

44 "*Nullifici*"

45 "*schiere infinite di nullifici*"

46 "*Quelli della 1° schiera sono quei numeri bicomplessi che si annullerebbero ove in luogo del simbolo i vi si ponesse $-h$: essi sono i prodotti di numeri bicomplessi qualunque* [...] *pel numero $h+i$* [...]. *Quelli della 2° schiera sono quei numeri bicomplessi che s'annullerebbero per $i=h$; ossia i prodotti di numeri bicomplessi qualunque* [...] *pel numero $-h+i$* [...]."

47 *"due schiere*"

48 *protofili*

49 "*infiniti fili*"

50 In Segre's language projective geometry strings ("*fili*") are constituted by complex points, which are preimages, in the correspondence of Argand and Gauss, of curves, surfaces, etc. So in $P_1(C)$, what is denominated as a chain is the preimage of a circumference or a straight line and is therefore a string ("*filo*"). In geometry of bicomplex numbers, "strings", defined similarly, consist of two complex strings ("*fili*"), called proto-strings ("*protofili"*).

51 "*i punti della retta oggettiva che stanno su uno stesso protofilo della 1° schiera sono quelli che hanno per coordinate numeri bicomplessi differenti per nullifici della 2° schiera"*

52 "*ogni numero bicomplesso $x+iy$ si può scomporre in un modo ben determinato nella somma di due nullifici l'uno (Zg) della 1° schiera, l' altro $(Z'g')$ della 2°*"

$kg = gk = k$ (and likewise for g' and k'),[53] in modern terms, that the bicomplex numbers of the form $Xg + Yk(X'g' + Y'k')$ constitute a sub-algebra isomorphic to ordinary complex numbers[54] and therefore the algebra of bicomplex numbers is isomorphic to the direct sum of two copies of the field of complex numbers: "*The nullifics of the 1st set are therefore reduced to complex numbers* (*with real coefficients* X, Y) *with the two unities g, k; and likewise the nullifics of the 2nd set with the two unities* g', k' [...]"[55] [Segre, 1892, p. 460]. Segre immediately determined the geometrical meaning of the real coefficients X, Y and X', Y' of the new representation of the bicomplex number, that is to say: "*the new real coefficients X, Y, X' and Y', which we thus come to put in place of the original ones of the bicomplex number, are nothing but the coefficients of the complex points $X + iY$, $X' + iY$ lying respectively on the two proto-strings that contain the corresponding bicomplex point* [...]"[56] [Segre, 1892, p. 461], on which we will not dwell.

As already observed, moreover, Segre inserted his work in the more general context of the study of hypercomplex numbers systems[57] by Weierstrass [Weierstrass, 1884] and Dedekind [Dedekind, 1885] and noticed that: "*The possibility of the decomposition of bicomplex numbers undertaken by us is a particular case of an analogous decomposition that Weierstrass and Dedekind undertake* [*for hypercomplex numbers* ...]"[58] [Segre, 1892, p. 461].

Subsequently, bearing in mind once more the formulation by Weierstrass, he applied the decomposition found to the determination of the roots of an algebraic equation of degree m to a bicomplex unknown [Segre, 1892, p. 462]. He observed, precisely, that taking into account the aforesaid decomposition, this equation can be written $\sum_{l=0}^{m}(a_l + a_l')(z + z') = 0$, and be reduced, since the product of two "nullifics" of different sets is equal to zero, to $\sum_{l=0}^{m} a_l z^l + \sum_{l=0}^{m} a_l' z'^l = 0$, and therefore to the pair $\sum_{l=0}^{m} a_l z^l = 0$, $\sum_{l=0}^{m} a_l' z'^l = 0$. From this he therefore deduced that (if all the coefficients are not equal to zero) the equation has infinitely many solutions if and only if all its coefficients are zero divisors ("*nullifics*") of the same type Otherwise, denoting as m and m' the degrees of the two polynomials in z and in z', he deduced that the roots are mm' and that if the given equation has degree m and the coefficient of the term of degree m is not a zero divisor ("*nullific*"), the equation has m^2 roots. Finally, Segre concluded this part with the analogous geometric result: "*In this way the preceding results are found analytically, that a straight line intersects a plane curve or complex algebraic surface of order m into m^2 bicomplex points, that two complex algebraic plane curves of orders m* and m_1 *intersect in $m^2 * m_1^2$ bicomplex points, etc.*"[59] [Segre, 1892, p. 463]. Segre concluded the work hypothesizing further extensions, to *tricomplex numbers, s – complex numbers* and consequently *s – hyperalgebraic* varieties, motivating their necessary introduction (paraphrasing Hamilton[60]) as follows: "*These subsequent extensions of the elements* (*geometric and analytical*) *and*

[53] That is to say, g and k (g' and k') satisfy the same relationships as 1 and i.

[54] Remarks that g and k (g' and k') are the unities in the sub-algebra isomorphic to the complex numbers but they are not unities in the bicomplex numbers. In fact, $gg' = 0$ and $kk' = 0$.

[55] "*I nullifiei della* 1ª *schiera sono dunque ridotti a numeri complessi* (*a coefficienti reali X*, *Y*) *con le due unità g, k; e similmente i nullifici della* 2ª *schiera con le due unità g', k'* [...]"

[56] "*i nuovi coefficienti reali X, Y, X', Y', che così veniamo a sostituire a quelli primitivi del numero bicomplesso, non sono altro che i coefficienti dei punti complessi $X + iY$, $X' + iY'$ giacenti risp. sui due protofili che contengono il corrispondente punto bicomplesso* [...]"

[57] For an in-depth examination in this direction see the work cited in note 38.

[58] "*La possibilità della scomposizione dei numeri bicomplessi da noi operata rientra come caso particolare di un'analoga scomposizione che il Weierstrass ed il Dedekind fanno per numeri a quante si vogliano unità* [...]"

[59] "*Pel tal modo si ritrovano analiticamente i risultati precedenti, che una retta taglia una curva piana o superficie algebrica d'ordine m complessa in m^2 punti bicomplessi, che due curve piane algebriche complesse di ordini m e m_1 si tagliano in $m^2 * m_1^2$ punti bicomplessi, ecc.*"

[60] Segre refers to Hamilton before underlining the necessity of these extensions: "*To this, and particularly to the theory of bicomplex numbers* (*tricomplex ones, etc.*) *one can almost entirely apply the considerations that Hamilton made on biquaternions* (*we have already noticed the link between them and those numbers*)" [Segre, 1892, p. 465]. ("*A ciò, e in particolare alla teoria*

of the varieties formed with them appear in this way natural, spontaneous, and at the same time useful, indeed necessary. For older studies on algebraic entities real elements were sufficient: but for deeper studies it was necessary to introduce (*ordinary*) *complex elements. In the conception of these new elements there was certainly some discretion, and real elements could also have been generalized in other ways. The choice that was made was however the most appropriate for the algebraic field* (*in that for it an algebraic equation came to always have as many roots as the degree, etc.*)*. Once this choice is made the introduction of hyperalgebraic varieties, bicomplex elements, and so forth, is not an artifice anymore: it is, as we have said, a necessity*."[61] [Segre, 1892, p. 465].

Probably the biggest lacuna in Segre's work was the absence of any reference to the British school,[62] apart from Hamilton. Segre only refers to B. Peirce and W. Clifford, who are marginally mentioned. Examination of their works would perhaps have made him go deeper into the difference between Hamilton's biquaternions and those of Clifford and notice the fact that his decomposition of bicomplex numbers in linear combination of idempotents is a particular case of that already pointed out in Peirce's 1870 work, published posthumously in 1881.[63]

It is to be stressed that Segre's work was not immediately followed up, even among his direct students. It is to be noticed, however, that in 1898 an ex-alumnus of Segre's, Angelo Ramorino, devoted the last part of a historical work of his precisely to the study of bicomplex numbers.[64] We should also notice a late use of bicomplex numbers (in 1941) by one of Segre's greatest students, Gino Fano (1871–1952), with reference to an elementary geometry problem.[65]

However, within Italian geometry (and above all analysis) the most important development to be noticed, in the 1930s and in the framework of the school of Gaetano Scorza (1876–1939), is the study of the analytical functions of bicomplex variables (and hypercomplex ones in general). Without going further into the matter, in this connection one can consider the works of Giuseppe Scorza Dragoni (1908–1996) and Nicolò Spampinato (1892–1971).[66] This theme has been developed more recently, also in the framework of the study of the analytical functions of two complex variables.[67]

Segre's paper too, though fitting perfectly into the algebraic processes of the late nineteenth century, and going in the direction of more and more general structures, had virtually no influence on the development of algebra of structures. This stems in part from the indifference of the Italian algebraic geometers (whose

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dei numeri bicomplessi (*tricomplessi ecc.*) *si possono applicare quasi completamente le considerazioni che Hamilton faceva sui biquaternioni* (*dei quali abbiam già rilevato il legame con quei numeri*)") See the preceding section for Hamilton's considerations.

61 "*Queste successive estensioni degli elementi* (*geometrici ed analitici*) *e delle varietà formate con essi appaiono per tal modo naturali, spontanee, ed in pari tempo utili, anzi necessarie. Per gli antichi studi degli enti algebrici bastavano gli elementi reali: ma per studi più profondi fu necessaria l'introduzione degli elementi complessi* (*ordinari*)*. Nel concetto di questi nuovi elementi vi era certo dell'arbitrio, e gli elementi reali si sarebbero potuti generalizzare anche in altri modi. La scelta fatta fu però la più opportuna pel campo algebrico* (*in quanto per essa un'equazione algebrica veniva ad avere sempre tante radici quanto è il grado, ecc.*)*. Una volta che tal scelta si è fatta l'introduzione delle varietà iperalgebriche, degli elementi bicomplessi, e così via, non è più un artificio: è, come abbiam detto, una necessità*"

62 In the nineteenth century there were both scientific and friendly relations between some important British mathematicians and the major Italian mathematicians. For example we know of the relations between Betty and Sylvester, Cayley and Cremona, Hirst and Cremona, as we see from the published and unpublished correspondence.

63 [Peirce, 1881].

64 On Angelo Ramorino (1869–?) see the biography in http://www.peano2008.unito.it/scuola/ramorino.pdf. He had been an assistant of D'Ovidio until 1896 and in 1898 became an assistant of Peano. This seems to us one of the many connections between two seemingly opposite schools like those of Segre and of Peano. The work mentioned is [Ramorino, 1898].

65 [Fano, 1941].

66 [Scorza Dragoni, 1934]; [Spampinato, 1935, 1936].

67 For a modern treatment of the matter from the algebraic point of view see, for instance: [Rochon and Shapiro, 2004]; as regards aspects concerning complex analysis see [Luna-Elizarraras et al., 2015]. A general review of the state of the art on hypercomplex algebras can be found in [Olariu, 2002].

acknowledged intellectual leader was Segre) to the language of abstract algebra, so that his proposals were unknown at home first of all. We have to wait, as mentioned, for the period after World War One to find in Gaetano Scorza, largely influenced by Segre himself, acting as an interlocutor in the study of algebraic structures.

6. Conclusions

The analysis of the origins of algebra of "*bicomplex numbers*" confirms that in the middle of the nineteenth century in Great Britain there was great interest in studies concerning new algebraic structures and the meaning of symbolic algebra. These studies then became part of the general line of study of algebras that centred on the American mathematical community (A. Albert, L. Dickson, S. Epsteen, E. Moore, O. Veblen, J.M. Wedderburn[68]) and the German one (R. Brauer, R. Dedekind, G. Frobenius, H. Hasse, E. Noether, E. Study). In comparison to this natural development, an exception in a sense, is Segre, who introduced this subject in his studies on complex geometry, then studying their algebraic properties. In this connection, Segre believed that analysis could not only furnish some tools for geometry but that the opposite could also happen: "*hyperalgebraic entities, as I call them, were found in the analysts and not in the geometers, but vice versa it can happen that with the researches started by me and that I hope will be continued by other geometers, geometry will come in turn to be in the front line and furnish some new results that are useful for analysis. Do you not think so? In-depth synthetic study of an entity often leads to results which the analyst had not reached and that he will only reach after they are thus obtained by the geometer*."[69] [Segre to Klein 1890 in [Luciano and Roero, 2012]].

The case of Cockle's "*Tessarines*", rediscovered and then called "*bicomplex numbers*" by Segre within the algebras of hypercomplex numbers, highlights the fact that algebraists/mathematicians rediscovered a structure introduced in the context of strictly algebraic studies within complex geometry and applied it in the study of the analytical functions of complex variables.

Finally, further supporting what has been highlighted, we notice that the study of bicomplex numbers is today largely used in problems relating to generalizations of Mandelbrot sets and, closely connected to this, in the study of dynamic systems and, in physics, in quantum theory of fields and Solitons. But in relation to all of this we can only give some bibliographical references,[70] limiting ourselves to ascertaining the confirmation of what was foreseen by Segre on the "natural need" to introduce them[71] and in general maintained by Ian Stewart, according to whom the history of mathematics had repeatedly shown that discarding a fine and profound theory just because it had no immediate applications was a very bad move [Stewart, 2008].

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[68] He is from Scotland but he took a position at Princeton University in 1909.

[69] "*gli enti iperalgebrici com'io li chiamo, si trovavano negl'analisti e non nei geometri, viceversa può accadere che colle ricerche da me avviate e che spero vengano proseguite da altri geometri, la geometria venga a trovarsi a sua volta in prima linea e fornire dei risultati nuovi ed utili all'analisi. Non le pare*? *Lo studio sintetico approfondito di un ente conduce spesso a risultati cui l'analista non era giunto e giungerà solo dopo che essi sono così ottenuti dal geometra*"

[70] See for example [Rochon, 2000]; [Matteau and Rochon, 2015].

[71] In this connection see the preceding section.

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