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SIGN-PRESERVING SOLUTIONS FOR A CLASS OF ASYMPTOTICALLY LINEAR SYSTEMS OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We study multiplicity of solutions to an asymptotically linear Dirichlet problem associated with a planar system of second order ordinary differential equations. The existence of two sign-preserving component-wise solutions is guaranteed when the Morse indexes of the linearizations at zero and at infinity do not coincide, and one of the asymptotic problems has zero-index. The proof is developed in the framework of topological and shooting methods and it is based on a detailed analysis and characterization of the phase angles in a two-dimensional setting.

1. Introduction

This paper is devoted to the study of existence of a pair of sign-preserving component-wise solutions to the planar Dirichlet problem

$$(1.1)\qquad \begin{cases} u''(t) + A(t, u(t))u(t) = 0, & t \in [0, \pi], \\ u(0) = u(\pi) = 0, \end{cases}$$

where $A : [0, \pi] \times \mathbb{R}^2 \to GL_s(\mathbb{R}^2)$ is a continuous function satisfying asymptotically linear conditions at the origin and at infinity, and $GL_s(\mathbb{R}^2)$ denotes

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the group of real symmetric matrices of order 2. In particular, there exist two continuous functions $A_i(\cdot) : [0,\pi] \to GL_s(\mathbb{R}^2)$, $i \in \{0,\infty\}$, such that

$$\lim_{|x|\to 0} A(t,x) = A_0(t) \text{ uniformly in } t \in [0,\pi], \tag{1.2}$$

$$\lim_{|x|\to \infty} A(t,x) = A_\infty(t) \text{ uniformly in } t \in [0,\pi]. \tag{1.3}$$

We also assume uniqueness of solutions of Cauchy problems associated with the system

$$u''(t) + A(t,u(t))u(t) = 0. \tag{1.4}$$

Multiplicity of sign-preserving solutions occurs when the nonlinearity crosses the first eigenvalue, and a sign-preserving condition on the non-diagonal entries of the nonlinearity $A(t,x)$ is imposed. As expected, the existence of one positive and one negative solution is guaranteed in the cooperative setting, while the existence of two solutions with opposite-sign components is ensured when the non-diagonal entries are negative. Our result is stated in terms of the Morse indexes. Denoting by $i(A_0)$ and $i(A_\infty)$ the Morse indexes associated with the linearizations of problem (1.4) at zero and infinity, respectively, multiplicity is attained when only one of the indexes is non-zero.

This paper represents a first step in the direction of studying multiplicity of solutions to asymptotically linear planar Dirichlet systems, in absence of a Hamiltonian structure and in absence of symmetric assumptions on the space variable, via phase angles and conjugate points theory. The ultimate goal is considerably more ambitious. Instead of focusing on pairs of sign-preserving solutions, we would hope to be able to determinate at least $2|i(A_0) - i(A_\infty)|$ nontrivial solutions of problem (1.1), under suitable extra assumptions involving the sign of the non-diagonal entries of the matrix $A(t,x)$. Our aim consists also in characterizing the solutions by their nodal properties. This paper should be considered as a first step in this program.

We emphasize that, in the scalar setting, no extra hypotheses beyond the asymptotically linear assumptions (1.2)-(1.3) and a nondegenerancy condition are needed to ensure the existence of $2|i(A_0) - i(A_\infty)|$ solutions for problem (1.1) (cf., among others, [7, 17, 52] and reference therein; see also [48] and [30] dealing with first order planar periodic Hamiltonian systems).

On the contrary, in the higher dimensional situation additional assumptions (such as convexity, symmetry in the space variable, sign conditions on all the entries of the matrix $A(t,x)$, or restriction to an autonomous context) have been imposed in the literature in order to prove that $|i(A_0)-i(A_\infty)|$ represents a lower bound for the number of non-trivial solutions of (1.1). Actually, in the vectorial case, the presence of a gap between the Morse indexes of the asymptotical problems can just ensure the existence of at most two solutions.

In this context, we wish to mention the pioneering works [1] and [12], dealing with periodic solutions of asymptotically linear Hamiltonian systems in $\mathbb{R}^{2N}$. In [1] the authors show the existence of at least one periodic solution assuming that the Maslov-type indexes of the autonomous linearizations at zero and at infinity, are different. The presence of a second periodic solution under a nondegenerancy condition has been proved in [12] in a non autonomous setting. Starting with the innovative papers [1] and [12], a vast literature has arisen. Existence of one or two solutions for asymptotically linear elliptic systems has been extensively investigated. Among the various possible references regarding two-point boundary value problems, we wish to quote the contributions of [25] dealing with second order Hamiltonian systems, [24, 41, 44, 45] related to first order Hamiltonian systems, [18, 27, 28, 46] focusing on planar cooperative gradient partial differential systems, and [15, 34, 36, 38, 43, 55] concentrating on variational planar elliptic systems of PDEs. Very recent contributions can be found in the papers [32, 33], which, according to a suitable choice of the index, are able to guarantee existence of nontrivial solutions to asymptotically linear systems, without imposing the symmetry condition on the asymptotic matrices $A_0(t)$ and $A_\infty(t)$.

Inspired by the classical papers [10] and [20], dealing with suitable classes of elliptic systems, an extensive literature has been developed on the study of *positivity* of solutions for the Dirichlet problem associated with planar second order partial differential systems, under the crucial assumption that the nonlinearity crosses the first eigenvalue. In this direction, we wish to quote [22], [40] and [58] providing the existence of a positive solution for non-variational planar elliptic systems under some additional structure conditions, in the absence of the symmetry assumption on the matrices $A_0(t)$ and $A_\infty(t)$. Note that in [40] a positive and a negative solution have been found simultaneously. Positivity of solutions for a planar system of ODEs has been investigated in [21] under a non-negativity assumption on both the components of the nonlinear vector. Of particular relevance are the papers [56, 57] by Sirakov and [9] by Chang, which prove the existence of a positive solution for elliptic systems of n second-order partial differential equations via first eigenvalues, adopting a fixed-point theorem and sub-supersolutions methods, respectively. It is worth noting that in [56, 57] the main assumptions involve either the first eigenvalues of the scalar differential operators of each equation of the system or the largest and the smallest among the first eigenvalues of the irreducible blocks of the vectorial differential operator of the system in a cooperative setting. We also emphasize that in [9] the nonlinearity is assumed to be quasi-monotonic, and each asymptotic matrix A_i, $i \in \{0, \infty\}$, is cooperative, fully coupled and satisfies a suitable hypothesis which somehow involves the sign of one of its eigenvalues.

As mentioned at the beginning of the Introduction, as many as $|i(A_0)-i(A_\infty)|$ solutions can be attained under suitable additional conditions. The most commonly considered extra-assumptions are symmetric conditions in the space variable on the potential for both first order and second order Hamiltonian systems (as far as two-point boundary value problems are concerned, see, among others, [44, 53, 54] and [23], respectively). In the PDE's setting, symmetric assumptions lead to multiple solutions of Dirichlet problems associated with planar second order systems (cf., among others, [42, 50, 59]).

No symmetry conditions have been required in the works [5, 8, 16, 47], which, by means of topological and shooting methods, provide multiplicity results for asymptotically linear, not necessarily Hamiltonian systems. In particular, the paper [8] adopts the notions of generalized polar coordinates associated with linear systems to get multiple solutions for a class of Dirichlet problems in $\mathbb{R}^N$ satisfying some diagonal and non-emptiness assumptions, while the paper [16] focuses on the planar problem (1.1) and, taking into account the bidimensional structure of the system, obtains multiplicity results when all the entries of the nonlinearity $A(t,x)$ preserve their sign, and both the diagonal entries are negative (which implies that A cannot be positive definite).

Inspired by [8], we use the concepts of generalized polar coordinates and their relation with the Malsov index in order to get multiple solutions to (1.1) in a no diagonal context. To our knowledge, very few studies have explored the feature of generalized polar coordinates and, especially, their role to solve nonlinear boundary value problems for second order elliptic systems. As observed in [4] and recalled in Remark 2.28 below, the generalized phase angles associated with an uncopuled linear system differ from the "natural" angular functions obtained as the angular coordinates of the solutions of each uncoupled equation in the phase plane. In order to gain a more in-depth undestanding of the generalized polar coordinates, we carry out a detailed study of the phase angles associated with the linear problems through basic tools of phase plane analysis. Taking advantage of the bidimensional setting, we determine an explicit expression of the two phase angles in terms of suitable auxiliary functions (cf. Proposition 2.22 and Definitions (2.21)-(2.23) below). This intrinsic characterization of the phase angles allows us to visualize their reciprocal motions in the phase plane and interpret their behaviour. Moreover, taking into account the relation between polar coordinates and the auxiliary functions, we show that the associated linear Dirichlet problem does not admit any solution whose initial data belong to certain regions.

The idea of finding out forbidden regions for initial data in order to solve nonlinear problems of the form (1.1) is due to [16] and [47] (cf. [16, Proposition 2.6 and Lemma 2.7] for its implicit use and [47, assumption (**H**)] for its straight formalization). Once we have detected the forbidden initial data for Dirichlet linear problems, in the spirit of [16] we embed the nonlinear system (1.4) into a two-parameters family of linear equations and we extract the initial data corresponding to the required sign-preserving solution of (1.1) from the continuum of parameters whose associated linear problems have first eigenvalue equal to zero, via Leray-Schauder continuation theorem combined with a shooting type argument.

Our final aim would consist in improving the results achieved in the paper [16] by exploiting the properties of the generalized phase angles, so that multiplicity results for (1.1) could be achieved in absence of negativity assumptions on the diagonal terms of the matrix $A(t,x)$. Although, for now, we have only focused on the crossing of the first eigenvalue (cf. condition (3.3) or (3.5)), we are eager to extend the result to any eigenvalue crossing, obtaining as many solutions as the number of eigenvalues which have been crossed.

Now we are in position to state our main result. Consider a continuous path of symmetric matrices $A : [0,\pi] \times \mathbb{R}^2 \to GL_s(\mathbb{R}^2)$ of the form

$$A(t,x) = \begin{bmatrix} a_{11}(t,x) & a_{12}(t,x) \\ a_{12}(t,x) & a_{22}(t,x) \end{bmatrix}.$$

Theorem 1.1. *Assume that A satisfies* (1.2)-(1.3). *Suppose, moreover, that*

$$\begin{gathered} i(A_0) = \nu(A_0) = 0 \quad \textit{and} \quad i(A_\infty) \geq 1 \\ \textit{or} \\ i(A_\infty) = \nu(A_\infty) = 0 \quad \textit{and} \quad i(A_0) \geq 1. \end{gathered} \tag{1.5}$$

If $a_{12}(t,x) > 0$ for every $(t,x) \in [0,\pi] \times \mathbb{R}^2$, then the Dirichlet problem (1.1) *admits two solutions $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$ satistying*

$$x_1(t) > 0\,,\ y_1(t) > 0, \qquad x_2(t) < 0\,,\ y_2(t) < 0 \qquad \forall t \in (0,\pi).$$

If $a_{12}(t,x) < 0$ for every $(t,x) \in [0,\pi] \times \mathbb{R}^2$, then the Dirichlet problem (1.1) *admits two solutions $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$ satistying*

$$x_1(t) > 0 > y_1(t)\,, \qquad x_2(t) < 0 < y_2(t) \qquad \forall t \in (0,\pi).$$

Note that in the cooperative case, $a_{12}(t,x) > 0$ in $[0,\pi] \times \mathbb{R}^2$, we just require the nonlinearity to cross the first eigenvalue in order to get one positive and one negative solution of the asymptotically linear problem (1.1). No sign assumptions on the diagonal terms $a_{ii}(t,x)$ are needed. Analogously, in the case $a_{12}(t,x) < 0$ in $[0,\pi] \times \mathbb{R}^2$ we can find two solutions whose components have opposite signs without imposing any restiction on the diagonal terms $a_{ii}(t,x)$.

To our knowledge, very few results concerning the existence of sign-preserving solutions are known in the setting $a_{12}(t,x) < 0$. An interesting result in this direction has been obtained by Liu in [40], providing the simultaneous existence of different types of sign-definite solutions for an autonomous planar system of PDEs. The algebraic sign of the solution components of weakly coupled bidimensional linear elliptic systems has been determined in the valuable work [14] by Cosner and Schaefer, by decoupling techniques.

The paper is organized as follows. Section 2 is devoted to the study of the linear Dirichlet problem

$$\begin{cases} u''(t) + B(t)u(t) = 0, & t \in (0,\pi) \\ u(0) = u(\pi) = 0, \end{cases} \tag{1.6}$$

where $B : [0,\pi] \to GL_s(\mathbb{R}^2)$ represents a path of symmetric 2×2 matrices.

The first part of the Section collects the preliminary definitions of Morse index, conjugate points and phase angles, the respective properties and reciprocal relations.

The second part of the Section contains original results which originate from a detailed, specific analysis of the phase angles in the planar context. For any two given linearly independent solutions of the system in (1.6), we define five auxiliary functions a, b, c, d, l (see Definitions (2.21)-(2.25)), from which the explicit expression of the cotangent of the phase angles (2.30) follows. Elementary, but useful properties of these five functions are highlighted in order to illustrate their mutual interactions with the angular rotations. We prove that the phase angles obtained through the expression (2.30) are ordered between them (cf. Proposition 2.27) and turn together closely, chasing each other according to Atkinson's Definition [3], in particular, they do not need to be arranged in increasing order to satisfy (2.11). Taking into account that clockwise half turns of a phase angle correspond to the zeros of $a + c$, we finally show the non-existence of nontrivial solutions u of the Dirichlet problem (1.6) corresponding to eigenfunctions of the first zero eigenvalue when $u'(0)$ lies in the first or third (resp. second or fourth) quadrant and the non-diagonal entries of the matrix $B(t)$ are negative (resp. positive), cf. Proposition 2.30.

In Section 3 we prove our main theorem, by combining the analysis developed in Section 2, with degree theory and shooting techniques.

Notation. By Id_n we mean the $n \times n$ identity matrix, and by A^T we mean the transpose of the matrix A. We denote by $\overset{\circ}{Y}$ the interior of a set $Y \subseteq \mathbb{R}$. Moreover, we set $\overline{\mathcal{Q}}_1 := [0,+\infty) \times [0,+\infty)$ and $\mathcal{Q}_1 := (0,+\infty) \times (0,+\infty)$, representing the first quadrant and its interior part, respectively. Analogous notations are used for the third quadrant $\overline{\mathcal{Q}}_3$ and its interior part $\mathcal{Q}_3$.

2. Linear Problem

The first part of this Section is devoted to review some preliminary definition and well-known results concerning the second order linear problem (1.6).

Let us first recall the definitions of index and of nullity of a path of symmetric matrices. We reformulate Proposition 2.1 proved in [23], according to the version stated in [16].

Proposition 2.1 ([16, 23]). *Given $B \in L^\infty([0,\pi]; GL_s(\mathbb{R}^2))$ there exists a sequence of eigenvalues of B, $\lambda_1(B) \leq \lambda_2(B) \leq \ldots \leq \lambda_j(B) \to +\infty$ as $j \to +\infty$ such that, for each $j \in \mathbb{N}$, there exists a space of dimension one of nontrivial solutions of the problem*

$$\begin{cases} u''(t) + (B(t) + \lambda_j(B) I_2) u(t) \;=\; 0 \\ u(0) = u(\pi) = 0. \end{cases} \tag{2.1}$$

Moreover $H_0^1([0,\pi]; \mathbb{R}^2) := \{u : [0,\pi] \to \mathbb{R}^2 \mid u(\cdot)$ is continuous on $[0,\pi]$, satisfies $u(0) = 0 = u(\pi)$, and $u' \in L^2([0,\pi]; \mathbb{R}^2)\}$ admits a basis of eigenvectors of B.

Definition 2.2. Given $B \in L^\infty([0,\pi]; GL_s(\mathbb{R}^2))$, its index $i(B)$ is defined as the number of negative eigenvalues and its nullity $\nu(B)$ the number of zero eigenvalues.

Observe that in the sequence of the eigenvalues of a matrix B we cannot have the same value repeated more than twice. In the case it is repeated twice, we say that the corresponding eigenvalue $\lambda(B) = \lambda_j(B) = \lambda_{j+1}(B)$, for some $j \in \mathbb{N}$, is double and has a space of eigenvectors of dimension two. Otherwise, we say that the eigenvalue is simple and the corresponding space of eigenvectors has dimension one.

Remark 2.3. We remark that $i(B)$ is the Morse index associated with the linear boundary value problem (1.6), cf. e.g. [23, 26], since it coincides with the sum of the dimensions of the eigenspaces of negative eigenvalues.

Note that each eigenvalue λ_j depends continuously from the path B, due to its variational characterization (cf. [16, 19]).

Proposition 2.4 ([16]). *Fixed $M > 0$, for each $j = 1, \ldots, +\infty$, $B \to \lambda_j(B)$ is continuous in $\{B \in L^1([0,\pi]; GL_s(\mathbb{R}^2)) : \|B(t)\| < M$ for a.e. $t \in (0,\pi)\}$.*

In case the linear system depends continuously on a parameter, there exists a continuous branch of eigenvectors when the eigenspace has dimension one, and some initial data are forbidden for the eigenfunction.

Proposition 2.5 ([16]). *Let $\mathcal{C}$ be a continuum of $\mathbb{R}^2$ and assume that $B : [0,\pi] \times \mathcal{C} \to GL_s(\mathbb{R}^2)$ is continuous. Suppose that zero is an eigenvalue of $B(\cdot, \bar\alpha)$ for each $\bar\alpha \in \mathcal{C}$, and that there exists $(a,b) \in \mathbb{S}^1$ such that the solution* *of* $u'' + B(t,\bar{\alpha})u = 0$ *with* $u(0) = 0$ *and* $u'(0) = (a,b)$ *does not vanish at* $t = \pi$. *Then, it is possible to define a continuous function* $\mathcal{V} : \mathcal{C} \to (C^1([0,\pi],\mathbb{R}^2))^2$ *such that* $\mathcal{V}(\bar{\alpha}) = (v_{\bar{\alpha}}(\cdot), v'_{\bar{\alpha}}(\cdot))$, *where* $v_{\bar{\alpha}}$ *is an eigenfunction of* $B(\cdot,\bar{\alpha})$ *associated with the zero eigenvalue for every* $\bar{\alpha} \in \mathcal{C}$.

To portray the Morse index in geometrical terms, we need to introduce the notion of conjugate point. Consider the second-order linear system

$$u''(t) + B(t)\,u(t) = 0, \qquad t \in (0,\pi), \tag{2.2}$$

where $B : [0,\pi] \to GL_s(\mathbb{R}^2)$ represents a path of symmetric 2×2 matrices.

DEFINITION 2.6. A point $t_0 \in [0,\pi]$ is conjugate to 0 for (2.2) with multiplicity ν_{t_0} if the Dirichlet problem

$$\begin{cases} u''(t) + B(t)\,u(t) = 0, & t \in [0,t_0], \\ u(0) = u(t_0) = 0 \end{cases} \tag{2.3}$$

admits ν_{t_0} linearly independent solutions.

REMARK 2.7. We remark that a conjugate point t_0 with respect to 0 can be equivalently called "moment of verticality" (see e.g. [8]) or "crossing instant" (see e.g. [6]).

It is easy to check that conjugate points to 0 are isolated, hence finite in number on any bounded interval. We are now ready to provide a geometric interpretation of the Morse index.

THEOREM 2.8 ([26], THEOREM 6; [31], THEOREM 8.2). *The Morse index* $i(B)$ *is equal to the number of conjugate points to* 0 *for* (2.2) *in the interval* $(0,\pi)$, *each counted with multiplicity:*

$$i(B) = \sum_{0<t_0<\pi} \nu_{t_0}, \tag{2.4}$$

where the summation runs over all conjugate points t_0.

REMARK 2.9. The Maslov index is a semi-integer homotopy invariant with fixed endpoints of paths l of Lagrangian subspaces of a standard symplectic vector space $(\mathbb{R}^{2N},\omega)$ which gives an algebraic count of nontransverse intersections of the family $\{l(t)\}_{t\in[a,b]}$ with a given Lagrangian subspace l_0 (we refer to [51] for a detailed definition). In particular, the Maslov index $\mu(B)$ associated with the linear boundary value problem (1.6) counts the nontransverse intersections in $[0,\pi]$ of the symplectic path $\Psi(\cdot)$ given by the fundamental matrix corresponding to (1.6) with the "vertical Lagrangian" $l_0 := \{0\} \times \mathbb{R}^2$.

It is possibile to establish a relation between the Masolv index $\mu(B)$ and number of conjugate points to 0 (see, e.g., [2, 4]). Let us emphasize that the difference between Morse and Maslov indexes consists in the fact that the Maslov index $\mu(B)$ computes the conjugate points in the closed interval $[0,\pi]$, while the Morse index $i(B)$ computes the conjugate points in the open interval $(0,\pi)$.

We conclude this note by mentioning the papers [6, 49] which define the Maslov index $m(B)$ associated with (1.6) as the semi-integer homotopy invariant counting the nontransverse intersections between $\Psi(\cdot)$ and l_0 in $[\varepsilon,\pi]$, where $\varepsilon>0$ is chosen in such a way that there are no conjugate instants in $[0,\varepsilon]$. According to this definition, in the non-degenerate case in which the nullity of B vanishes (i.e. $\nu(B)=0$), the Morse and Maslov indexes coincide: $m(B)=i(B)$, see [6, Remark 3.7].

A phase angle analysis.
This paragraph presents the notion of phase angles and illustrates their relation with the Morse index. Our basic references are [3, 8, 31]. We wish to underline the contribution of the paper [39], which has introduced in the literature the concept of generalized polar coordinates for linear systems (cf. also [29, 35]) and of the already mentioned work [3], which has proposed the definition of angular coordinate we are going to adopt.

We now concentrate on the second-order linear system (2.2). Setting

$$v(t):=u'(t), \qquad z(t):=(u(t),v(t)),$$

we can rewrite system (2.2) into the following form

$$Jz'=S_B(t)z, \tag{2.5}$$

where

$$J=\begin{pmatrix}0 & -\mathrm{Id}_2\\ \mathrm{Id}_2 & 0\end{pmatrix} \qquad \text{and} \qquad S_B(t)=\begin{pmatrix}B(t) & 0\\ 0 & \mathrm{Id}_2\end{pmatrix}.$$

Consider two linearly independent solutions $z_1(t):=(u_1(t),v_1(t))$ and $z_2(t):=(u_2(t),v_2(t))$ of (2.5) satisfying the initial condition

$$u_1(0)=0=u_2(0). \tag{2.6}$$

For each $j\in\{1,2\}$, the components of z_j can be explicited by the following expression:

$$u_j(t)=(x_j(t),y_j(t)), \qquad v_j=(x_j'(t),y_j'(t)). \tag{2.7}$$

We now define the two 2×2 matrices

$$X(t)=\begin{pmatrix}x_1(t) & x_2(t)\\ y_1(t) & y_2(t)\end{pmatrix} \qquad \text{and} \qquad X'(t)=\begin{pmatrix}x_1'(t) & x_2'(t)\\ y_1'(t) & y_2'(t)\end{pmatrix}. \tag{2.8}$$

Since $X'(t)-iX(t)$ is invertible, we can set

$$\Theta(t):=\big(X'(t)+iX(t)\big)\big(X'(t)-iX(t)\big)^{-1}. \tag{2.9}$$

Note that $\Theta(0) = \mathrm{Id}_2$ as an immediate consequence of the initial condition (2.6)-(2.7). It is also well-known that $\Theta(t)$ is a unitary symmetric 2×2 matrix, whose spectrum is given by $\mathfrak{s}(\Theta(t)) = \{\tilde{\lambda}_1(t), \tilde{\lambda}_2(t)\}$. There exists a unique continuous map $\vartheta_j : [0, \pi] \to \mathbb{R}$ such that for every $t \in [0, \pi]$ and $j \in \{1, 2\}$

$$\tilde{\lambda}_j(t) = e^{2i\vartheta_j(t)} \qquad \text{and} \qquad \vartheta_j(0) = 0. \tag{2.10}$$

The choice of arranging continuously $\vartheta_1(t)$ and $\vartheta_2(t)$ in increasing order leads to a rearrangement of the eigenvalues $\tilde{\lambda}_1(t)$ and $\tilde{\lambda}_2(t)$, from which the definition of angular coordinates follows.

DEFINITION 2.10 ([3]). We term phase angles of the system (2.2) the unique two continuous functions $\theta_j \colon [0, \pi] \to \mathbb{R}$ satisfying for every $t \in [0, \pi]$ and $j \in \{1, 2\}$

$$\lambda_j(t) = e^{2i\theta_j(t)}, \qquad \theta_j(0) = 0,$$

$$\theta_1(t) \le \theta_2(t) \le \theta_1(t) + \pi, \tag{2.11}$$

where $\mathfrak{s}(\Theta(t)) = \{\lambda_1(t), \lambda_2(t)\}$.

The relation between conjugate points to 0 and phase angles is established by the following proposition:

PROPOSITION 2.11 ([8], PROPOSITION 3.13). *The following facts are equivalent:*

(1) $t_0 \in [0, \pi]$ *is a conjugate point to* 0 *of geometric multiplicity* $\nu \in \{1, 2\}$*;*
(2) $\dim \ker X(t_0) = \nu$*;*
(3) 1 *is an eigenvalue of algebraic multiplicity* ν *for the matrix* $\Theta(t_0)$*;*
(4) *if* $\nu = 2$*, there exist* $h_1, h_2 \in \mathbb{N}$ *such that*

$$\theta_1(t_0) = h_1\pi, \qquad \theta_2(t_0) = h_2\pi,$$

if $\nu = 1$*, there exist* $i \in \{1, 2\}$ *and* $h \in \mathbb{N}$ *such that*

$$\theta_i(t_0) = h\pi \quad \text{and} \quad \theta_l(t_0) \notin \pi\mathbb{N} \;\text{ when } l \neq i,\; l \in \{1, 2\}.$$

For each $t \in [0, \pi]$ and $j \in \{1, 2\}$, we write the phase angles in the following form

$$\theta_j(t) = k_j(t)\pi + \alpha_j(t), \qquad \text{with } \; k_j(t) \in \mathbb{N} \; \text{ and } \; \alpha_j(t) \in (0, \pi]. \tag{2.12}$$

We are finally in position to express the link between the Morse index and the angular coordinates.

PROPOSITION 2.12 ([8, 13, 31]). *The Morse index* $i(B)$ *is given by*

$$i(B) \;=\; k_1(\pi) \;+\; k_2(\pi), \tag{2.13}$$

where the natural numbers k_j *have been defined in formula* (2.12)*.*

REMARK 2.13. Let $\theta_1(\cdot, \lambda_j)$ and $\theta_2(\cdot, \lambda_j)$ be the phase angles of the eigenvalue problem (2.1). Combining (2.11) with Proposition 2.12, we easily see that

$$\theta_2(\pi, \lambda_{2k-1}) = k\pi, \qquad \theta_1(\pi, \lambda_{2k}) = k\pi \qquad \forall\, k \in \mathbb{N}.$$

Thus, if θ_1 and θ_2 are the phase angles of (2.2), it follows that

$$\begin{aligned} \lambda_{2k-1}(B) = 0 \quad &\Longleftrightarrow \quad \theta_2(\pi) = k\pi \\ \lambda_{2k}(B) = 0 \quad &\Longleftrightarrow \quad \theta_1(\pi) = k\pi. \end{aligned}$$

Taking into account (2.8), we introduce the auxiliary 2×2 matrix M, by setting

$$M(t) := X'(t)\, X^{-1}(t). \tag{2.14}$$

We are interested in proving that the eigenvalues of M correspond to the cotangents of the phase angles.

Proposition 2.11 ensures that $M(t)$ is well-defined whenever t is not a conjugate point to 0.

It is well-known that the fundamental solution associated with (2.5)-(2.6) is a Lagrangian plane, i.e.

$$X^{'\,T}(t)\, X(t) = X^T(t)\, X'(t) \qquad \forall\, t \in [0, \pi], \tag{2.15}$$

or, equivalently,

$$-y_1'(t)y_2(t) + y_2'(t)y_1(t) - x_1'(t)x_2(t) + x_2'(t)x_1(t) = 0 \qquad \forall\, t \in [0, \pi]. \tag{2.16}$$

As an immediate consequence of (2.15), we deduce that M is a symmetric matrix:

$$M^T = (X^{-1})^T X^{'\,T} = (X^{-1})^T X^{'\,T} X\, X^{-1} = (X^{-1})^T X^T\, X'\, X^{-1} = M.$$

The eigenvalues of the matrices M and Θ are related to each other.

LEMMA 2.14. *Let η be an eigenvalue of M, then $\tilde{\lambda} := \dfrac{(\eta + i)^2}{|\eta + i|^2}$ is an eigenvalue of Θ.*

PROOF. Let η_1 and η_2 be the eingevalues of the matrix M. There exists an orthogonal matrix U satisfying

$$M = U^{-1} D U, \qquad \text{where} \qquad D = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}.$$

According to the definition (2.9) of Θ, we obtain

$$\Theta = \big(M + i\,\mathrm{Id}\big)\,\big(M - i\,\mathrm{Id}\big)^{-1} = U^{-1}\,(D + i\,\mathrm{Id})(D - i\,\mathrm{Id})^{-1}\, U.$$

Taking into account that

$$(D + i\,\mathrm{Id})(D - i\,\mathrm{Id})^{-1} = \begin{pmatrix} \frac{\eta_1 + i}{\eta_1 - i} & 0 \\ 0 & \frac{\eta_2 + i}{\eta_2 - i} \end{pmatrix} = \begin{pmatrix} \frac{(\eta_1 + i)^2}{|\eta_1 + i|^2} & 0 \\ 0 & \frac{(\eta_2 + i)^2}{|\eta_2 + i|^2} \end{pmatrix},$$

the proof is complete. □

The link between the eigenvalues of M and the phase angles follows by combining Lemma 2.14 with (2.10).

PROPOSITION 2.15. *Let* ϑ_1, ϑ_2 *be the phase angles of* (2.2) *and* η_1, η_2 *be the eigenvalues of* M*, then*

$$\cot \vartheta_1(t) = \eta_1(t), \qquad \cot \vartheta_2(t) = \eta_2(t) \qquad \forall\, t \in [0, \pi]. \tag{2.17}$$

Let us now recall some monotonicity properties of the phase angles.

On the lines of [11], we first observe that the auxiliary matrix M associated with two linearly independent solutions of $u''(t)+B(t)\,u(t) = 0$, $u(0) = 0$ satisfies the Riccati matrix equation:

$$M'(t) = -B(t) - M^2(t).$$

For each $j \in \{1, 2\}$, let η_j be an eigenvalue of M and let $\boldsymbol{v}_j$ be the column matrix of a corresponding unit eigenvector (i.e. $M\boldsymbol{v}_j = \eta_j \boldsymbol{v}_j$, $\|\boldsymbol{v}_j\| = 1$), then

$$\eta_j'(t) = -\mu_j(t) - \eta_j^2(t), \tag{2.18}$$

where

$$\mu_j(t) = \boldsymbol{v}_j^T(t)\, B(t)\, \boldsymbol{v}_j(t). \tag{2.19}$$

Taking into account (2.17), we obtain the corresponding equations for the phase angles.

PROPOSITION 2.16 ([11]). *For each* $j \in \{1, 2\}$*, let* ϑ_j *be a phase angle of* (2.2) *and* $\boldsymbol{v}_j$ *be the eigenvector of* M *corresponding to the eigenvalue* η_j*. Then,*

$$\vartheta_j'(t) = \mu_j(t) \sin^2 \vartheta_j(t) \,+\, \cos^2 \vartheta_j(t), \tag{2.20}$$

where $\mu_j(t)$ *is defined in* (2.19).

The monotonicity of the phase angles in a neighbourhood of a conjugate point to 0 immediately follows.

COROLLARY 2.17 ([11, 31]). *Let* ϑ_j *be a phase angle of* (2.2)*. Then,* $\theta_j(t_0)$ *is strictly increasing if* $\theta_j(t_0)$ *is a multiple of* π*.*

REMARK 2.18. Let ϑ_j be a phase angle of (2.2). Then, $\theta_j(t_0)$ is strictly increasing if the matrix $B(t_0)$ is positive definite.
Note that in [16] the authors obtain multiplicity results for an asymptotically linear problem of the form (1.1) by assuming that one eigenvalue of $A(t, x)$ is negative for every $(t, x) \in [0, \pi] \times \mathbb{R}^2$, which means that A is never positive definite in [16].

COROLLARY 2.19. *If the matrix* $B(t)$ *is negative definite in* $[0,\pi]$*, then* (1.6) *does not admit any solution.*
Analogously, if the path of matrixes $A(t,x)$ *is negative definite for every* $(t,x) \in [0,\pi] \times \mathbb{R}^2$*, then* (1.1) *does not admit any solution.*

Our next aim consists in providing an explicit expression for the cotangents of the phase angles, taking advantage of the planar context.

To this purpose, we introduce some auxiliary continuous functions depending on two linearly independent solutions z_1 and z_2 of (2.5)-(2.6):

$$a_{z_1,z_2}(t) := \ x_1'(t)y_2'(t) + x_1(t)y_2(t) - x_2'(t)y_1'(t) - x_2(t)y_1(t), \tag{2.21}$$

$$c_{z_1,z_2}(t) := -x_1'(t)y_2'(t) + x_1(t)y_2(t) + x_2'(t)y_1'(t) - x_2(t)y_1(t), \tag{2.22}$$

$$b_{z_1,z_2}(t) := \ x_1'(t)y_2(t) + x_1(t)y_2'(t) - x_2'(t)y_1(t) - x_2(t)y_1'(t), \tag{2.23}$$

$$d_{z_1,z_2}(t) := -x_1'(t)y_2(t) + x_1(t)y_2'(t) + x_2'(t)y_1(t) - x_2(t)y_1'(t). \tag{2.24}$$

According to (2.16), we also set

$$l_{z_1,z_2}(t) := x_1(t)x_2'(t) - x_1'(t)x_2(t) = y_1'(t)y_2(t) - y_1(t)y_2'(t). \tag{2.25}$$

We will omit the subscripts when no ambiguity arises. Note that

$$\left(\frac{a+c}{2}\right) = x_1y_2 - x_2y_1 = \det X, \qquad b = \left(\frac{a+c}{2}\right)'. \tag{2.26}$$

The next lemma easily follows.

LEMMA 2.20. *Let* $t_0 \in (0,\pi]$ *be a conjugate point to* 0 *of multiplicity* ν_{t_0}*.*

$$\begin{aligned} \nu_{t_0} = 1 \qquad &\Longleftrightarrow \qquad a(t_0) + c(t_0) = 0, \\ \nu_{t_0} = 2 \qquad &\Longleftrightarrow \qquad a(t_0) + c(t_0) = 0 \quad \textit{and} \quad b(t_0) = 0. \end{aligned}$$

We are now interested in writing the phase angles in function of a, b and c. As a first step, we state the following result.

LEMMA 2.21. *The functions*

$$\eta_1 := \frac{b + \sqrt{d^2 + 4l^2}}{a+c} \qquad \textit{and} \qquad \eta_2 := \frac{b - \sqrt{d^2 + 4l^2}}{a+c}$$

are eigenvalues of M*.*

PROOF. From (2.8) and (2.14), it is immediate to check that

$$M = \frac{2}{a+c}\begin{pmatrix} x_1' & x_2' \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix} = \frac{1}{a+c}\begin{pmatrix} b-d & 2l \\ 2l & b+d \end{pmatrix}, \tag{2.27}$$

whose eingenvalues η satisfy the equation

$$\left(\frac{b}{a+c}-\eta\right)^2-\frac{d^2+4l^2}{(a+c)^2}=0.$$

The thesis easily follows. □

An easy calculation leads to

$$b^2+c^2-a^2=d^2+4(x_1x_2'-x_1'x_2)(y_1'y_2-y_1y_2'),$$

and, consequently,

$$b^2+c^2-a^2=d^2+4l^2\geq 0. \tag{2.28}$$

Thus, the eigenvalues of M might be expressed in terms of the functions a, b, c as follows:

$$\eta_1=\frac{b+\sqrt{b^2+c^2-a^2}}{a+c}\quad\text{and}\quad\eta_2=\frac{b-\sqrt{b^2+c^2-a^2}}{a+c}. \tag{2.29}$$

According to Proposition 2.15, we achieve our goal.

Proposition 2.22. *Let ϑ_1, ϑ_2 be the phase angles of* (2.2) *defined according to* (2.10)*, then*

$$\cot\vartheta_1=\frac{b+\sqrt{b^2+c^2-a^2}}{a+c}\quad\textit{and}\quad\cot\vartheta_2=\frac{b-\sqrt{b^2+c^2-a^2}}{a+c}. \tag{2.30}$$

From (2.27) and (2.28), we easily observe that

$$\det M=\frac{a-c}{a+c}.$$

Expressions (2.30) show us how the sign of the functions b, $a+c$, $a-c$ does affect the rotation of the phase angles.

Remark 2.23. Consider an interval $I\subset(0,\pi]$ such that $b(t)>0$ for every $t\in I$. Due to the positivity of $b+\sqrt{b^2+c^2-a^2}$ in I, we can rewrite the angular coordinates in the following way:

$$\cot\vartheta_1=\frac{b+\sqrt{b^2+c^2-a^2}}{a+c}\quad\text{and}\quad\cot\vartheta_2=\frac{a-c}{b+\sqrt{b^2+c^2-a^2}}. \tag{2.31}$$

In particular, $\vartheta_1(t_0)\notin\frac{\pi}{2}+\pi\mathbb{N}$ and $\vartheta_2(t_0)\notin\pi\mathbb{N}$ whenever $t_0\in I$. This means that I admits at most one conjugate point to 0.

Analogously, consider an interval $J\subset(0,\pi]$ such that $b(t)<0$ for every $t\in J$. Due to the negativity of $b-\sqrt{b^2+c^2-a^2}$ in J, we can rewrite the angular coordinates in the following way:

$$\cot\vartheta_1=\frac{a-c}{b-\sqrt{b^2+c^2-a^2}}\quad\text{and}\quad\cot\vartheta_2=\frac{b-\sqrt{b^2+c^2-a^2}}{a+c}. \tag{2.32}$$

In particular, $\vartheta_1(t_0) \notin \pi\mathbb{N}$ and $\vartheta_2(t_0) \notin \frac{\pi}{2} + \pi\mathbb{N}$ whenever $t_0 \in J$. This means that J admits at most one conjugate point to 0.

We finally note that

$$b(t_0) = 0 \quad \textit{and} \quad a(t_0)+c(t_0) \neq 0 \quad \Longrightarrow \quad \cot\vartheta_1 = \operatorname{sign}(a+c)\sqrt{\frac{c-a}{a+c}} = -\cot\vartheta_2 \text{ in } t_0,$$

which, according to (2.11), implies that $\theta_1(t_0)$ and $\theta_2(t_0)$ lies in two different and consecutive quadrants provided that $c(t_0) \neq a(t_0)$. In particular,

$$\frac{(k-1)\pi}{2} \leq \theta_1 \leq \frac{(k+1)\pi}{2} \quad \Longleftrightarrow \quad \frac{k\pi}{2} \leq \theta_2 \leq \frac{(k+2)\pi}{2}, \quad k \in \mathbb{N}. \tag{2.33}$$

Taking into account Corollary 2.17 and the relation between the Morse index and the phase angles given in (2.13), it is possible to prove the existence of at least $i(B)$ zeros in $(0,\pi)$ of the function b, at which b changes sign.

From (2.31)-(2.32), we immediately see that

$$\exists\, j \in \{1,2\}: \quad \vartheta_j(\tau_0) \in \frac{\pi}{2} + \pi\mathbb{N} \quad \Longleftrightarrow \quad a(\tau_0) - c(\tau_0) = 0,$$

$$\forall\, j \in \{1,2\}: \quad \vartheta_j(\tau_0) \in \frac{\pi}{2} + \pi\mathbb{N} \quad \Longleftrightarrow \quad a(\tau_0) - c(\tau_0) = 0 \quad \text{and} \quad b(\tau_0) = 0.$$

We need some preliminary lemmas to show that the angles ϑ_1 and ϑ_2 defined in (2.30) are ordered, i.e. one of the following two alternatives holds:

$$\vartheta_1(t) \leq \vartheta_2(t) \leq \vartheta_1(t) + \pi \quad \text{or} \quad \vartheta_2(t) \leq \vartheta_1(t) \leq \vartheta_2(t) + \pi \qquad \forall\, t \in [0,\pi]. \tag{2.34}$$

LEMMA 2.24. *It does not exist* $s_* \in [0,\pi]$ *such that* $b(s_*) = a(s_*) = c(s_*) = 0$.

PROOF. Assume, by contradiction, that $b(s_*) = a(s_*) = c(s_*) = 0$, for some $s_* \in [0,\pi]$. From (2.28), we get $d(s_*) = l(s_*) = 0$. According to (2.21)-(2.25), we notice that

$$b + d = 0 \implies x_1 y_2' = x_2 y_1', \qquad b - d = 0 \implies x_1' y_2 = x_2' y_1,$$

$$a + c = 0 \implies x_1 y_2 = x_2 y_1, \qquad a - c = 0 \implies x_1' y_2' = x_2' y_1',$$

$$l = 0 \quad \implies \quad x_1 x_2' = x_2 x_1', \qquad y_1 y_2' = y_2 y_1'.$$

Thus, $u_1 = (x_1, y_1, x_1', y_1')$ is proportional to $u_2 = (x_2, y_2, x_2', y_2')$ in s_* and, consequently, in $[0,\pi]$, which contradicts the linear independence of u_1 and u_2.□

Taking into account that $u_i = (x_i, y_i)$ solves $u''(t) + B(t)\,u(t) = 0$, where $B(t)$ is a symmetric matrix of the form

$$B(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}, \tag{2.35}$$

and recalling the definitions given in (2.21)-(2.25), by a simple computation we get

$$b' = -(\alpha + \gamma)\frac{a+c}{2} + (a-c). \tag{2.36}$$

As immediate consequence, we can state the following result.

LEMMA 2.25. *The zeros of the function* $(a+c)$ *are isolated and they have, at most, multiplicity* 2.

PROOF. Proposition 2.11 and Lemma 2.20 guarantee that the zeros of $(a+c)$ are the conjugate points to 0 for (2.2). They are isolated by Corollary 2.17.

Let s^* be a zero of $(a+c)$. From (2.26) and(2.36), we know that

$$(a+c)'(s^*) = 2b(s^*) \qquad \text{and} \qquad (a+c)''(s^*) = 2b'(s^*) = (a-c)(s^*). \tag{2.37}$$

Thus, according to Lemma 2.24, the first and second derivative of $(a+c)$ cannot simultaneously vanish in s^*. This completes the proof. □

REMARK 2.26. Observe that the Morse index $i(B)$ of (1.6) is given by the sum of the zeros of $(a+c)$ in $(0,\pi)$, counted with their multiplicity.

Combining (2.21)-(2.23) with (2.6)-(2.7), we observe that

$$b(0) = a(0) + c(0) = 0. \tag{2.38}$$

Hence, $b'(0) = a(0) - c(0)$ never vanishes, by Lemma 2.24.

We are finally in a position to prove inequalities (2.34).

PROPOSITION 2.27. *Let* ϑ_1 *and* ϑ_2 *be the phase angles of* (2.2) *defined by* (2.30).
Assume that $b'(0) > 0$, *then* $\vartheta_1(t) \le \vartheta_2(t) \le \vartheta_1(t) + \pi$ *for every* $t \in [0,\pi]$.
Assume that $b'(0) < 0$, *then* $\vartheta_2(t) \le \vartheta_1(t) \le \vartheta_2(t) + \pi$ *for every* $t \in [0,\pi]$.

According to Definition 2.10, we emphasize that $\theta_i \equiv \vartheta_i$ if $b'(0) > 0$, while $\theta_i \equiv \vartheta_j$ with $i \ne j \in \{1,2\}$ if $b'(0) < 0$.

PROOF. We prove the Proposition under the assumption $b'(0) > 0$. Being the complementary case analogous, we will omit the details of the corresponding proof.

The idea of the proof consists in dividing $[0,\pi]$ in subintervals in which the function $(a+c)$ preserve its sign, and estimating the phase angles in each subinterval according to (2.30).

According to Lemma 2.25, we can enumerate the m zeros of $(a+c)$. Adopting the convention that a zero with multiplicity 2 is counted twice, we denote by s_j

the j-th zero of the function $(a+c)$ in $(0,\pi]$. Define the intervals having two consecutive zeros as endpoints, by setting

$$J_1 := (0,s_1], \quad J_2 := [s_1,s_2], \quad \dots\,, \quad J_j := [s_{j-1},s_j], \quad \dots\,, \quad J_{m+1} := [s_m,\pi]. \tag{2.39}$$

If $s_{j-1}=s_j$ is a double zero, then $J_j \equiv \{s_j\}$.

Suppose that $b'(0)>0$. Taking into account that $(a+c)(0)=b(0)=0$ and $(a+c)'=2b$, it follows that

$$(a+c)(t)>0 \qquad \forall\, t \in \mathring{J}_1 = J_1 \setminus \{s_1\}. \tag{2.40}$$

We wish to prove that the sign of $(a+c)$ alternates between positive in the interior part of the even intervals and negative in the interior part of the odd ones, by showing that

$$(a+c)(t)<0 \quad \forall\, t \in \mathring{J}_{2k} \qquad \text{and} \qquad (a+c)(t)>0 \quad \forall\, t \in \mathring{J}_{2k+1}. \tag{2.41}$$

We proceed by induction on n. Suppose that for every $k\in\mathbb{N}$, with $k\le n$,

$$(-1)^k(a+c)(t)<0 \qquad \forall\, t \in \mathring{J}_k. \tag{2.42}$$

We are interested in proving that

$$(-1)^{n+1}(a+c)(t)<0 \qquad \forall\, t \in \mathring{J}_{n+1}. \tag{2.43}$$

The interior part of J_{n+1} is non-empty if one of the following two alternatives occurs: either $s_{n-1}=s_n$, or s_n is a simple zero of $(a+c)$.
If $s_{n-1}=s_n$, then $s_{n-2}<s_{n-1}$, and $(a+c)(s_{n-1})=b(s_{n-1})=(a+c)'(s_{n-1})=0$. Thus, combining Lemma 2.24 with (2.37), we deduce that $(a+c)''(s_{n-1})\neq 0$. In particular, there exists a neighborhood U of s_{n-1} such that $(a+c)(a+c)''>0$ in $U\setminus\{s_{n-1}\}$. Since $(-1)^{n-1}(a+c)(t)<0$ for every $t\in(s_{n-2},s_{n-1})$, we infer that $(-1)^{n-1}(a+c)(t)<0$ for every $t\in(s_{n-1},s_{n+1})=(s_n,s_{n+1})$, which proves (2.43).

Alternatively, if s_n is a simple zero of $(a+c)$, then $s_{n-1}<s_n$, and $2b(s_n)=(a+c)'(s_n)\neq 0$. Since $(-1)^n(a+c)(t)<0$ for every $t\in(s_{n-1},s_n)$, we deduce that $(-1)^n(a+c)'(s_n)>0$. Consequently, $(-1)^n(a+c)(t)>0$ for every $t\in(s_n,s_{n+1})$, which proves (2.43).

This completes the proof of (2.41).

Our final aim consists in demonstrating that

$$\vartheta_1(t)\le\vartheta_2(t)\le\vartheta_1(t)+\pi \qquad \forall\, t\in[0,\pi]. \tag{2.44}$$

Let us proceed by iteration. Let us, first, concentate our attention on J_1. We claim that

$$0<\vartheta_1(t)\le\vartheta_2(t)<\pi<\vartheta_1(t)+\pi \qquad \forall\, t\in\mathring{J}_1. \tag{2.45}$$

Taking into account (2.10) and Corollary 2.17, we begin by observing that

$$\vartheta_j(0) = 0 \qquad \text{and} \qquad \vartheta_j'(0) > 0 \qquad \forall\, j \in \{1, 2\}. \tag{2.46}$$

As an immediate consequence of (2.40), we obtain that

$$\frac{b(t) + \sqrt{b^2(t) + c^2(t) - a^2(t)}}{a(t) + c(t)} \geq \frac{b(t) - \sqrt{b^2(t) + c^2(t) - a^2(t)}}{a(t) + c(t)} \qquad \forall\, t \in \mathring{J}_1. \tag{2.47}$$

Hence, according to (2.30), we conclude that

$$0 < \vartheta_1(t) < \pi, \qquad 0 < \vartheta_2(t) < \pi \quad \text{and} \quad \cot \vartheta_1(t) \geq \cot \vartheta_2(t) \qquad \forall\, t \in \mathring{J}_1,$$

which leads to the required claim (2.45). If $J_1 = (0, \pi]$, the proof of the Proposition is complete, and $i(B) = 0$ by Remark 2.26.

Our ultimate goal consists in proving the following inequalities

$$\begin{gathered} (k-1)\pi < \vartheta_1(t) < k\pi < \vartheta_2(t) < (k+1)\pi, \quad \vartheta_2(t) \leq \vartheta_1(t) + \pi \qquad \forall\, t \in \mathring{J}_{2k}, \\ s_{2k-1} = s_{2k} \qquad \Longrightarrow \qquad \vartheta_1(s_{2k}) = \vartheta_2(s_{2k}) = k\pi; \end{gathered} \tag{2.48}$$

$$\begin{gathered} k\pi < \vartheta_1(t) \leq \vartheta_2(t) < (k+1)\pi < \vartheta_1(t) + \pi \qquad \forall\, t \in \mathring{J}_{2k+1}, \\ s_{2k} = s_{2k+1} \qquad \Longrightarrow \qquad \vartheta_1(s_{2k}) = k\pi \quad \text{and} \quad \vartheta_2(s_{2k}) = (k+1)\pi. \end{gathered} \tag{2.49}$$

Note that (2.49) represents the extension of (2.45) to any odd interval.

Let us follow an inductive approach. We first concentrate on the even intervals. We are interested in deducing (2.48) from the inductive assumption:

$$\begin{gathered} (k-1)\pi < \vartheta_1(t) \leq \vartheta_2(t) < k\pi < \vartheta_1(t) + \pi \qquad \forall\, t \in \mathring{J}_{2k-1}, \\ s_{2k-2} = s_{2k-1} \quad \Longrightarrow \quad \vartheta_1(s_{2k-1}) = (k-1)\pi \quad \text{and} \quad \vartheta_2(s_{2k-1}) = k\pi. \end{gathered} \tag{2.50}$$

If $J_{2k} = \{s_{2k-1}\} = \{s_{2k}\}$, then (2.50) and Corollary 2.17 lead to $\vartheta_1(J_{2k}) = \vartheta_2(J_{2k}) = k\pi$, which verify (2.48).

Otherwise, assume that $\mathring{J}_{2k} \neq \emptyset$. Two alternatives arise: either $s_{2k-2} = s_{2k-1}$ or s_{2k-1} is a simple zero of $(a + c)$.

Let us first observe that in both cases $\vartheta_2(s_{2k-1}) = k\pi$. In particular, if $s_{2k-2} = s_{2k-1}$, from (2.50) we know that $\vartheta_1(s_{2k-1}) = (k-1)\pi$ and $\vartheta_2(s_{2k-1}) = k\pi$.

If s_{2k-1} is a simple zero of $(a + c)$, from (2.30), (2.41), (2.50), and Corollary 2.17, we deduce that $b(s_{2k-1}) < 0$, and $\vartheta_1(s_{2k-1}) < k\pi = \vartheta_2(s_{2k-1})$.

Moreover, in both cases it follows that

$$(k-1)\pi < \vartheta_1(t) < k\pi, \qquad k\pi < \vartheta_2(t) < (k+1)\pi \qquad \forall\, t \in \mathring{J}_{2k}. \tag{2.51}$$

To complete the proof of (2.48), it remains to show that $\vartheta_2 \le \vartheta_1 + \pi$ in $\mathring{J}_{2k}$.

Due the negativity of $(a+c)$ in $\mathring{J}_2$ established by (2.41), we immediately see that

(2.52)
$$\frac{b(t) + \sqrt{b^2(t) + c^2(t) - a^2(t)}}{a(t) + c(t)} \le \frac{b(t) - \sqrt{b^2(t) + c^2(t) - a^2(t)}}{a(t) + c(t)} \quad \forall\, t \in \mathring{J}_{2k}.$$

According to (2.30), we notice that

$$\cot(\vartheta_1(t)) \le \cot(\vartheta_2(t)) = \cot\left(\vartheta_2(t) - \pi\right) \quad \forall\, t \in \mathring{J}_{2k},$$

which combined with (2.51) leads to $\vartheta_1(t) \ge \vartheta_2(t) - \pi$ for every $t \in \mathring{J}_2$. This completes the proof of (2.48).

We now focus on the odd intervals. Our next aim consists in deducing (2.49) from the inductive assumption (2.48).

If $J_{2k+1} = \{s_{2k}\} = \{s_{2k+1}\}$, then by combining Corollary 2.17 with (2.48), we infer that $\vartheta_1(J_{2k+1}) = k\pi$ and $\vartheta_2(J_{2k+1}) = (k+1)\pi$, which satisfy (2.49).

Otherwise, assume that $\mathring{J}_{2k+1} \neq \emptyset$. Two alternatives arise: either $s_{2k} = s_{2k-1}$ or s_{2k} is a simple zero of $(a+c)$.

Let us first observe that in both cases $\vartheta_1(s_{2k}) = k\pi$. In particular, if $s_{2k} = s_{2k-1}$, inductive assumption (2.48) immediately ensures that $\vartheta_1(s_{2k}) = \vartheta_2(s_{2k}) = k\pi$.

If s_{2k} is a simple zero of $(a+c)$, from (2.30), (2.41), (2.48), and Corollary 2.17, we obtain that $b(s_{2k}) > 0$, and $\vartheta_1(s_{2k}) = k\pi < \vartheta_2(s_{2k}) < (k+1)\pi$.

Moreover, in both cases it follows that

$$k\pi < \vartheta_1(t) < (k+1)\pi, \qquad k\pi < \vartheta_2(t) < (k+1)\pi \qquad \forall\, t \in \mathring{J}_{2k+1}. \tag{2.53}$$

Due the positivity of $(a+c)$ in $\mathring{J}_{2k+1}$, we immediately see that (2.47) holds true in the odd interval $\mathring{J}_{2k+1}$. Hence, (2.30) guarantees that

$$\cot\vartheta_1(t) \ge \cot\vartheta_2(t) \quad \forall\, t \in \mathring{J}_{2k+1},$$

which, combined with (2.53), proves (2.49) and completes the proof of the Proposition.

We remark that denoting by $J_{m+1} = [s_m, \pi]$ the last interval with $s_m < \pi$, then $i(B) = m$ by Remark 2.26. □

Remark 2.28. We wish to mention the interesting Remark 2.3 in [4] which shows an example of a planar system of uncoupled second order equations, where the phase angles ϑ_i obtained through Atkinson's construction and satisfying one of the altenative inequalities in (2.34) do not coincide with the "natural" angular functions obtained as the angular coordinates φ_i of the solutions of each uncoupled equation in the phase plane. Actually, the polar angles φ_1 and φ_2 can be obtained by "interchanging" ϑ_1 and ϑ_2 when $b^2 + c^2 - a^2 = 0$.

The last part of the section is devoted to the study of sign-preserving solutions to the Dirichlet problem associated with the linear system (2.2). Let $z_i = (u_i, v_i)$ be two linearly independent solutions of (2.5) satisfying the initial conditions

$$u_i(0) = (x_i(0), y_i(0)) = 0 \quad \text{and} \quad v_i(0) = (x_i'(0), y_i'(0)) := \bar{\alpha}_i \in \mathbb{S}^1. \tag{2.54}$$

Due to the uniqueness of solutions of Cauchy problems associated with the system (2.5), z_1 and z_2 are linearly independent if and only if $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are linearly independent. Therefore, without loss of generality, we may replace the subscripts z_1 and z_2 in the definitions (2.21)-(2.25) by the corresponding initial slopes $\bar{\alpha}_1$ and $\bar{\alpha}_2$.

According to (2.23), it is not restrictive to choose $\bar{\alpha}_1$ and $\bar{\alpha}_2$ such that

$$b'_{\bar{\alpha}_1,\bar{\alpha}_2}(0) > 0. \tag{2.55}$$

As an immediate consequence of Proposition 2.27, $\vartheta_2(t) = \theta_2(t)$ for every $t \in [0, \pi]$, where we refer to (2.30) for the definition of ϑ_2. Taking into account (2.36), (2.38) and (2.21)-(2.22), we easily see that

$$b'_{\bar{\alpha}_1,\bar{\alpha}_2}(0) = (a - c)(0) = 2\left(x_1'(0)y_2'(0) - x_2'(0)y_1'(0)\right).$$

Remark 2.29. The positivity of $b'_{\bar{\alpha}_1,\bar{\alpha}_2}(0)$ establishes a relation between the reciprocal positions of the initial slopes: if we fix $\bar{\alpha}_1$ in $\mathbb{S}^1$, then $\bar{\alpha}_2$ lies in the unitary semicircle which originates in $\bar{\alpha}_1$ and ends at $-\bar{\alpha}_1$ moving counterclockwise.

Recalling that $B(t)$ is a symmetric matrix of the form (2.35), and taking into account (2.25), a simple computation leads to

$$l'_{\bar{\alpha}_1,\bar{\alpha}_2}(t) = x_1(t)x_2''(t) - x_1''(t)x_2(t) = -\beta(t)\,\left(x_1(t)y_2(t) - x_2(t)y_1(t)\right),$$

which, combined with (2.26), implies that

$$l' = -\beta\,\frac{a+c}{2}. \tag{2.56}$$

Observe that l is strictly monotone whenever the non-diagonal entries β of the matrix B preserve their sign and $\theta_2 < \pi$. By (2.54), it is also clear that $l_{\bar{\alpha}_1,\bar{\alpha}_2}(0) = 0$.

The following crucial proposition ensures the non-existence of sign-preserving component-wise solutions to the Dirichlet problem associated with the system (2.2) for initial slopes belonging to suitable regions.

Proposition 2.30. *Consider the problem*

$$\begin{cases} u'' + B(t)\,u = 0, & t \in [0, \pi] \\ u(0) = u(\pi) = 0, \end{cases} \tag{2.57}$$

where $B \in L^\infty([0,\pi]; GL_s(\mathbb{R}^2))$ *is a symmetric matrix of the form* (2.35)*. Assume that*

$$\beta(t) > 0 \quad \forall\, t \in [0,\pi] \qquad (\text{ or } \ \beta(t) < 0 \quad \forall\, t \in [0,\pi] \ ). \tag{2.58}$$

Then, $\nu(B) \le 1$ *and problem* (2.57) *does not admit any eigenfuncion* u *associated with the first zero eigenvalue such that* $u'(0)$ *lies in the second or fourth (resp. first or third) quadrant.*

Moreover, both the components of the solution of (2.57) *corresponding to the first zero eigenvalue preserve their sign.*

Note that an analogous of Proposition 2.30 can be found in [14] (cf. [14, Theorem 2.4 and Theorem 2.7]) under additional decoupling conditions on the matrix $B(t)$. In particular, Cosner and Schaefer in [14] focus on planar linear partial differential systems, where either the diagonal entries of B coincide (i.e. $\alpha = \gamma$ in (2.35)) or the non-diagonal entries of B (which could differ from each other) are multiples of the difference $(\alpha - \gamma)$ between the diagonal terms.

Proof. Let $u = (x_1, y_1)$ be a nontrivial solution of (2.57) with $u'(0) = \bar\alpha_1 \in \mathbb{S}^1$, corresponding to the first eigenfunction associated with the eigenvalue $\lambda_1(B) = 0$. According to Remark 2.13, $\theta_2(\pi) = \pi$, and $\theta_1(\pi) \le \pi$.

Consider an arbitrary $\bar\alpha_2 \in \mathbb{S}^1$ verifying (2.55). Since $i(B) = 0$, there are no conjugate points in $(0,\pi)$, and, consequently, by Lemma 2.20, $a_{\bar\alpha_1,\bar\alpha_2} + c_{\bar\alpha_1,\bar\alpha_2}$ preserves its sign in $(0,\pi)$. Taking into account (2.26) and (2.38), we immediately conclude that

$$a_{\bar\alpha_1,\bar\alpha_2}(t) + c_{\bar\alpha_1,\bar\alpha_2}(t) > 0 \qquad \forall\, t \in (0,\pi). \tag{2.59}$$

• Let us treat the case $\beta(t) > 0$ for every $t \in [0,\pi]$. By (2.56), it follows that

$$l_{\bar\alpha_1,\bar\alpha_2}(t) < 0 \qquad \forall\, t \in (0,\pi]. \tag{2.60}$$

Denote, as usual, $u_2 = (x_2, y_2)$ the solution of (2.2) satisfying $u_2(0) = 0$ and $u_2'(0) = \bar\alpha_2$.

Recalling the definition of l in (2.25) and (2.60), we calculate $l_{\bar\alpha_1,\bar\alpha_2}$ at the time π:

$$l_{\bar\alpha_1,\bar\alpha_2}(\pi) = -x_1'(\pi)\, x_2(\pi) < 0, \qquad l_{\bar\alpha_1,\bar\alpha_2}(\pi) = y_1'(\pi)\, y_2(\pi) < 0. \tag{2.61}$$

Note that the zeros of each component of the solution u are simple at $t = \pi$, and $\nu(B) = 1$. Hence, by combining Lemma 2.20 with (2.26) and (2.59) we conclude that

$$b_{\bar\alpha_1,\bar\alpha_2}(\pi) < 0. \tag{2.62}$$

Moreover, (2.61) implies that

$$y_2(\pi) = -\frac{x_1'(\pi)}{y_1'(\pi)}\, x_2(\pi), \tag{2.63}$$

which means that all the solutions u_2 of (2.2) with $u_2(0) = 0$ whose initial slope $\bar{\alpha}_2$ verifies (2.55) lie on a half-line passing through the origin. Finally, from an easy combination of the previous relations (2.62)-(2.63) with the definition of b (2.23), it follows that

$$-\frac{x_2(\pi)}{y_1'(\pi)}\left((x_1'(\pi))^2 + (y_1'(\pi))^2\right) < 0,$$

which leads to

$$\frac{x_2(\pi)}{y_1'(\pi)} > 0. \tag{2.64}$$

According to (2.61), $y_2(\pi)$ has opposite sign with respect to $x_2(\pi)$, $x_1'(\pi)$ and $y_1'(\pi)$.

Assume now, by contradiction, that $\bar{\alpha}_1$ belongs to the fourth quadrant (the second quadrant case reduces to the fourth quadrant one, due to the linearity of the problem).

We claim that $u(t)$ remains in the fourth quadrant for every $t \in (0, \pi)$.

Suppose, by contradiction, that there exists $t_0 \in [0, \pi)$ such that

$$x_1(t_0) > 0, \quad y_1(t_0) = 0, \quad y_1'(t_0) > 0.$$

From (2.59) and (2.26), we obtain $x_1(t_0)\, y_2(t_0) > 0$, which implies that

$$y_2(t_0) \ > \ 0. \tag{2.65}$$

Furthermore, (2.60) and (2.25) lead to $y_1'(t_0)\, y_2(t_0) \ < \ 0$, whence it follows $y_2(t_0) < 0$, which contradicts (2.65).

Analogously, assume, by contradiction, that there exists $\tau_0 \in [0, \pi)$ such that

$$x_1(\tau_0) = 0, \quad x_1'(\tau_0) < 0, \quad y_1(\tau_0) < 0.$$

By (2.59) and (2.26), we get $-x_2(\tau_0)\, y_1(\tau_0) > 0$, and, consequently,

$$x_2(\tau_0) \ > \ 0. \tag{2.66}$$

Moreover, (2.60) and (2.25) ensures that $-x_1'(\tau_0)\, x_2(\tau_0) < 0$, whence it follows that $x_2(\tau_0) < 0$, contradicting (2.66). This proves the claim.

Therefore, taking into account that $u = (x_1, y_1)$ remains in the fourth quadrant in $(0, \pi)$, and that, according to (2.61), the zeros of each component of the solution u are simple at $t = \pi$, we deduce that

$$x_1'(\pi) < 0 \qquad \text{and} \qquad y_1'(\pi) > 0. \tag{2.67}$$

If we combine (2.67) with (2.61) we get $x_2(\pi) < 0$, whereas if we combine (2.67) with (2.64) we obtain $x_2(\pi) > 0$, an absurd.

We have so proved that problem (2.57) with $\beta > 0$ does not admit any eigenfuncion u associated with the first zero eigenvalue such that $u'(0)$ lies in the fourth or second quadrant.

Let $\bar{\alpha}_2$ be a point of the fourth (resp. second) quadrant which satisfies (2.55). Observe that (2.55) ensures that $\bar{\alpha}_2$ is forbidden to lie in the second (resp. fourth) quadrant if $\bar{\alpha}_1$ lies in the third (resp. first) quadrant. Exactly as before, by a symmetric argument, we see that $u_2(t)$ remains in the fourth (resp. second) quadrant for every $t \in (0, \pi)$.

In particular, if $\bar{\alpha}_1$ lies in the third quadrant and $\bar{\alpha}_2$ in the fourth one, then $x_2(\pi) > 0$ and $y_2(\pi) < 0$, which, according to (2.61), imply that $x_1'(\pi) > 0$ and $y_1'(\pi) > 0$. It turns out that both the components of the solution u of the Dirichlet problem (2.57) are negative in $(0, \pi)$: indeed if, by contradiction, u entered the second or fourth quadrant, it would remain inside, and, consequently, $x_1'(\pi)\, y_1'(\pi) < 0$, an absurd.

Analogously, by the linearity of the problem, if $\bar{\alpha}_1$ lies in the first quadrant and $\bar{\alpha}_2$ in the second one, then both the components of u are positive in $(0, \pi)$.

• We now briefly focus on the case $\beta(t) < 0$ for every $t \in [0, \pi]$, which can be handled in an analogous way. First of all, relations (2.56) and (2.59) guarantee that

$$l_{\bar{\alpha}_1, \bar{\alpha}_2}(t) > 0 \qquad \forall\, t \in (0, \pi], \tag{2.68}$$

whence it follows that

$$-x_1'(\pi)\, x_2(\pi) > 0 \qquad \text{and} \qquad y_1'(\pi)\, y_2(\pi) > 0.$$

In particular, $\nu(B) = 1$, and the zeros of each component of u are simple at $t = \pi$. Moreover, (2.62) holds, which, in turns, implies (2.64). As an easy consequence,

$$x_1'(\pi) \text{ has opposite sign with respect to } x_2(\pi),\ y_2(\pi) \text{ and } y_1'(\pi). \tag{2.69}$$

As before, from the positive sign of $(a + c)$ and l, it is easy to show that once u is in the first or third quadrant, it should remain inside. The same property holds true for u_2.

Proceeding as in the previous case, we can prove that problem (2.57) does not admit any eigenfuncion u associated with the first zero eigenvalue such that $u'(0)$ lies in the first or third quadrant: indeed if, by contradiction, $\bar{\alpha}_1$ belonged to the first or third quadrant, it would follow that $x_1'(\pi)\, y_1'(\pi) > 0$, contradicting (2.69).

Furthermore, notice that if $\bar{\alpha}_1$ lies in the second quadrant and $\bar{\alpha}_2$ in the third one, then $x_2(\pi) < 0$, $y_2(\pi) < 0$, and, consequently, $x_1'(\pi) > 0$ and $y_1'(\pi) < 0$, whence we easily conclude that $x_1(t) > 0$ and $y_1(t) < 0$ for every $t \in (0, \pi)$. This completes the proof. □

Note that the proof of Proposition 2.30 is based on the simple study of the signs of $(a + c)$ and l. It can be adapted to describe the situation at the *first conjugate point* $t_0 \in (0, \pi)$ to 0 for (2.2), illustrating the evolution in $[0, t_0]$ of all the solutions of the Cauchy problems associated with (2.2).

Taking into account the relation (2.4) between Maslov index and conjugate points, we would be interested in extending Proposition 2.30 to the next conjugate points. Careful attention should be devoted to the possible changes of sign of the function l.

3. Main result

We consider the Cauchy problem associated with (1.4)

$$\begin{cases} u''(t) + A(t, u(t))\, u(t) = 0 \\ u(0) = 0 \\ u'(0) = \bar{\alpha} \in \mathbb{R}^2, \end{cases} \tag{3.1}$$

and we denote by $u_{\bar{\alpha}}(\cdot)$ its unique solution. Condition (1.3) guarantees the boundedness of A and, consequently, the continuability of the solutions of the Cauchy problem (3.1).

Inspired by [8, 16, 47], we focus on the linear, parameter-dependent system

$$u''(t) + A(t, u_{\bar{\alpha}}(t))\, u(t) = 0, \tag{3.2}$$

where $\bar{\alpha} \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

The following lemma, adopted in [8, 16, 47] to obtain multiplicity results in an asymptotically linear setting, describes the asymptotic behaviour of the parameter-dependent matrix of (3.2).

Lemma 3.1. (*cf.* [8]) *Suppose that the continuous function* $A : [0, \pi] \times \mathbb{R}^2 \to GL_s(\mathbb{R}^2)$ *satisfies assumptions* (1.2) *and* (1.3)*, then*

$$A(t, u_{\bar{\alpha}}(t)) \to A_\infty(t) \quad \text{in} \quad L^1([0, \pi]) \quad \text{if} \quad |\bar{\alpha}| \to +\infty,$$

$$A(t, u_{\bar{\alpha}}(t)) \to A_0(t) \quad \text{in} \quad L^1([0, \pi]) \quad \text{if} \quad |\bar{\alpha}| \to 0.$$

We are now ready to prove Theorem 1.1. Since the proof is similar in spirit to the one exhibited in [16], some details will be omitted. The interested reader is referred to [16] for a more detailed proof.

Proof of Theorem 1.1 We concentrate on the case $i(A_0) = \nu(A_0) = 0$ and $i(A_\infty) \geq 1$. By the Definition 2.2 of the Morse index and nullity, $\lambda_i(A_0) > 0$ for all $i \in \mathbb{N}$, and there are exactly $i(A_\infty)$ negative eigenvalues $\lambda_j(A_\infty)$ with $j \in \{1, \ldots, i(A_\infty)\}$. In particular,

$$\lambda_1(A_\infty) < 0 < \lambda_1(A_0). \tag{3.3}$$

• Assume first that $a_{12}(t,x) > 0$ for every $(t,x) \in [0,\pi] \times \mathbb{R}^2$.

We focus on the search of solutions of (1.1) whose initial slope belongs to $\mathcal{Q}_1$, the interior part of the first quadrant.

Lemma 3.1, Proposition 2.4 and inequalities (3.3) lead to

$$\lim_{|\bar\alpha|\to+\infty} \lambda_1(A(\cdot,u_{\bar\alpha}(\cdot))) \,<\, 0 \,<\, \lim_{|\bar\alpha|\to 0} \lambda_1(A(\cdot,u_{\bar\alpha}(\cdot))). \tag{3.4}$$

In particular, there exist R_1, R_2 satisfying $0 < R_1 < R_2$ such that $\lambda_1(A(\cdot,u_{\bar\alpha}(\cdot))) < 0$ for every $\alpha \in \overline{\mathcal{Q}}_1$ with $|\bar\alpha| = R_2$ and $\lambda_1(A(\cdot,u_{\bar\alpha}(\cdot))) > 0$ for every $\bar\alpha \in \overline{\mathcal{Q}}_1$ with $|\bar\alpha| = R_1$. By a simple application of the Leray-Schauder continuation theorem [37] (cf. the proof exhibited in [16] for more details), we deduce the existence of a closed connected set $\mathcal{C} \subset \{\bar\alpha \in \overline{\mathcal{Q}}_1 \,:\, R_1 < |\bar\alpha| < R_2\}$ such that $\mathcal{C} \cap (\{0\} \times (R_1,R_2)) \neq \emptyset$, $\mathcal{C} \cap ((R_1,R_2) \times \{0\}) \neq \emptyset$ and

$$\lambda_1(A(\cdot,u_{\bar\alpha}(\cdot))) \,=\, 0 \qquad \forall \bar\alpha \in \mathcal{C}.$$

Proposition 2.30 ensures that there are no eigenfuntions u associated with the eigenvalue $\lambda_1(A(\cdot,u_{\bar\alpha}(\cdot))) \,=\, 0$ such that $u'(0)$ lies in the second or the fourth quadrant. This enables us to employ Proposition 2.5 with $B(t,\bar\alpha) = A(t,u_{\bar\alpha}(t))$. Thus, we infer the existence of a continuous function $\mathcal{V}$ defined on $\mathcal{C}$ such that $\mathcal{V}(\bar\alpha) \,=\, (v_{\bar\alpha}(\cdot), v'_{\bar\alpha}(\cdot))$, where $v_{\bar\alpha}$ is an eigenfunction $A(\cdot,u_{\bar\alpha}(\cdot))$ associated with the zero eigenvalue.

Setting $\sigma(\bar\alpha) := v'_{\bar\alpha}(0) \in \mathcal{Q}_1 \cup \mathcal{Q}_3$, we notice that $v_{\bar\alpha}$ is a nontrivial solution of the system

$$\begin{aligned} &u'' \,+\, A(t,u_{\bar\alpha}(t))\, u \,=\, 0 \\ &u(0) \,=\, u(\pi) \,=\, 0 \end{aligned}$$

with $u'(0) \,=\, \sigma(\bar\alpha)$. By linearity of the problem, we can restrict ourselves to the case $\sigma(\bar\alpha) \in \mathcal{Q}_1$.

Let us write any $\gamma \in \mathbb{R}^2$ in the polar coordinates $(\omega(\gamma), \rho(\gamma))$, given by $\gamma_1 = \rho\cos\omega$, $\gamma_2 = \rho\sin\omega$. Consider the continuous function $g : \mathcal{C} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ defined by $g(\bar\alpha) := \omega(\sigma(\bar\alpha)) - \omega(\bar\alpha)$. As a consequence of the Leray-Schauder theorem, there exist $\tilde\alpha = (0,\tilde\alpha_2)$ and $\hat\alpha = (\hat\alpha_1, 0) \in \mathcal{C}$. Observe that $g(\tilde\alpha) = \omega(\sigma(\tilde\alpha)) - \frac{\pi}{2} < 0$ and $g(\hat\alpha) \,=\, \omega(\sigma(\hat\alpha)) \,>\, 0$. Hence, recalling that $\mathcal{C}$ is a connected set, we infer the existence of $\bar\alpha_1 \in \mathcal{C} \cap \mathcal{Q}_1$ such that $\omega(\sigma(\bar\alpha_1)) = \omega(\bar\alpha_1)$. In particular, there exists $C \,>\, 0$ such that $\sigma(\bar\alpha_1) \,=\, C\,\bar\alpha_1$, from which we get $u_{\bar\alpha_1} \,=\, \frac{u_{\sigma(\bar\alpha_1)}}{C}$ and, consequently, $u_{\bar\alpha_1}(\pi) = 0$.

Notice that $u_{\bar\alpha_1}$ solves the Dirichlet problem (1.1).

Moreover, since $\lambda_1(A(\cdot,u_{\bar\alpha_1}(\cdot))) = 0$, by Proposition 2.30 we conclude that both the components of $u_{\bar\alpha_1}$ are positive in $(0,\pi)$.

With an analogous procedure, we find $\bar\alpha_3 \in \mathcal{C} \cap \mathcal{Q}_3$ such that $\lambda_1(A(\cdot,u_{\bar\alpha_3}(\cdot))) = 0$, and $u_{\bar\alpha_3}(\pi) = 0$. In particular, $u_{\bar\alpha_3}$ is a solution of the Dirichlet problem (1.1), whose components are both negative in $(0,\pi)$. The cooperative case has been demonstrated.

• Analogous arguments apply to the study of the case $a_{12}(t,x) < 0$ for every $(t,x) \in [0,\pi] \times \mathbb{R}^2$. We are able to find two solutions of the Dirichlet problem (1.1), whose initial slopes belong to the second and fourth quadrant, respectively. The components of each solution have opposite sign in $(0,\pi)$.
This completes the proof in the case $i(A_0) = \nu(A_0) = 0$ and $i(A_\infty) \geq 1$.

The case $i(A_\infty) = \nu(A_\infty) = 0$ and $i(A_0) \geq 1$ yields the reverse inequalities

$$\lambda_1(A_0) < 0 < \lambda_1(A_\infty), \tag{3.5}$$

and it can be treated analogously; the details of the proof will be omitted for brevity. □

We believe that our approach might allow extensions to the case of multiple eigenvalue crossings.

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