SOME NOTES ON A SUPERLINEAR SECOND ORDER HAMILTONIAN SYSTEM

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Abstract. Variational methods are used in order to establish the existence and the multiplicity of nontrivial periodic solutions of a second order dynamical system. The main results are obtained when the potential satisfies different superquadratic conditions at infinity. The particular case of equations with a concave-convex nonlinear term is covered.

1. Introduction

For over thirty years many authors have shown great interest in the study the second order dynamical system

$$-\ddot{u} = \nabla F(t,u) \quad \text{a.e. in } [0,T], \tag{1.1}$$

where $T > 0$, $N \geq 1$ is an integer and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a smooth function (a brief history is given in the next section).

Because of the variational structure, direct methods as well as min-max methods of critical point theory have been widely exploited in several papers (see for example [9]-[23], [29], [32], [33], [35]-[55], [57]) with the aim of establishing different existence and multiplicity results of periodic solutions of problem (1.1).

In [32] the existence of one non-constant periodic solution has been proved requiring a very known superquadratic condition on F, namely, there exist $\mu > 2$, $L > 0$ such that for all $|\xi| > L$ and $t \in [0,T]$

$$0 < \mu F(t,\xi) \leq \nabla F(t,\xi) \cdot \xi. \tag{1.2}$$

Moreover, existence results have been investigated in [33, 44] when F, in addition to a suitable coercivity condition, satisfies subquadratic behavior with respect to the second variable, in Rabinowitz's sense, i.e., that there exist $0 < \mu < 2$, $L > 0$ such that for all $|\xi| > L$ and $t \in [0,T]$

$$\nabla F(t,\xi) \cdot \xi \leq \mu F(t,\xi).$$

In the papers [36, 37], non constant periodic solutions of (1.1) have been obtained under the following superquadratic conditions on F

$$\liminf_{|\xi|\to\infty} \frac{H_\mu(t,\xi)}{|\xi|^2} \geq 0,$$

where $H_\mu(t,\xi) = \nabla F(t,\xi) \cdot \xi - \mu F(t,\xi)$, with $\mu > 2$. In particular, the analysis pointed out in these articles allows one to consider different nonlinearities, improving the results of the preceding literature. In [35] the case when, among the others, $\nabla F(t,\xi) \cdot \xi - \mu F(t,\xi) \to +\infty$ as $|\xi| \to \infty$ is considered. More recently, in [29, 38], problem (1.1) has been studied when the right hand side is perturbed by a linear

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term $B(t)u$ where $B(t)$ is a symmetric matrix whose components are integrable functions.

It is interesting to note that in almost all these papers the potential F is required to satisfy some conditions that are strictly related to superquadratic or quadratic growth at zero with respect to the vectorial variable.

In this note, the aim is to consider problem (1.1) when

$$F(t,\xi) = \frac{1}{2}A(t)\xi \cdot \xi - \lambda b(t)G(\xi),$$

where $A : [0,T] \to \mathbb{R}^{N\times N}$ ($N \geq 1$) is a suitable matrix-valued function with components in $L^\infty([0,T],\mathbb{R})$, $b \in L^1([0,T],\mathbb{R})$, $G \in C^1(\mathbb{R}^N)$ and λ is a positive parameter, so that (1.1) reduces to

$$\begin{cases} -\ddot{u} + A(t)u = \lambda b(t)\nabla G(u) \quad \text{a.e. in } [0,T] \\ \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0. \end{cases} \tag{P$_\lambda$}$$

To be precise, first we only require that G satisfies an Ambrosetti-Rabinowitz condition of type (1.2) ; see assumption (4.1) of Theorem 4.1. There we prove the existence of an explicit positive interval of parameters λ for which (P$_\lambda$) admits at least one non trivial solution. Then, adding an algebraic condition on G, see (4.9) in Theorem 4.2, we obtain a second nontrivial solution.
It is relevant to point out that (4.9) is not a local condition at zero. Indeed, it is perfectly compatible both with a quadratic or superquadratic growth near zero. Moreover, in Theorem 6.1, it is emphasized that the subquadracity of G at zero is a sufficient condition for (4.9). Hence, we can obtain results for (P$_\lambda$) when the right hand side is of the form $\lambda|u|^{q-2}u + |u|^{r-2}u$, with $1 < q < 2 < r$ (see Corollary 6.1). A further multiplicity result is obtained in Theorem 6.2 when the Ambrosetti-Rabinowitz condition is replaced by a different assumption involving a suitable behavior at infinity of the function $\nabla G(\xi) \cdot \xi - 2G(\xi)$.

The approach adopted in this paper is variational and it is based on two different critical point theorems due to Bonanno [7] and Bonanno-D'Aguì [8] that have been already fruitfully used by other authors for studying differential problems of very different nature (see, for instance, [13, 15, 16]). For other critical point results and applications we refer for instance to [30, 31, 56].

In the next section we present a brief history of the research connected with the problem. Background material is given in Section 3. We state and prove our main results in Section 4. In Section 5 we compare our results obtaining nontrivial solutions with those obtained with other methods. Examples and further results are discussed in Section 6.

2. History

The periodic non-autonomous problem

$$\ddot{x}(t) = \nabla_x V(t, x(t)), \tag{2.1}$$

has an extensive history in the case of singular systems (cf., e.g., Ambrosetti-Coti Zelati [1]). The first to consider it for nonsingular potentials were Berger and the third author [5] in 1977. They proved the existence of solutions to (2.1) under the condition that

$$V(t,x) \to \infty \quad \text{as} \quad |x| \to \infty$$

uniformly for a.e. $t \in I$. Subsequently, Willem [50], Mawhin [27], Mawhin-Willem [28], Tang [42, 43], Tang-Wu [46, 47], Wu-Tang [51] and others proved existence under various conditions (cf. the references given in these publications).

Most previous work considered the case when $A(t) = 0$. Ding and Girardi [17] considered the case of (5.1) when the potential oscillates in magnitude and sign,

$$-\ddot{x}(t) \;+\; B(t)x(t) \;=\; b(t)\nabla W(x(t)) \tag{2.2}$$

and found conditions for solutions when the matrix $B(t)$ is symmetric and positive definite and the function $W(x)$ grows superquadratically and satisfies a homogeneity condition. Antonacci [3, 4] gave conditions for existence of solutions with stronger constraints on the potential but without the homogeneity condition, and without the negative definite condition on the matrix. Generalizations of the above results are given by Antonacci and Magrone [2], Barletta and Livrea [6], Guo and Xu [22], Li and Zou [26], Faraci and Livrea [20], Bonanno and Livrea [9, 10], Jiang [24, 25], Shilgba [39, 40], Faraci and Iannizzotto [19] and Tang and Xiao [48].

3. Preliminaries

Following the notation of [28], let H^1_T be the Sobolev space of functions $u \in L^2([0,T],\mathbb{R}^N)$ having a weak derivative $\dot{u} \in L^2([0,T],\mathbb{R}^N)$. It is well known that H^1_T, endowed with the norm

$$\|u\|_{H^1_T} := \left(\int_0^T |u(t)|^2\,dt + \int_0^T |\dot{u}(t)|^2\,dt\right)^{1/2}, \tag{3.1}$$

is a Hilbert space, compactly embedded in $C^0([0,T],\mathbb{R}^N)$ and $C^\infty_T \subset H^1_T$.

Let us describe the assumptions on the function A involved in problem (P_λ). Let $A : [0,T] \to \mathbb{R}^{N\times N}$ be a matrix-valued function such that

$(\mathscr{A})_1$ *$A(t) = (a_{ij}(t))$ is a symmetric matrix with $a_{ij} \in L^\infty([0,T])$ for every $t \in [0,T]$.*

$(\mathscr{A})_2$ *There exists a positive constant ν such that*

$$A(t)\xi\cdot\xi \ge \nu|\xi|^2$$

for every $\xi \in \mathbb{R}^N$ and a.e. in $[0,T]$.

Clearly, if $\lambda_{\max}(t)$ denotes the biggest eigenvalue of $A(t)$, one has that $\lambda_{\max} \in L^\infty([0,T])$. Hence, if we put

$$\Lambda = \|\lambda_{\max}\|_\infty \tag{3.2}$$

one has

$$\nu|\xi|^2 \le A(t)\xi\cdot\xi \le \Lambda\,|\xi|^2, \tag{3.3}$$

for every $t \in [0,T]$ and $\xi \in \mathbb{R}^N$. Because of the previous conditions, it is possible to introduce on H^1_T the following inner product

$$\langle u, v\rangle := \int_0^T A(t)u(t)\cdot v(t)\,dt + \int_0^T \dot{u}(t)\cdot\dot{v}(t)\,dt, \tag{3.4}$$

for every $u, v \in H^1_T$. The norm induced by $\langle\cdot,\cdot\rangle$ is

$$\|u\| := \left(\int_0^T A(t)u(t)\cdot u(t)\,dt + \int_0^T |\dot{u}(t)|^2\,dt\right)^{1/2}. \tag{3.5}$$

It is also simple to verify that $\|\cdot\|$ is equivalent to $\|\cdot\|_{H^1_T}$. Indeed, from (3.3) on has

$$\sqrt{m}\|u\|_{H^1_T} \leq \|u\| \leq \sqrt{M}\|u\|_{H^1_T}, \tag{3.6}$$

where $m = \min\{1, \nu\}$ and $M = \max\{1, \Lambda\}$. If we denote by $\bar{k}$ the constant of the embedding $(H^1_T, \|\cdot\|_H) \hookrightarrow (C^0, \|\cdot\|_\infty)$ and put

$$k := \begin{cases} \sqrt{\frac{2}{m}} \max\left\{\sqrt{T}, \frac{1}{\sqrt{T}}\right\} & \text{if } \sqrt{2}-1 \leq T \leq \frac{1}{\sqrt{2}-1}, \\ \sqrt{\frac{T}{m}}\left(1+\frac{1}{T}\right) & \text{otherwise,} \end{cases} \tag{3.7}$$

by a simple computation one can obtain that (see [20] and [28])

$$\bar{k} \leq k. \tag{3.8}$$

Assuming that

($\mathscr{B}$) $b \in L^1([0,T]) \setminus \{0\}$ *is such that* $b \geq 0$ *a.e. in* $[0,T]$

and

($\mathscr{G}$) $G \in C^1(\mathbb{R})$*, with* $G(0) = 0$,

let $\Phi, \Psi : H^1_T \to \mathbb{R}$ be defined as follows

$$\Phi(u) := \frac{1}{2}\|u\|^2, \qquad \Psi(u) := \int_0^T b(t)G(u(t))\, dt, \tag{3.9}$$

for every $u \in H^1_T$. Standard arguments show that Φ and Ψ are continuously Gâteaux differentiable, with

$$\Phi'(u)(v) = \int_0^T \dot{u}(t)\cdot\dot{v}(t)\, dt + \int_0^T A(t)u(t)\cdot v(t)\, dt, \tag{3.10}$$

and

$$\Psi'(u)(v) = \int_0^T b(t)\nabla G(u(t))\cdot v(t)\, dt, \tag{3.11}$$

for every $u, v \in H^1_T$. Moreover, by the Sobolev embedding theorem, Ψ' is a compact operator. It is also useful to point out that estimate (3.8) provides the following property of the sublevels of Φ

$$\Phi^{-1}(]-\infty, r]) \subseteq \left\{u \in C^0([0,T], \mathbb{R}^N) : \ \|u\|_{C^0} \leq k\sqrt{2r}\right\}, \tag{3.12}$$

for every $r \geq 0$.

From the fact that a solution of problem (P_λ) is any function $u_0 \in C^1([0,T], \mathbb{R}^N)$ such that $\dot{u}_0$ is absolutely continuous

$$-\ddot{u}_0(t) + A(t)u_0 = \lambda b(t)\nabla G(u_0) \quad \text{a.e. in } [0,T], \tag{3.13}$$

and

$$u_0(T) - u_0(0) = \dot{u}_0(T) - \dot{u}_0(0) = 0, \tag{3.14}$$

it was shown in [11] that

$$\text{a critical point of the functional } \ I_\lambda := \Phi - \lambda\Psi \text{ is a solution of } (\mathrm{P}_\lambda). \tag{3.15}$$

The existence of multiple solutions will be obtained exploiting the well known Palais-Smale condition that here we recall for the reader convenience.

If X is a Banach space and $I : X \to \mathbb{R}$ is a Gâteaux differentiable function, we say that I satisfies the Palais-Smale condition (briefly (PS)) if

(PS) *every sequence* $\{x_n\}$ *in* X *such that*

$$\{I(x_n)\} \text{ is bounded and } I'(x_n) \to 0 \text{ in } X^*$$

admits a strongly convergent subsequence.

We conclude this section clarifying that the main results of the present note will be proved by applying in a suitable way the following two critical point theorems due to Bonanno [7, Theorem 3.2] and Bonanno-D'Aguì [8, Theorem 2.1]

Theorem 3.1. [7, Theorem 3.2] *Let* X *be a real Banach space and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functions such that* Φ *is bounded from below and* $\Phi(0) = \Psi(0) = 0$. *Fix* $r > 0$ *such that* $\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) < +\infty$ *and assume that for each*

$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$$

the functional $I_\lambda = \Phi - \lambda\Psi$ *satisfies the (PS)-condition and it is unbounded from below. Then, for each*

$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$$

the functional I_λ *admits at least two distinct critical points.*

Theorem 3.2. [8, Theorem 2.1] *Let* X *be a real Banach space and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functions such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there are* $r \in \mathbb{R}$ *and* $\tilde{u} \in X$, *with* $0 < \Phi(\tilde{u}) < r$ *such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$ *the functional* $I_\lambda = \Phi - \lambda\Psi$ *satisfies the (PS)-condition and it is unbounded from below.*
Then, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$ *the functional* I_λ *admits at least two non-zero critical points* $u_{\lambda,1}$, $u_{\lambda,1}$ *such that* $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

Remark 3.1. In Remark 2.1 of [8] it is pointed out that Theorem 3.2 still holds whenever the (PS) is replaced by the so called (C), provided that Φ is coercive.

For the sake of completeness, let us recall that if X is a Banach space and $I : X \to \mathbb{R}$ is a Gâteaux differentiable function, we say that I satisfies the Cerami condition (briefly (C)) if

(C) *every sequence* $\{x_n\}$ *in* X *such that*

$$\{I(x_n)\} \text{ is bounded and } (1 + \|x_n\|)I'(x_n) \to 0 \text{ in } X^*$$

admits a strongly convergent subsequence.

4. Main results

Here is a first existence result.

Theorem 4.1. *Assume that* $(\mathscr{A})_1 - (\mathscr{A})_2$ *and* $(\mathscr{G})$ *hold. Moreover, suppose that there exist* $R > 0$ *and* $\mu > 2$ *such that for every* $|\xi| \geq R$

$$0 < \mu G(\xi) \leq \nabla G(\xi) \cdot \xi. \tag{4.1}$$

Then, for every $b : [0, T] \to \mathbb{R}$ *satisfying* $(\mathscr{B})$ *and for every*

$$\lambda \in \frac{1}{2k^2\|b\|_1} \left]0, \sup_{c>0} \frac{c^2}{\max_{|\xi|\leq c} G(\xi)}\right[$$

problem (P_λ) *admits at least one nontrivial solution.*

Proof. Fix $b : [0, T] \to \mathbb{R}$ satisfying $(\mathscr{B})$ and $\lambda \in \frac{1}{2k^2\|b\|_1} \left]0, \sup_{c>0} \frac{c^2}{\max_{|\xi|\leq c} G(\xi)}\right[$. We wish to apply Theorem 3.1 with $X = H^1_T$ and Φ, Ψ as defined in the previous section 3.

First, we verify that $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition. If $\{u_n\}$ is such that $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n) \to 0$ in X^* one can verify that $\{u_n\}$ is bounded. Indeed, if for every $n \in \mathbb{N}$ we put

$$\mathcal{G}(u_n(t)) = \frac{1}{\mu}\nabla G(u_n(t)) \cdot u_n(t) - G(u_n(t)),$$

$$T_n = \{t \in [0, T] : |u_n(t)| > R\},$$

from assumptions $(\mathscr{G})$ and (4.1) there exist $M_1, M_2 > 0$ independent from n, such that for every $n \in \mathbb{N}$ large enough one has

$$\begin{aligned}
M_1 + \frac{1}{\mu}\|u_n\| &\geq I_\lambda(u_n) - \frac{1}{\mu}I'_\lambda(u_n)(u_n) \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 + \lambda\int_0^T b(t)\mathcal{G}(u_n(t))\, dt \qquad (4.2)\\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 + \lambda\left[\int_{T_n} b(t)\mathcal{G}(u_n(t))\, dt + \int_{[0,T]\setminus T_n} b(t)\mathcal{G}(u_n(t))\, dt\right] \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 - \lambda M_2. \qquad (4.3)
\end{aligned}$$

Because $\mu > 2$, from (4.2) one can conclude that $\{u_n\}$ is bounded. At this point, since Φ' is a homomorphism and Ψ' is a compact operator the (PS)-condition holds (see also [23, Proposition 3.8]).

From condition (4.1) there exist α, $\beta > 0$ such that

$$G(\xi) \geq \alpha|\xi|^\mu - \beta \tag{4.4}$$

for every $\xi \in \mathbb{R}$. Hence, if $\{\xi_n \in \mathbb{R}^N\}$ is such that $|\xi_n| \to +\infty$ and put $w_n(t) = \xi_n$ one has that $w_n \in H^1_T$ and, also in view of (3.3),

$$\begin{aligned}
I_\lambda(w_n) &= \frac{1}{2}\int_0^T A(t)\xi_n \cdot \xi_n\, dt - \lambda\int_0^T b(t)G(\xi_n) \\
&\leq \frac{\Lambda T}{2}|\xi_n|^2 - \lambda\alpha\|b\|_1|\xi_n|^\mu + \lambda\beta\|b\|_1, \qquad (4.5)
\end{aligned}$$

namely I_λ is unbounded from below.

Let $c > 0$ such that

$$0 < \lambda < \frac{1}{2k^2\|b\|_1}\frac{c^2}{\max_{|\xi|\leq c} G(\xi)}$$

and put

$$r = \frac{c^2}{2k^2}. \tag{4.6}$$

From (3.12) one has

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) \le \|b\|_1 \max_{|\xi|\le c} G(\xi) < +\infty \tag{4.7}$$

and, in particular

$$0 < \lambda < \frac{1}{2k^2\|b\|_1} \frac{c^2}{\max_{|\xi|\le c} G(\xi)} \le \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}.$$

Hence, all the assumptions of Theorem 3.1 are satisfied and I_λ admits at least two critical points, possibly one being zero. The proof is completed taking in mind claim (3.15). □

Remark 4.1. We explicitly point out the proof of the previous theorem furnishes two solutions of problem (P_λ). However, it is not possible to assure that both these solutions are nontrivial.

Remark 4.2. Condition (4.1) has been used by other authors in order to establish the existence of at least one solution for the more general problem (1.1), for example see [18, 23, 32]. It is interesting to emphasize that in the cited papers the results are obtained requiring some further assumption on F, namely it is superquadratic at zero with respect to the second variable and/or it satisfies a global sign condition.

The next result represents a multiplicity theorem and throughout it we will make use of the following constant

$$L = \frac{1}{k^2 T\Lambda}, \tag{4.8}$$

where Λ is the number introudced in (3.2).

Theorem 4.2. *Suppose that the assumptions of Theorem 4.1 are satisfied. Moreover, assume that there exist $c > 0$ and $\tilde{\xi} \in \mathbb{R}^N$, with $|\tilde{\xi}| < c$, such that*

$$\frac{\max_{|\xi|\le c} G(\xi)}{c^2} < L\frac{G(\tilde{\xi})}{|\tilde{\xi}|^2}. \tag{4.9}$$

Then, for every $b : [0, T] \to \mathbb{R}$ satisfying $(\mathscr{B})$ and for every

$$\lambda \in \frac{1}{2k^2\|b\|_1} \left] \frac{1}{L}\frac{|\tilde{\xi}|^2}{G(\tilde{\xi})}, \frac{c^2}{\max_{|\xi|\le c} G(\xi)} \right[$$

problem (P_λ) admits at least two nontrivial solutions.

Proof. Fix $b : [0, T] \to \mathbb{R}$ satisfying $(\mathscr{B})$ and $\lambda \in \frac{1}{2k^2\|b\|_1} \left] \frac{1}{L}\frac{|\tilde{\xi}|^2}{G(\tilde{\xi})}, \frac{c^2}{\max_{|\xi|\le c} G(\xi)} \right[$. In this case we wish to apply Theorem 3.2 with $X = H^1_T$ and Φ, Ψ as introduced in section 3. From $|\tilde{\xi}| < c$ and (4.9) one has

$$|\tilde{\xi}| < \sqrt{L}c. \tag{4.10}$$

Indeed, arguing by contradiction, assume $c \le \frac{1}{\sqrt{L}}|\tilde{\xi}|$. It follows

$$\frac{\max_{|\xi|\le c} G(\xi)}{c^2} \ge \frac{G(\tilde{\xi})}{c^2} \ge L\frac{G(\tilde{\xi})}{|\tilde{\xi}|^2},$$

in contradiction with (4.9).
Let r be as defined in (4.6) and put

$$\tilde{u}(t) = \tilde{\xi} \quad \forall t \in [0, T].$$

From (4.9) one has that $\tilde{\xi} \neq 0$. Hence, because of (4.10)

$$0 < \frac{1}{2} \int_0^T A(t)\tilde{\xi} \cdot \tilde{\xi}\, dt \leq \frac{1}{2} T\Lambda |\tilde{\xi}|^2 = \frac{1}{2} \frac{1}{Lk^2} |\tilde{\xi}|^2 \leq \frac{c^2}{2k^2},$$

namely,

$$0 < \Phi(\tilde{u}) < r.$$

Moreover, from assumption (4.9) and taking in mind (3.12) one has

$$\begin{aligned}
\frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} &\leq 2k^2 \|b\|_1 \frac{\max_{|\xi| \leq c} G(\xi)}{c^2} \\
&< 2k^2 \|b\|_1 L \frac{G(\tilde{\xi})}{|\tilde{\xi}|^2} \\
&= 2\|b\|_1 \frac{1}{T\Lambda} \frac{G(\tilde{\xi})}{|\tilde{\xi}|^2} \\
&\leq 2\|b\|_1 \frac{G(\tilde{\xi})}{\int_0^T A(t)\tilde{\xi} \cdot \tilde{\xi}}\, dt \\
&= \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.
\end{aligned}$$

Arguing as in Theorem 4.1, it is possible to verify that $I_\lambda = \Phi - \lambda\Psi$ is unbounded from below an satisfies the (PS)-condition. Hence, all the assumptions of Theorem 3.2 are satisfied and I_λ admits two nontrivial critical points $u_{1,\lambda},\ u_{2,\lambda}$ with $I_\lambda(u_{1,\lambda}) < 0 < I_\lambda(u_{2,\lambda})$. This, in conjunction with claim (3.15), concludes the proof. □

5. Comparison with other results

In [38] a study was made of the system:

$$-\ddot{x}(t) + A(t)x(t) = \lambda \nabla_x V(t, x(t)), \tag{5.1}$$

where $V(t,x) \in C([0,T] \times \mathbb{R}^N)$. A special case is

$$-\ddot{u} + A(t)u = \lambda b(t) \nabla G(u). \tag{5.2}$$

The elements of the symmetric matrix $A(t)$ were only assumed to be integrable functions on $I = [0,T]$, i.e., for each j and k, $a_{jk}(t) \in L^1(I)$. This implies that there is an extension D of the operator

$$D_0 x = -\ddot{x}(t) + A(t)x(t)$$

having the same essential spectrum as $-\ddot{x}(t)$. In particular, it has a discrete, countable spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound λ_0 :

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_l < \ldots. \tag{5.3}$$

The hypotheses $(\mathscr{A})_1$ and $(\mathscr{A})_2$ were not assumed there. The counterparts of the functionals $\Phi, \Psi : H^1_T \to \mathbb{R}$ are

$$\Phi(u) := d(u), \qquad \Psi(u) := \int_0^T 2V(t,u)\; dt, \tag{5.4}$$

Let

$$G(x) = d(x) - 2\int_I V(t,x)\, dt, \tag{5.5}$$

where $d(x) = (Dx, x)$. The counterpart of $I_\lambda := \Phi - \lambda\Psi$ is

$$G_\lambda(x) = d(x) - 2\lambda \int_I V(t,x)\, dt, \quad 0 < \lambda < \infty. \tag{5.6}$$

It was not required that the Ambrosetti-Rabinowitz condition (1.2) be satisfied. Neither the (PS)-condition nor the (C)-condition held. One consequence of the results proved there is

Theorem 5.1. *Assume* $\lambda_0 > 0$ *and*

(1) $V(t,x)/|x|^2 \to \infty$ as $|x| \to \infty$.
(2) *There are positive constants* μ *and* m *such that*

$$2V(t,x) \le \mu |x|^2, \quad |x| \le m,\ x \in \mathbb{R}^N.$$

Then the system (5.1) has a nontrivial solution for almost all $\lambda \in]0, \lambda_0/\mu[$. *If we add the hypothesis: There is a function* $W(t) \in L^1(I)$ *such that*

$$\begin{aligned} 2V(t,x+y) - 2V(t,x) - (2ry - (r-1)^2 x)\cdot \nabla_x V(x,t) \\ \ge -W(t), \quad t \in I,\ x, y \in \mathbb{R}^N,\ r \in [0,1], \end{aligned} \tag{5.7}$$

then the system (5.1) has a nontrivial solution for each $\lambda \in]0, \lambda_0/\mu[$.

Theorem 5.2. *The conclusions of Theorem 5.1 hold if we replace Hypothesis (5.7) with: There are a constant* C *and a function* $W(t) \in L^1(I)$ *such that*

$$H(t, \theta x) \le C(H(t,x) + W(t)), \quad 0 \le \theta \le 1,\ t \in I,\ x \in \mathbb{R}^N,$$

where

$$H(t,x) = \nabla_x V(t,x)\cdot x - 2V(t,x).$$

Note that there were no results concerning multiple nontrivial solutions.

6. Examples

Example 6.1. The following problem

$$\begin{cases} -\ddot{u} + u = \frac{1}{18} u e^{\frac{|u|^2}{36}} \left(|u|^6 + 91|u|^4 - 1208|u|^2 + 576\right) & \text{a. e. in } [0,1] \\ u(1) - u(0) = \dot{u}(1) - \dot{u}(0) = 0 \end{cases}$$

admits at least two nontrivial solutions.

Indeed, one can apply Theorem 4.2 when $T = 1$, $A(t) = I_{N\times N}$, $\lambda = 1$, $b \equiv 1$ and $G(\xi) = |\xi|^2(|\xi|^2 - 1)(|\xi|^2 - 16)e^{|\xi|^2/36}$. In this case $k = \sqrt{2}$, $L = 1/2$ and choosing $c = 4$ and $\bar{\xi} \in \mathbb{R}^N$, with $|\bar{\xi}| < 1$ such that $\max_{|\xi|\le 4} G(\xi) = \max_{|\xi|\le 1} G(\xi) = G(\bar{\xi})$, some calculations show that

$$\frac{\max_{|\xi|\le 4} G(\xi)}{16} = \frac{G(\bar{\xi})}{16} < \frac{G(\bar{\xi})}{2} < \frac{1}{2}\frac{G(\bar{\xi})}{|\bar{\xi}|^2},$$

namely (4.9) holds. Condition (4.1) is clearly satisfied. Moreover, one can verify that $\frac{1}{2} < G(\bar{\xi}) < 4$, hence

$$\frac{1}{2k^2\|b\|_1}\frac{1}{L}\frac{|\bar{\xi}|^2}{G(\bar{\xi})} = \frac{1}{2G(\bar{\xi})} < 1 < \frac{4}{\max_{|\xi|\leq 4} G(\xi)} = \frac{1}{2k^2\|b\|_1}\frac{16}{\max_{|\xi|\leq 4} G(\xi)}$$

and the conclusion is achieved by applying Theorem 4.2 with $\lambda = 1$.

We explicitly wish to point out that condition (4.9), that is the crucial assumption of Theorem 4.2, is not a local condition at zero on the potential G. Note that in the above example G has a quadratic behaviour at zero. However, in general, a sub-quadratic growth at zero represents a sufficient condition to satisfy (4.9). On account of this remark, as a particular case of Theorem 4.2, it is possible to state the following

Theorem 6.1. *Suppose that the assumptions of Theorem 4.1 are satisfied. Moreover, assume that*

$$\limsup_{\xi\to 0}\frac{G(\xi)}{|\xi|^2} = +\infty. \tag{6.1}$$

Then, for every $b : [0,T] \to \mathbb{R}$ satisfying ($\mathscr{B}$) and for every

$$\lambda \in \frac{1}{2k^2\|b\|_1}\left]0, \sup_{c>0}\frac{c^2}{\max_{|\xi|\leq c} G(\xi)}\right[$$

problem (P_λ) admits at least two nontrivial solutions.

Proof. Fix $b : [0,T] \to \mathbb{R}$ satisfying ($\mathscr{B}$) and $\lambda \in \frac{1}{2k^2\|b\|_1}\left]0, \sup_{c>0}\frac{c^2}{\max_{|\xi|\leq c} G(\xi)}\right[$. Let $c > 0$ be such that

$$0 < \lambda < \frac{1}{2k^2\|b\|_1}\frac{c^2}{\max_{|\xi|\leq c} G(\xi)}. \tag{6.2}$$

Because of assumption (6.1) there exists $\tilde{\xi} \in \mathbb{R}^N \setminus \{0\}$, with $|\tilde{\xi}| < c$, such that

$$0 < \frac{1}{2k^2\|b\|_1}\frac{1}{L}\frac{|\tilde{\xi}|^2}{G(\tilde{\xi})} < \lambda, \tag{6.3}$$

where L is the constant introduced in (4.8). Hence, putting together (6.2) and (6.3) one has that

$$\frac{\max_{|\xi|\leq c} G(\xi)}{c^2} < L\frac{G(\tilde{\xi})}{|\tilde{\xi}|^2}$$

and

$$\lambda \in \frac{1}{2k^2\|b\|_1}\left]\frac{1}{L}\frac{|\tilde{\xi}|^2}{G(\tilde{\xi})}, \frac{c^2}{\max_{|\xi|\leq c} G(\xi)}\right[,$$

Namely, all the assumptions of Theorem 4.2 are satisfies and the proof is complete. □

Remark 6.1. We can observe that Theorem 6.1 is a generalization of Theorem 4.1 of [11] where the extra conditions $\nabla G(0) \neq 0$ and $b \equiv 1$ are exploited.

As a consequence of Theorem 6.1 we can study the following class of Hamiltonian systems

$$\begin{cases} -\ddot{u}+u=\lambda|u|^{q-2}u+|u|^{r-2}u \quad \text{a.e. in } [0,1] \\ \\ u(1)-u(0)=\dot{u}(1)-\dot{u}(0)=0, \end{cases} \tag{6.4}$$

with $1<q<2<r$.

Corollary 6.1. *Put*

$$\lambda^*=\left(\frac{r}{r-2}\frac{2-q}{q}\right)^{\frac{2-q}{r-2}}\left[\frac{q}{2}\cdot\frac{r-2}{r-q}\right]^{\frac{r-q}{r-2}}.$$

Then, for every $\lambda\in]0,\lambda^*[$ problem (6.4) admits at least two nontrivial solutions.

Proof. Fix $\lambda\in]0,\lambda^*[$ and put

$$T=1,\quad A(t)=I_{N\times N},\quad b\equiv 1\quad \text{and}\quad G(\xi)=\lambda\frac{|\xi|^q}{q}+\frac{|\xi|^r}{r}.$$

It is possible to verify that all the assumptions of Theorem 6.1 are satisfied, with $k=1$ and $\|b\|_1=1$. Hence, for every $\nu\in\left]0,\frac{1}{2}\sup_{c>0}\frac{c^2}{\max_{|\xi|\le c}G(\xi)}\right[$ problem

$$\begin{cases} -\ddot{u}+u=\nu\left(\lambda|u|^{q-2}u+|u|^{r-2}u\right) \quad \text{a.e. in } [0,1] \\ \\ u(1)-u(0)=\dot{u}(1)-\dot{u}(0)=0, \end{cases} \tag{6.5}$$

admits at least two nontrivial solutions. At this point, it is clear that

$$\sup_{c>0}\frac{c^2}{\max_{|\xi|\le c}G(\xi)}=\sup_{c>0}\frac{c^2}{\frac{\lambda}{q}c^q+\frac{c^r}{r}},$$

and, since the function $\varphi:]0,+\infty[\to]0,+\infty[$ defined by putting

$$\varphi(c)=\frac{c^2}{\frac{\lambda}{q}c^q+\frac{c^r}{r}}$$

for every $c>0$ is continuous and such that

$$\lim_{c\to 0^+}\varphi(c)=\lim_{c\to+\infty}\varphi(c)=0,$$

there exists $c_{\max}>0$ such that

$$\varphi(c_{\max})=\sup_{c>0}\frac{c^2}{\max_{|\xi|\le c}G(\xi)}.$$

In particular, simple calculations show that

$$c_{\max}=\left(\lambda\frac{r}{r-2}\frac{2-q}{q}\right)^{\frac{1}{r-q}}$$

and

$$\frac{1}{2}\sup_{c>0}\frac{c^2}{\max_{|\xi|\le c}G(\xi)}=\frac{1}{\lambda^{\frac{r-2}{r-q}}}\frac{q(r-2)}{2(r-q)}\left(\frac{r}{r-2}\frac{2-q}{q}\right)^{\frac{2-q}{r-q}}=\left(\frac{\lambda^*}{\lambda}\right)^{\frac{r-2}{r-q}}.$$

Thus, since $\lambda\in]0,\lambda^*[$ and taking in mind that $0<q<2<r$, one has that

$$\frac{1}{2}\sup_{c>0}\frac{c^2}{\max_{|\xi|\le c}G(\xi)}>1.$$

Finally, we can conclude observing that (6.5) reduces to (6.4) when $\nu=1$. □

Thanks to Remark 3.1, in analogy with Theorem 6.1, it is possible to obtain another multiplicity theorem by requiring a different condition with respect to the Ambrosetti-Rabinowitz type condition (4.1).

Theorem 6.2. *Assume that $(\mathscr{A})_1 - (\mathscr{A})_2$, $(\mathscr{G})$ and (6.1) hold. Moreover, suppose that*

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^2} = +\infty \tag{6.6}$$

and that there exist $M, \tau,\ \alpha > 0$ with $1 \le \alpha \le \tau$ such that

$$G(\xi) \le M(1 + |\xi|^{\alpha+1}) \tag{6.7}$$

and

$$\liminf_{|\xi|\to+\infty} \frac{\nabla G(\xi)\cdot\xi - 2G(\xi)}{|\xi|^\tau} > 0 \tag{6.8}$$

Then, for every $b \in L^\infty([0,T]) \setminus \{0\}$ such that $\underline{b} = \operatorname{essinf}\ b > 0$ and for every

$$\lambda \in \frac{1}{2k^2\|b\|_1}\left]0, \sup_{c>0} \frac{c^2}{\max_{|\xi|\le c} G(\xi)}\right[$$

problem (P_λ) admits at least two nontrivial solutions.

Proof. Fix $b \in L^\infty([0,T]) \setminus \{0\}$ such that $\underline{b} > 0$ and let λ such that

$$0 < \lambda < \frac{1}{2k^2\|b\|_1} \sup_{c>0} \frac{c^2}{\max_{|\xi|\le c} G(\xi)}.$$

If $X = H^1_T$ and $\Phi,\ \Psi$ are as defined in the previous section 3, since Φ is coercive, we wish to apply Theorem 3.2 with (PS) replaced by (C).

Arguing exactly as in the proof of Corollary 6.1 it is possible to find $r \in \mathbb{R}$ and $\tilde{u} \in X$ such that $0 < \Phi(\tilde{u}) < r$ and

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

From (6.6) there exist a sequence $\{\xi_n\}$ in $\mathbb{R}^N$ and a positive number η with

$$\eta > \frac{T\Lambda}{2\lambda\|b\|_1} \tag{6.9}$$

such that $|\xi_n| \to +\infty$ and

$$G(\xi_n) > \eta|\xi_n|^2 \tag{6.10}$$

for every $n \in \mathbb{N}$. Hence, if we put $w_n(t) = \xi_n$ for every $t \in [0,T]$, $n \in \mathbb{N}$, one has that $w_n \in X$ and, from (3.3) and (6.10) one obtains

$$\begin{aligned} I_\lambda(w_n) &= \Phi(w_n) - \lambda\Psi(w_n) \\ &= \frac{1}{2}\int_0^T A(t)\xi_n\cdot\xi_n\, dt - \lambda\|b\|_1 G(\xi_n) \\ &\le \frac{T\Lambda}{2}|\xi_n|^2 - \lambda\|b\|_1 G(\xi_n) \\ &\le \left(\frac{T\Lambda}{2} - \lambda\|b\|_1\eta\right)|\xi_n|^2. \end{aligned}$$

Hence, passing to the limit in the previous inequality and taking in mind (6.9), it follows that I_λ is unbounded from below.

Let us now verify the I_λ satisfies (C). Assume that $\{u_n\}$ in X is such that

$$\{I_\lambda(u_n)\} \text{ is bounded and } (1+\|u_n\|)I'(u_n) \to 0 \text{ in } X^*. \tag{6.11}$$

We claim that

$$\{u_n\} \text{ is bounded.} \tag{6.12}$$

By contradiction, if (6.12) does not hold, passing to a subsequence if needed, one has that

$$\|u_n\| \to +\infty. \tag{6.13}$$

By assumption (6.8) there exist $M_1,\ M_2 > 0$ such that

$$\nabla G(\xi)\cdot\xi - 2G(\xi) \geq M_1|\xi|^\tau - M_2 \tag{6.14}$$

for every $\xi \in \mathbb{R}^N$. Condition (6.11) assure the existence of a positive constant M_3 as well of a sequence $\{\varepsilon_n\}$ with $\varepsilon \downarrow 0^+$ such that

$$\|u_n\|^2 - 2\lambda \int_0^T b(t)G(u_n(t))\ dt \leq M_3 \tag{6.15}$$

and

$$\langle u_n, h\rangle - \lambda \int_0^T b(t)\nabla G(u_n(t))\cdot h\ dt \leq \varepsilon_n \frac{\|h\|}{1+\|u_n\|} \tag{6.16}$$

for every $n \in \mathbb{N}$ and every $h \in X$. Acting in (6.16) with $h = -u_n$ and adding (6.15) to (6.16) one has

$$\lambda \int_0^T b(t)[\nabla G(u_n(t))\cdot u_n(t) - 2G(u_n(t))]\ dt \leq M_3 + \varepsilon_n \frac{\|u_n\|}{1+\|u_n\|} \tag{6.17}$$

for every $n \in \mathbb{N}$. On the other hand, since $\underline{b} > 0$, by (6.14) one has

$$\begin{aligned}\lambda \int_0^T b(t)[\nabla G(u_n(t))\cdot u_n(t) - 2G(u_n(t))]\ dt &\geq \lambda \int_0^T b(t)[M_1|u_n(t)|^\tau - M_2]\ dt \\ &\geq \lambda \underline{b} M_1\|u_n\|_\tau^\tau - \lambda M_2\|b\|_1 \quad (6.18)\end{aligned}$$

for every $n \in N$. Putting together (6.17) and (6.18) one has that $\{u_n\}$ is bounded in $L^\tau([0,T],\mathbb{R}^N)$. Hence, because $1 \leq \alpha \leq \tau$ it is clear that

$$\{u_n\} \text{ is bounded in } L^\alpha([0,T],\mathbb{R}^N). \tag{6.19}$$

Condition (3.8) assures that

$$\frac{|u_n(t)|}{\|u_n\|} \leq k \tag{6.20}$$

for every $t \in [0,T]$ and every $n \in \mathbb{N}$. Hence, putting together (6.15), (6.7) one has

$$\begin{aligned}\|u_n\| &\leq 2\lambda \int_0^T b(t)\frac{G(u_n(t))}{\|u_n\|}\ dt + \frac{M_3}{\|u_n\|} \\ &\leq 2M\lambda \int_0^T b(t)\left(\frac{1}{\|u_n\|} + \frac{|u_n(t)|^{\alpha+1}}{\|u_n\|}\right)\ dt + \frac{M_3}{\|u_n\|} \qquad (6.21) \\ &\leq 2M\lambda\left(\frac{\|b\|_1}{\|u_n\|} + k\|b\|_\infty\|u_n\|_\alpha^\alpha\right) + \frac{M_3}{\|u_n\|}\end{aligned}$$

for every $n \in \mathbb{N}$. At this point it is clear that (6.21), in view of (6.19), is in contradiction with (6.13). Hence (6.12) holds and recalling again that Φ' is a homomorphism and Ψ' is a compact operator, since from (6.11) it follows that $I'_\lambda(u_n) \to 0$ in X^*, one has that $\{u_n\}$ admits a convergent subsequence, namely I_λ

satisfies (C). We are now in the position to apply the alternative version of Theorem 3.2 where condition (PS) is replaced by (C), see Remark 3.1, that concludes the proof. □

Remark 6.2. The previous Theorem 6.2 extends Theorem 4.2 of [11] because it does not require that $\nabla G(0) \neq 0$ and, in addition, it considers problems where the nonlinearity can explicitly depend on the variable t.

Example 6.2. Let $A : [0,T] \to \mathbb{R}^{N\times N}$ be a matrix-valued function satisfying conditions $(\mathscr{A})_1 - (\mathscr{A})_2$ and $b \in L^1([0,T])$ with $\bar{b} = \text{essinf } b > 0$. Then, for every $\lambda \in \frac{1}{2k^2\|b\|_1}\,]0, e^2[$ problem

$$\begin{cases} -\ddot{u} + A(t)u = 2\lambda b(t)u \ln|u| \left(\ln|u| + 1\right) & \text{a.e. in } [0,T] \\ \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0 \end{cases}$$

admits at least two non trivial solutions. Indeed, if one considers the function $G(\xi) = |\xi|^2 \ln^2|\xi|$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$, with $G(0) = 0$, direct calculations show that $\nabla G(\xi) = \xi \ln|\xi| \left(\ln|\xi| + 1\right)$ and $\max_{|\xi|\leq 1} G(\xi) = e^{-2}$. Hence,

$$\sup_{c>0} \frac{c^2}{\max_{|\xi|\leq c} G(\xi)} \geq e^2$$

and the conclusion is achieved by applying Theorem 6.2.

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