Historical notes on star geometry in mathematics, art and nature

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- Gamma: "I can. Look at this Counterexample 3: a star-polyhedron -I shall call it urchin. This consists of 12 star-pentagons. It has 12 vertices, 30 edges, and 12 pentagonal faces-you may check it if you like by counting. Thus the Descartes-Euler thesis is not true at all, since for this polyhedron V - E + F = - 6".
- Delta: "Why do you think that your 'urchin' is a polyhedron?"
- Gamma: "Do you not see? This is a polyhedron, whose faces are the twelve star-pentagons".
- Delta: "But then you do not even know what a polygon is! A star-pentagon is certainly not a polygon!"

In the above dialogue from [25], Imre Lakatos used the example of star polyhedra to describe the complex path *definition – proof – refutation – new definition* that mathematical thought threaded before reaching a consensus on fundamental points of the main mathematical topics. In this paper we will try to examine how in the history of polyhedra (and in particular star polyhedra), a long period of «discovery» of individual types due to the observation of natural objects or due to artistic imagination preceded (and was connected with) the mathematical solution fixing the "right" definitions. Such long period of discovery—we will argue—influenced further investigations on nature and art. The paper will start from the thirteen century and will end with the publications of Pappo's work [30] and Kepler's "Harmonices Mundi" which provided solid mathematical foundations to the subject. We want to describe the geometric ideas and the theories of Adelard of Bath, Thomas Bradwardine, Luca Pacioli, Albrecht Dürer, Simon Stevin, Daniele Barbaro, Jan Brożek. We conclude with some short notes about the subsequent developments (Johannes Kepler, Louis Poinsot and Albert Badoureau).
A star polygon may be constructed by connecting with straight lines every h-th point out of n regularly spaced points lying on a circumference. We call order the number n and species the number h of the star polygon. It is obviously possible to have a star polygon only if n and h are co-prime integers.
In western culture, the mathematicians' interest for star figures originated as a result of the common use of these shapes by artists and artisans in their architectural decorations and mosaics, especially Islamic, in the medieval period. Symmetries and fascinating combinations of elementary figures, essentially inspired by the observation of shapes in nature and intuition, led artists to create increasingly articulated patterns, persuading mathematicians to progressively develop a study of their regularity and properties. The key to formalize these new concepts was the interest by artists-mathematicians as Pacioli, Piero della Francesca, Leonardo da Vinci and

Dürer who influenced the mathematicians Stevin and Barbaro. From that time on, the systematic exploration and classification of the star figures through mathematical theories, allowed to suppose the existence of new shapes that neither nature nor art had previously shown. This is why it is important to study less known medieval treatises where star polygons and polyhedra appeared, even if their completed classifications occurred only in the 17-th century by Kepler and in the 19-th century by Augustin-Louis Cauchy and Poinsot.

In this paper, we do not analyse the well-known Greek period; similarly, we give only a brief mention of the very interesting contributions of Arab and Islamic art and mathematics. It seems that the influence of Arabic mathematicians on the western mathematicians of the Middle Ages was not crucial. We did not find manifest or declared Arab influence on the authors we present here. It is important however, to mention that in the texts of practical geometry by the mathematicians of Islamic culture, the needs of artists and artisans were significantly linked to the abilities of the mathematicians. The fact that star figures arose and were mathematically studied starting from artistic necessities is perfectly showed by a quote by Abū'l-Wafā' al-Būzjānī (c. 940-998)[1] and by a figure.[2]

> The geometer knows the correctness of what he wants by means of proofs, since he is the one who has derived the notions on which the artisan and the surveyor base their work. However, it is difficult for him to transform what he has proved into a [practical] construction, since he has no experience with the practical work of the artisan.

Boethius (c. 480-524) was probably the first to talk about star pentagon, in his *Geometria*, showing a pentagram (a well-known figure) inscribed in a circle. In his very popular translation of Euclid's "Element" (realized starting from a translation by Adelard of Bath [7]), Giovanni Campano (1220-1296) described the pentagram and the property that the sum of its interior angles is equal to two right angles. A first systematic study on star polygons is contained in Adelard of Bath's (1080-1152) Latin translation of Euclid's text (1120), from the Arabic text by al-Ḥajjāj ibn Yūsuf ibn Maṭar. Adelard had been in close contact with Arab culture, staying for many years in Spain, Cilicia and Syria.[3] In his text, he showed that in a convex polygon with n angles, the sum of the measures of the interior angles is: $(n\text{-}2)2R$ (where R means a right angle). He deduced the sum of the interior angles of a star polygon starting from the number of intersections of each side with the others [20].

The results of Adelard were probably developed independently in [4] (printed in 1496) by the English theologian and mathematician Thomas le Byer, known as de Bradwardine (c. 1290-1349). Bradwardine was one of the Oxford Calculators, a

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[1] *Kitāb fīmā yahtāju ilayhi al-sāni min al-a'māl al-handasiya* (*On the Geometric Constructions Necessary for the Artisan*). We got it from [28].

[2] The picture is in the anonymous text (dating to the 14th century) *Fī tadākhul al-ashkāl al-mutashābiha aw al-mutawāfiqa* (*On Interlocks of similar or Corresponding Figures*) and in [12], p. 774. See also [13] and [32].

[3] See http://turnbull.mcs.st-and.ac.uk

group of thinkers devoted to natural science, mainly physics, astronomy and mathematics. In 1331 Bradwardine was ordained sub-deacon; in 1337 he became Chancellor of St. Paul's Cathedral and finally, he was elected Archbishop of Canterbury. At the end of the first part of his work, Bradwardine included a chapter in which he studied figures of egredient angles (*figurae egredientium angulorum*), distinguishing them from the convex ones (*simplex*):

> I shall speak about figures of egredient angle […]. Discussion about them is rare, and I have not seen a discussion of them, except only by Campanus, who only casually touches on pentagon a little. A figure is said to be of egredient angles when the sides of some polygonal figure from among the simple ones are produced until they meet outside.[4]

Star polygons produced in this way from simple polygons are those of the second species, which Bradwardine called "of the first order" (i.e. the first constructible, given the number of its sides). Bradwardine's first conclusion was that the star pentagon is the first figure of egredient angles; then he showed that the pentagon of egredient angles has five angles equal to two right angles. His third deduction was that of figures of egredient angles of given specie, each figure of successive order adds two right angles over the figure of precedent order. He also affirmed that, while this is immediately evident for all figures having an even order, for each of them is composed of two simple figures mutually entwined (the hexagon is worth four right angles for it is composed of two triangles; the octagon is composed of two quadrangles, and so on), the same is less evident for figures having an odd order:

> it is likely, however that the heptagon adds two right angles over the hexagon and the nonagon two right angles over the octagon and so for the others.

Bradwardine considered as star polygons figures such as the star hexagon that in the nineteenth century would be regarded as two triangles rather than as a simple polygon. He showed how to construct figure of higher species, producing the sides further until they would meet. His fourth statement was that the heptagon is the first figure of egredient angles of the second species. His fifth conclusion was that the first figure of the following order is always taken from the third member of the preceding species. Bradwardine was also interested in the sum of the angles, but he said that:

> To investigate here the value of the angles of such figures would be more laborious than fruitful, and so I do not set about it. But it sometimes seemed to me that all orders of figures are agreed in this, that always the first has the value of two right angles, and each successor adds two right angles in value over its predecessor. But although this is near to it in reality, yet I do not assert it.

In 1542, the French mathematician and philosopher Charles de Bouvelles (1479-1567?) resumed Bradwardine's theory on star polygons in his "Géométrie pratique"[5]. He did not show great originality, but it is noteworthy that he used proofs

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[4] The quotes are from [27]. For Bradwardine's geometry see also [26].

[5] We used the reprint of 1551 [17].

from the composition or decomposition of polygons into triangles. For example, in order to calculate the sum of the angles of a pentagram, he inscribed it in a regular pentagon and then he said that any angle of the pentagram is the third part of any angle of the pentagon. So the sum of the angles of a pentagram is equal to two right angles. He argued that the star hexagon is composed by two equilateral triangles; the sum of its angles is four right angles and its area is twice the area of the original convex hexagon[6].

In the 17th century, Jan Brożek (1585-1652), a Polish mathematician, astronomer, physician, poet, writer, and musician who was also rector of the Kraków Academy and biographer of Copernicus, further developed the theory. In [6], Brożek showed, by simple arguments on angles at centre and at circumference, the property that it is possible to create an infinity number of star polygons so that the sum of their interior angles is two right angles. The problem of the not univocal definition of star figures, however, is still present: for Brożek, the hexagon with six egredient angles was a not convex figure with twelve sides; the star heptagon was a not convex figure with 14 sides, and so on. For the first time, Brożek clarified that a star hexagon is built by two triangles and a star octagon (of II species) by two squares. Brożek conceived a special procedure for the construction of star polygons starting from isoperimetric convex polygons. From the regular pentagon *SDCBY*, he built, by extending its sides, the star polygon *SEDICOBVYA* (fig. 1). He overturned the triangle *SAE* along the segment *AE* and made the same thing for all the triangles of the figure; finally, he proved that the new pentagon *HKLFG* is isoperimetric with the star pentagon. The transformations were even more for polygons with greater number of sides, and Brożek showed what was possible to do starting from a 14-sided figure.

It seems that in Europe the earliest representation of a star polyhedron (a small stellated dodecahedron) is in Venice, in a 1420 mosaic by Paolo Uccello situated on the floor of the St Mark's Basilica (fig. 2). The first systematic discussion of star polyhedra is [29] (completed as manuscript in 1498 and printed in 1509[7]) by Luca Pacioli (1445-1517)[8].

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[6] See [17] p. 23. See also [16]. For other short studies of this period on star polygons, see [11] pp. 478-481.

[7] Two manuscripts of this work are in Milan (at Biblioteca Ambrosiana) and in Geneva (at Bibliothéque Publique et Universitaire); a third manuscript, from which the print copy was probably edited, is lost (see [39]).

[8] On Pacioli there is abundant literature; see [39].

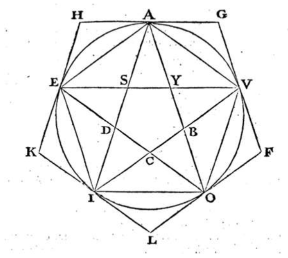

Fig. 1. Picture from [6], p. 47.

Fig. 2. Picture of a mosaic in the St Mark's Basilica, Venice

Pacioli was, as is well known, under the influence of Piero della Francesca, who was not explicitly interested in star polygons and polyhedra [15]. His work represented a milestone and was responsible for reawakening the interest of mathematicians and artists on the problems associated with the geometrical shapes, in particular Archimedean polyhedra. Here, we do not discuss this topic, but we refer to the large bibliography in [19]. In his treatise, which contains the famous drawings by Leonardo, Pacioli showed not only the five regular polyhedra, but also six polyhedra created by cutting solid angles from the regular solids, later known as Archimedean (*polyhedron abscissum*). Pacioli also presented for the first time some new star polyhedra that he built by adding regular and equilateral pyramids on the faces of regular polyhedra (*polyhedron elevatum*), or by adding pyramids on the faces of Archimedean solids (*polyhedron abscissum elevatum*). Sometimes this procedure has been called *Kleetope* (in honour of the mathematician Victor Klee). Most likely, Pacioli's book was the main source of inspiration for the subsequent developments of the topic.

The first of Pacioli's new polyhedra is the *elevatum tetrahedron* with 12 faces, 18 edges and 8 vertices (now called *kistetrahedron,* a Catalan solid since it is the dual of the truncated tetrahedron); this is a tetrahedron with triangular pyramids added to each face. The second polyhedron is the *elevatum cube* with 16 faces, 36 edges and 14 vertices (now called *tetrakis hexahedron,* dual of the truncated octahedron). Then Pacioli argued that it is possible to create an *abscissum elevatum cube* (the *disdyakis dodecahedron*) with 48 faces, 72 edges and 26 vertices, dual of the cuboctahedron. From the octahedron, he continued, it is possible to create the *ele-*

vatum octahedron (or *stella octangula*) with 24 faces, 36 edges and 8 vertices. He also showed the *elevatum icosahedron* (*triakis icosahedron*, dual of the truncated dodecahedron) and the *elevatum dodecahedron* (*pentakis dodecahedron*, dual of the truncated icosahedron), both with 60 faces, 90 edges and 32 vertices. Then Pacioli illustrated the *abscissum elevatum dodecahedron* (*elevatum icosidodecahedron*) with 120 faces, 180 edges and 62 vertices (fig. 3). Probably Pacioli used material models of the solids. He said:

> E cascando in piano questo sempre si ferma in 6 ponte o coni piramidali [in 6 vertici]. De li quali coni uno sia de pyramide pentagona e li altri 5 sono dele pyramidi triangule. La qual cosa […] pare a l'ochio absurda che simil ponte sieno ad un piano. […].[9]

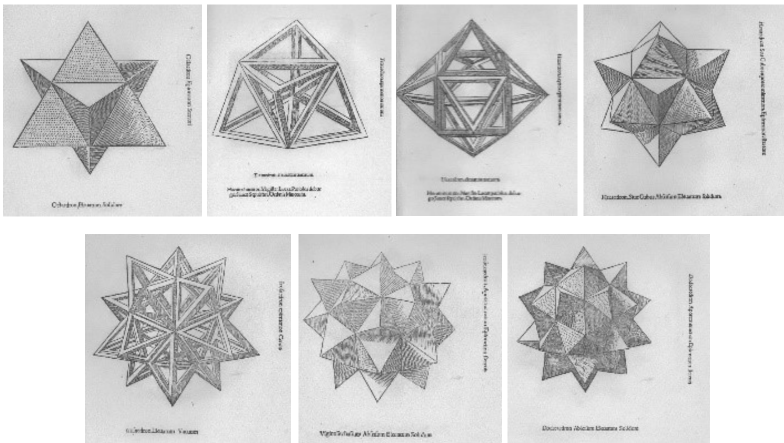

Fig. 3. Pictures from [29].

Pacioli's text influenced Fra' Giovanni da Verona (c. 1457-1525) who created wonderful intarsia in the Roman Catholic church Santa Maria in Organo in Verona; pictures of star polyhedra inspired from Leonardo's ones are also present.[10]
Without doubt, Pacioli influenced Albrecht Dürer (1471-1528), who lived in Venice between 1494 and 1495, and then again between 1505 and 1507.[11] His famous treatise [12, 18][12] includes some significant notes on star polyhedra. The book was

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[9] "And falling flat this always stops in 6 pyramidal cones. One is the vertex of the pyramid with pentagonal base and the other 5 are of the triangular pyramids. It seems absurd that they are on the same plane". This propriety observed by Pacioli is not correct: the vertex of the pyramid with pentagonal base is not on the same plane as the five vertices of the triangular pyramids. The distance from the vertex to the plane, however, is very small, which suggests that Pacioli made some tests with material models of the solids. On this point, we would like to acknowledge the precious help of Maria Dedò.

[10] An interesting text on the relationships between Giovanni and Leonardo is Hayashi, S., I Poliedri nelle Tarsie di fra Giovanni da Verona: l'Influenza delle Illustrazioni nel De Divina Proportione (1509) di Luca Pacioli. *Studi Italici*, 59 (2009), pp. 97-117.

[11] On the relationships between Pacioli and Dürer, see [31] at preface.

[12] We are referring here to [31].

intended to be a guide for young craftsmen and artists, giving them both practical and mathematical tools for their trade. Dürer, recalling the famous phrase of Leonardo, wrote:

> Many young people have been trained in the art of painting, and have been educated in them only with daily practice, without giving them the basis. They grew up in intelligence like the wild tree that was not carved.

In the second part of this book, Dürer provided compass and straight edge constructions (sometimes approximated) for the regular polygons from the triangle to the 16-gon. He also provided star figures made with arcs of circumference (not with line segments, so they are not star polygons), which are a variety of stars described within circles for ornamental purposes (fig. 4). He talked about star polygons built by crossing over each other and rotating regular polygons:

> Sometimes it is convenient to superpose the figures [...] or even to allow them to penetrate each other, as I indicated in the six figures I built as result.

It is interesting to note that this construction of star polygons is quite different from those of Bradwardine, who extended the sides of regular polygons; so it is not by chance that among the six examples of star polygons, the most famous, the pentagram, is missing. Although without drawings and his famous developments of surfaces, the great artist provided some notes on star polyhedra; he built star polyhedra by interpenetration of regular polyhedra:

> You can look for the intersections two by two of these solids of the same size, making sure that the vertices of one pierce the faces of the other. This gives good results in construction.

Or, like Pacioli and Leonardo, adding pyramids on the faces of regular polyhedra:

> So you may want to apply on each side of these solids a pyramid, more or less high, with as many sides as the face on which it is placed.

Dürer defined a class of polyhedra larger than Pacioli's one, since his pyramids were not only equilateral but they had different altitudes.

It is also thanks to the work of Dürer, that the decorative use of star polyhedra in texts on the perspective drawing spread in the environment of Nuremberg during the sixteenth century. The work [21] of Wenzel Jamnitzer[13] (1507-1585) stands out amongst all. Jamnitzer was a goldsmith of Nuremberg, and his work is full of beautiful geometric shapes on copper plates, but without explanations and mathematical theories, which could be used to build his solids. His book, however, develops a study of 120 regular and semi-regular solids, obtained by modifying, starring or cutting the five platonic polyhedra (fig. 5). Other significant texts are [37] by Lorenz Stoer (c. 1540-1620) and [3] by Daniele Barbaro (1513-1570). Coadjutor bishop of Aquileia (Italy), Barbaro, had held important roles in Venetian politics (ambassador in England, delegate of Venice at the Council of Trent etc.) and his treatise is one of the most significant of the century on perspective. Barbaro

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[13] On its diffusion in Germany and beyond, see [40].

used Bradwardine's construction of star polygons extending the sides of regular polygons, as well as drawing their diagonals:

> Et se prolungherai i lati della superficie di 5, 6, 7 e più lati […] farai simiglianti figure come appare nelle figure […] se tirerai le linee dagli anguli agli anguli & dai lati ai lati & dai lati agli anguli.[14]

Influenced by Dürer and Pacioli, Barbaro for the first time showed the developments of star polyhedra; the first description was related to Pacioli's elevatum tetrahedron:

> Spiegatura d'Alcuni corpi fondati sopra la soperficie dei corpi sì regulari come irregulari, & prima di quello, il quale è fondato sopra la piramide. Molto dilettevole è la pratica seguente & ha di belle considerationi, imperocchè ella trova il modo con la quale sopra le superficie piane de i corpi regulari come irregulari si fanno le piramidi di molti lati come si vede della spiegatura di dodici trianguli di lati uguali rinchiusa & posta insieme forma un corpo di molte punte fondate sopra la piramide & si hanno a ponere insieme secondo i numeri notati nelle soperficie triangulari come appare nella figura.[15]

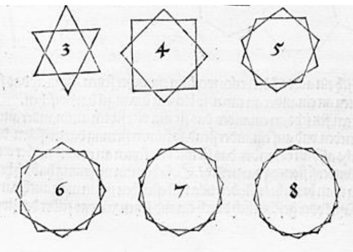

Fig. 4. Picture from [18], second part, Fig. 27.

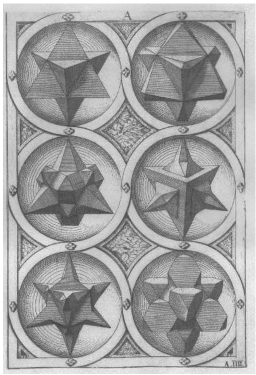

Fig. 5. Picture from [21], carta VIII r. Reproduced with permission from www.bncf.firenze.sbn.it

Barbaro also showed the elevatum cube, the stella octangula, the "Spiegatura d'un corpo sostenuto dallo icosaedro" (elevatum icosahedron) and the "spiegature" of other three star polyhedra among which the elevatum icosidodecahedron. Only for

[14] And if you will extend the sides of the surface of 5, 6, 7 […] and more sides you will make similar shapes as it appears in the figures […] if you draw the lines from the angles to the angles & sides to sides & sides to angles.

[15] Developments of some shapes built on the surface of both regular and irregular & before that is built on the pyramid. The following practice is of great pleasure and it has beautiful remarks because it explains how to build pyramids of many faces on both regular and irregular solids, as we can see the development of twelve triangles of equal sides enclosed & placed together forms a solid of many vertices placed on the pyramid & we have to put together according to the numbers noted in the triangular surfaces as it appears in the figure.

three polyhedra, Barbaro showed the perspective view (fig. 6). He had, like Pacioli, the conviction that he could proceed indefinitely. This conviction was inevitable, in the absence of a clear definition and classification of the examined objects. For a first attempt in this direction, we will have to wait for Kepler's work. Barbaro's book, however, was the first real step forward on star polyhedra. While strictly following Pacioli's constructive idea, the Venetian mathematician laid the foundations for further developments by artists and artisans, thanks to his precise indications for the actual construction of such polyhedra. The mathematician Simon Stevin (1548–1620), too, was profoundly influenced by Dürer; in the third of the five books of [37], he described ways to build Archimedean polyhedra by cutting off or augmenting Dürer's solids:

> Besides the five regular solids mentioned by mathematicians, we draw attention to some other solids which, though they do not have so great regularity as required in these five regular solids […] nevertheless would be full of Geometrical speculations and of a remarkable arrangement of the correlative faces. Now six of these solids have been mentioned by Albert Dürer in his Geometry.

Stevin, starting from regular polyhedra, divided all the edges of the solid into two or three parts and cut each solid angle by a plane passing through the points of division of the edges adjacent to it. Stevin also described augmented regular solids obtained by placing on top of each face of a regular polyhedron a pyramid with equal edges. He mentioned a conversation with Frans Cophart, the leader of the Collegium Musicum Leiden. In our opinion, this conversation is a rare testimony not only of the development of ideas born from the intertwining between mathematical knowledge and artistic intuition, but also of the effort of mathematicians to clarify fundamental definitions, such as the regular polyhedron:

> the extraordinary lover of Geometry wanted to persuade me that he happened to have found a sixth regular solid whose construction was as follows: draw all the diagonals of all the squares of a cube, and then draw planes from all the solid angles of the cube through two diagonals up to the mid-points of said diagonals, and in this way cut off all the sides of the faces of the cube, with the adjacent solid part of the cube included between two intersecting planes. And thus the cube (since it has twelve edges) will have twelve incisions; there remains an elegant solid included by twenty-four equal equilateral triangles.

The solid is what is now called the "stella octangula". Stevin, while admiring the discovery, had to deny this claim, because the vertices of Cophart's solid do not lie all on one sphere, but are distributed across two spheres. At the same time, he discovered another way of constructing the solid, by starting from an octahedron and then augmenting it by placing a pyramid on each face. By applying the same procedure to all regular bodies, he obtained four new polyhedra. Stevin then calculated the length of the edges of the polyhedra inscribed in the same sphere and showed their developments on the plane, adding a description of how to build the solids. For example, on the augmented tetrahedron he wrote:

> […] dispose four [equilateral] triangles, as in the preceding first section, each of whose sides be equal to the line *G*. Subsequently four times three triangles such as the three triangles 1, 2, 3, each of whose sides be equal to the said *G*.

Stevin showed the same "augmented" polyhedra of Pacioli, but apparently, he did not know him and in any case, he did not mention him.
With Stevin, we consider concluded what can be defined as the prehistory of star polyhedra. The first chapter of the real mathematical history of polyhedra begins with Kepler. In 1619, Johannes Kepler (1571-1630) in [24] showed two star polyhedra, calling them regular ("most perfect regular"). See also [1, 5, 22, 23]. Kepler obtained these polyhedra by "stellation" from the dodecahedron and the icosahedron, so as to have star polygons as faces. Defining regular star polygons the ones created by extending the sides of regular polygons, Kepler recognised as regular the polyhedra having pentagrams as faces, i.e. the *small stellated dodecahedron* and the *great stellated dodecahedron* (both with 12 faces) (fig. 7). These, with their duals discovered by Poinsot, are the only regular star polyhedra.

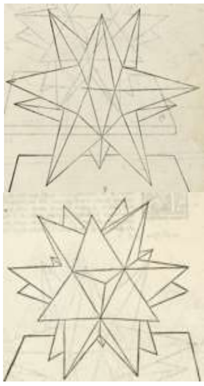

Fig. 6. Pictures from [3], pp. 111, 112.

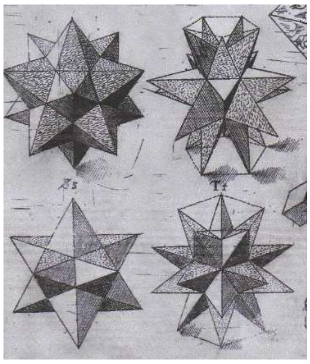

Fig. 7. Pictures from [24].

Following the discussion by Lakatos mentioned in the beginning, we could say that it is with Kepler that the dialectic *definition – proof – refutation – new definition* acquired a proper mathematical content. We should not forget, however, that these polyhedra already had illustrious predecessors in the field of art, with Uccello and Jamnitzer. It is only two centuries after Kepler, that we will have new studies (and rediscoveries) on star polyhedra. As rightly pointed out by Lakatos, the consistent application of the definitions led to explore this "new" mathematical topic. Louis Poinsot (1777-1859) was the first to study it in [33], see also [34]; he wrote:

> Cela posé, je dis que l'on peut construire de nouveaux solides parfaitement réguliers: ils ont tous leurs faces égales et régulières, également inclinées deux à deux, et assemblées en même nombre autour de chaque sommet. Ils peuvent être inscrits et circonscrits à la sphère; et quoiqu'ils présentent au dehors des cavités et des saillies, ils sont convexes

> suivant cette définition générale, que tous leurs angles dièdres sont au-dessous de deux angles droits. La différence essentielle de ces solides aux polyèdres ordinaires, est que, dans ceux-ci les faces étant projetées par des rayons sur la sphère inscrite ou circonscrite, les polygonés correspondant recouvrent une seule fois la sphère; au lieu que dans les autres, ces polygones la recouvrent exactement ou deux fois, ou trois fois, &c.; et cela d'une manière uniforme.[16]

As we can see, there are several points in the quote above, where definitions are clarified. For example, the concept of "cover", where the projection of a polyhedron is assimilated to the tiling of a sphere, or the different ways of conceiving the term "convex". Poinsot did not know Kepler's text, thus he rediscovered two of Kepler's regular polyhedra and added their duals: the *great icosahedron* (with 12 vertices, 30 edges, 20 faces)—dual of the great stellated dodecahedron—and the *great dodecahedron* (with 12 vertices, 30 edges, 12 faces)—dual of the small stellated dodecahedron. Some years later Cauchy, following Poinsot, proved that these four are the only possible regular star polyhedra [9, 10].
As we can see, the discussion on regular polyhedra, born with Plato and developed by Stevin's musician friend, continued to develop until very recent times, with some unexpected results. Examining semi-regular polyhedra (here we use this term generically, because different authors use it in different way), the situation becomes more difficult. We go from thirteen Archimedean solids classified by Kepler and rediscovered several times, to their duals, discovered and classified in [10]. As to star polyhedra, Jean Paul Albert Badoureau (1853-1923) in [2] took a decisive step forward at the end of the century. He gave a new and more precise definition (as well as a new name) to semi-regular polyhedra:

> Je désigne sous le nom de polyèdres isocèles des polyèdres formés par des polygones réguliers, convexes ou étoilés, et tels qu'on puisse les faire coïncider avec eux-mêmes ou avec leurs symétriques, en plaçant un sommet sur n'importe quel autre[17]

Symmetries gain a growing importance, in parallel with the development of group theory in mathematics. The inspiration for the most advanced mathematics came directly from the progress of crystallography:

> J'ai pu simplifier la théorie des polyèdres isocèles convexes, au moyen de considérations empruntées soit à la Géométrie élémentaire, soit à la Cristallographie, soit aux notions introduites dans la science par Bravais et développées par M. Jordan.[18]

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[16] I say that we can construct new perfectly regular solids: they have equal and regular, also inclined two by two, faces, and assembled in the same number around each vertex. They can be inscribed and circumscribed to the sphere and although they have cavities and protrusions outward, they are convex according to this general definition that all their dihedral angles are smaller than two right angles. The essential difference between these solids and the ordinary polyhedra is that, in these the faces projected by the rays on the inscribed or circumscribed sphere, the corresponding polygons cover the sphere only once; instead of the others, these polygons cover it exactly twice, or three times, etc.; always in a uniform way.
[17] I designated with the name of "isosceles polyhedra" the polyhedra composed by regular, convex or star polygons, and such that we can make them coincide with themselves or with their symmetries, placing a vertex on any other.

Badoureau also highlighted the potential connection with Arab art culture:

> Les assemblages isocèles étoilés se déduisent des assemblages convexes […] Les figures auxquelles il donnent lieu pourraient bien avoir été connues des géomètres arabes, si l'on en juge par leur analogie avec les dessins dont l'art oriental aime à orner ses créations.[19]

Afterwards, for many years, the history of the study of polyhedra went on in the absence of a clear mathematical strategy, but with the discovery of new individual figures; in 1881, Johann Pitsch added four new polyhedra to Badoureau's thirty-seven solids. Only in the 1950s, a new important progress occurred in connection with a mathematically coherent vision, thanks to the work of Miller, Longuet–Higgins and Coxeter [14]. The latter showed seventy-five uniform polyhedra, but could not yet prove that the list was complete, which was only done in the 1970s [35, 36].

This history, started with Platonic polyhedra, seems to be analogous to the history of the discovery and definition of "element" in physics and chemistry: from the discovery of single chemical elements, to the periodic table, up to the Bohr model of the atom, which allowed to frame new individual discoveries within an overall theory. In the end, we can assert that our Bohr's atom was the group theory.

In conclusion, we may ask whether the deepest beauty of this story lies in the aesthetics of forms or in the wonderful intellectual adventure that led to find a logic and rational order in the endless variety of these forms.

[18] I was able to simplify the theory of convex isosceles polyhedra by means of considerations from elementary geometry, crystallography, or notions introduced into science by Bravais and developed by M. Jordan.

[19] The assemblages of starred isosceles are derived from convex assemblies [...] The arising figures were well known by Arab geometers, if we judge by analogy with the drawings of which oriental art loves to adorn its creations.

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