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Local Spectral Theory for R and S Satisfying $R^nSR^n = R^j$

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Abstract: In this paper, we analyze local spectral properties of operators R, S and RS which satisfy the operator equations $R^nSR^n = R^j$ and $S^nRS^n = S^j$ for same integers $j \geq n \geq 0$. We also continue to study the relationship between the local spectral properties of an operator R and the local spectral properties of S. Thus, we investigate the transmission of some local spectral properties from R to S and we illustrate our results with an example. The theory is exemplified in some cases.

Keywords: local spectral subspaces; Dunford's property (C) and property (β); Drazin invertible operators

1. Introduction

In this paper, we continue the analysis undertaken in [1–6] on the general problem of study the local spectral properties for R, S, RS and $SR \in L(X)$ in the case R and S satisfy the operator equations $R^nSR^n = R^j$ for same integers $j \geq n \geq 0$. Following the procedure of [1], we study the relationship of Dunford property (C) for products R^nS and SR^n for operator $R^{j-n} \in L(X)$ which satisfy the operator equations

$$SR^n = R^{j-n} \text{ for same integers } j \geq n \geq 0, \tag{1}$$

and hence

$$R^nSR^n = R^j \text{ for same integers } j \geq n \geq 0. \tag{2}$$

The paper is organized as follows.

In Section 2, to keep the paper sufficiently self-contained, we collect some preliminary definitions and propositions that are used in what follows. In Section 3, we show some results concerning the transmission of some local spectral properties from R to S. In Section 4, we give an example that plays a crucial role for the theory. The final considerations are given in Section 5.

2. Notation and Complementary Results

A bounded operator $T \in L(X)$ on a complex infinite dimensional Banach space X is said to have *the single valued extension property* at $\lambda_o \in \mathbf{C}$. In short, T has the SVEP at λ_o, if for every open disc $\mathbf{D}_{\lambda_o}$ centered at λ_o the only analytic function $f : \mathbf{D}_{\lambda_o} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \tag{3}$$

is the constant function $f \equiv 0$.

T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbf{C}$.

To facilitate the reader, we remember that the SVEP is a typical tool of the local spectral theory. If $\rho_T(x)$ denote the local resolvent set of T at the point $x \in X$, defined as the union of all open subsets $\mathcal{U}$ of $\mathbf{C}$ for which there exists an analytic function $f : \mathcal{U} \to X$ that satisfies

$$(\lambda I - T)f(\lambda) = x \text{ for all } \lambda \in \mathcal{U} \tag{4}$$

then the local spectrum $\sigma_T(x)$ of T at x is defined by

$$\sigma_T(x) := \mathbf{C} \setminus \rho_T(x),$$

and, obviously, $\sigma_T(x) \subseteq \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T.

Remark 1. *Let $\lambda \in \rho_T(x)$ and $\mathcal{U}$ denotes an open neighborhood of λ. If $f : \mathcal{U} \to X$ satisfies the equation $(\lambda I - T)f(\mu) = x$ on $\mathcal{U}$, then $\sigma_T(f(\lambda)) = \sigma_T(x)$ for all $\lambda \in \mathcal{U}$ (see [7], Lemma 1.2.14). Moreover, $0 \in \sigma_{\lambda I - T}(x)$ if and only if $\lambda \in \sigma_T(x)$.*

Theorem 1. *Let $T \in L(X)$, X a Banach space. Then, T has SVEP if and only if every $0 \neq x \in X$ the local spectrum $\sigma_T(x)$ is non-empty.*

Proof. See ([7], Proposition 1.2.16). □

The SVEP has a decisive role in local spectral theory it has a certain interest to find conditions for which an operator has the SVEP.

Definition 1. *Let T is a linear operator on a vector space X. The* hyperrange *of T is the subspace*

$$T^\infty(X) := \bigcap_{n \in \mathbf{N}} T^n(X).$$

Generally, $T(T^\infty(X)) \subseteq T^\infty(X)$, thus we are interested in finding conditions for which $T(T^\infty(X)) = T^\infty(X)$. For every linear operator T on a vector space X, there corresponds the two chains:

$$\{0\} = \ker T^0 \subseteq \ker T \subseteq \ker T^2 \cdots .$$

and

$$X = T^0(X) \supseteq T(X) \supseteq T^2(X) \cdots .$$

The ascent of T is the smallest positive integer $p = p(T)$, whenever it exists, such that $\ker T^p = \ker T^{p+1}$. If such p does not exist, we let $p = +\infty$. Analogously, the descent of T is defined to be the smallest integer $q = q(T)$, whenever it exists, such that $T^{q+1}(X) = T^q(X)$. If such q does not exist, we let $q = +\infty$.

It is possible to prove that, if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$. Note that $p(T) = 0$ means that T is injective, and $q(T) = 0$ that T is surjective.

Theorem 2. *If $T \in L(X)$ and X is a Banach space, then*

$$T \text{ does not have the SVEP at } 0 \Rightarrow p(T) = \infty. \tag{5}$$

As noted in [1] (Lemma 1.1), the local spectrum of Tx and x may differ only at 0, i.e., For every $T \in L(X)$ and $x \in X$, we have

$$\sigma_T(Tx) \subseteq \sigma_T(x) \subseteq \sigma_T(Tx) \cup \{0\}. \tag{6}$$

Moreover, if T is injective, then

$$\sigma_T(Tx) = \sigma_T(x) \text{ for all } x \in X. \tag{7}$$

For every subset $F \subseteq \mathbf{C}$, the *analytic spectral subspace* of T associated with F is the set

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}$$

For every subset $F \subseteq \mathbf{C}$, the *global spectral subspace* $\mathfrak{X}_T(F)$ consists of all $x \in X$ for which there exists an analytic function $f : \mathbf{C} \setminus F \to X$ that satisfies

$$(\lambda I - T)f(\lambda) = x \text{ for all } \lambda \in \mathbf{C} \setminus F. \tag{8}$$

In general, $\mathfrak{X}_T(F) \subseteq X_T(F)$ for every closed sets $F \subseteq \mathbf{C}$. The identity $X_T(F) = \mathfrak{X}_T(F)$ holds for all closed sets $F \subseteq \mathbf{C}$ whenever T has SVEP, precisely. T has SVEP if and only if $X_T(F) = \mathfrak{X}_T(F)$ holds for all closed sets $F \subseteq \mathbf{C}$.

Definition 2. *The analytical core $K(\lambda I - T)$ of $\lambda I - T$ is the set*

$$K(\lambda I - T) := X_T(\mathbf{C} \setminus \{\lambda\}) = \{x \in X : \lambda \notin \sigma_T(x)\} \tag{9}$$

The analytic core of an operator T is an invariant subspace, which, in general, is not closed [8].

Definition 3. *An operator $T \in L(X)$ is said to be upper semi-Fredholm, $T \in \Phi_+(X)$, if T(X) is closed and the kernel kerT is finite-dimensional. An operator $T \in L(X)$ is said to be lower semi-Fredholm, $T \in \Phi_-(X)$ if the range $T(X)$ has finite codimension.*

Definition 4. *An operator $T \in L(X)$ is said to be Drazin invertible if there exist $C \in Ł(X)$ such that*

1. *$T^m(X) = T^{m+1}C$ for some integer $m \geq 0$;*
2. *$C = TC^2$; and*
3. *$TC = CT$*

In this case, C is called *Drazin inverse* of T and the smallest $m \geq 0$ in (4) is called the *index* $i(T)$ of T.

3. Operator Equation $R^nSR^n = R^j$

As mentioned in the Introduction, in this section, we show some results concerning the transmission of some local spectral properties from R to S.

We study the relationship between the local spectral properties of an operator R and the local spectral properties S, if this exists. In particular, we study a reciprocal relationship, analogous to that of (2). We also show that many local spectral properties, such as SVEP and Dunford property (C), are transferred from operator R to S somehow through a bond. While these properties are, in general, not preserved under sums and products of commuting operators, we obtain positive results in the case of our perturbations.

We suppose that $R, S \in Ł(X)$ satisfy $R^nSR^n = R^j$ for some integers $j \geq n \geq 0$. The case $n = 2$ and $j = 1$ is studied in [1,9,10]; if $n = j = 1$, the operators A and B are relatively regular.

Moreover, if $T \in Ł(X)$ is *Drazin invertible* operator with $i(T) = k$, then, by (4),

$$T^{2k+1} = T^{k+1}CT^{k+1}.$$

Therefore, in this case, $j = 2k + 1$ and $n = k + 1$.

Lemma 1. *For every $x \in X$, we have*

$$\sigma_{R^{j-n}}(R^n x) \subseteq \sigma_{SR^n}(x). \tag{10}$$

Moreover,

$$\sigma_{SR^n}(SR^n x) \subseteq \sigma_{R^{j-n}}(x), \qquad \sigma_{SR^n}(SR^n x) \subseteq \sigma_{R^{j-n}}(R^n x) \tag{11}$$

Proof. Suppose that $\lambda_0 \in \rho_{SR^n}(x)$; then, there exists an open neighborhood $\mathcal{U}_0$ if λ_0 and an analytic function $f : \mathcal{U}_o \to X$ such that

$$(\lambda I - SR^n)f(\lambda) = x \qquad \text{for all } \lambda \in \mathcal{U}_0. \tag{12}$$

From this, it then follows that

$$\begin{aligned} R^n x &= R^n(\lambda I - SR^n)f(\lambda) = (\lambda R^n - R^n SR^n)f(\lambda) \\ &= (\lambda R^n - R^j)f(\lambda) = (\lambda I - R^{j-n})R^n f(\lambda), \end{aligned}$$

for all $\lambda \in \mathcal{U}_0$. Hence, $\lambda_o \in \rho_{R^{j-n}}(R^n x)$; thus,

$$\sigma_{R^{j-n}}(R^n x) \subseteq \sigma_{SR^n}(x).$$

To show the first inclusion (11), let $\lambda_0 \in \rho_{R^{j-n}}(x)$; then, there exists an open neighborhood $\mathcal{U}_o$ of λ_0 and an analytic function $f : \mathcal{U}_o \to X$ such that

$$(\lambda I - R^{j-n})f(\lambda) = x \qquad \text{for all } \lambda_0 \in \mathcal{U}_0.$$

Consequently,

$$\begin{aligned} SR^n x &= SR^n(\lambda I - R^{j-n})f(\lambda) = (\lambda SR^n - SR^j)f(\lambda) \\ &= (\lambda SR^n - SR^n SR^n)f(\lambda) = (\lambda SR - [SR^n]^2)f(\lambda) \\ &= (\lambda I - SR^n)SR^n f(\lambda), \end{aligned}$$

for all $\lambda \in \mathcal{U}_0$, and since $SR^n f(\lambda)$ is analytic, we obtain $\lambda_0 \in \rho_{SR^n}(SR^n x)$. Hence, this shows the first inclusion of (11). To show the second inclusion, let $\lambda_0 \in \rho_{R^{j-n}}(R^n x)$; then, there exists an open neighborhood $\mathcal{U}_o$ of λ_0 and an analytic function $f : \mathcal{U}_o \to X$ such that

$$(\lambda I - R^{j-n})f(\lambda) = R^n x \qquad \text{for all } \lambda_0 \in \mathcal{U}_0.$$

Consequently, the argument is similar to that first part. □

Theorem 3. *Suppose that $\mathcal{F}$ is a closed subset of $\mathbf{C}$ and $0 \in \mathcal{F}$. Then, $X_{R^{j-n}}(\mathcal{F})$ is closed if and only if $X_{SR^n}(\mathcal{F})$ is closed.*

Proof. Suppose that $X_R^j(\mathcal{F})$ is closed and let (x_m) be a sequence of $X_{SR^n}(\mathcal{F})$ which converges to $x \in X$. Then, for every $m \in \mathbf{N}$, we have $\sigma_{SR^n}(x_m) \subseteq \mathcal{F}$. By (10), we have $\sigma_{R^{j-n}}(R^n x_m) \subseteq \mathcal{F}$. Since $0 \in \mathcal{F}$, by (6) where $T = R^{j-n}$, we have $\sigma_{R^{j-n}}(R^n x_m) \subseteq \sigma_{R^{j-n}}(R^j x_m) \cup \{0\} \subseteq \mathcal{F}$. Therefore, $R^j x_m \in X_{R^{J-n}}(\mathcal{F})$ i.e., $R^{j-n}R^n x_m \in X_{R^{J-n}}(\mathcal{F})$. By [9] (Lemma 2.3) , $R^n x_m \in X_{R^{J-n}}(\mathcal{F})$ and by assumption $X_{R^{j-n}}(\mathcal{F})$ is closed. We then have $R^n x \in X_{R^{j-n}}(\mathcal{F})$, i.e., $\sigma_{R^{j-n}}(x) \subseteq \mathcal{F}$. By (11),

$$\sigma_{SR^n}(SR^n x) \subseteq \sigma_{R^{j-n}}(x) \subseteq \mathcal{F}.$$

Then, $SR^n x \in X_{SR^n}(\mathcal{F})$, by [9] (Lemma 2.3) $x \in X_{SR^n}(\mathcal{F})$, thus $X_{SR^n}(\mathcal{F})$ is closed. Conversely, suppose that $X_{SR^n}(\mathcal{F})$ is closed and let (x_m) be a sequence of $X_{R^{j-n}}(\mathcal{F})$ which converges to $x \in X$;

then, $\sigma_{R^{j-n}}(x_m) \subseteq \mathcal{F}$ for every $m \in \mathbf{N}$. By (11), $\sigma_{SR^n}(SR^n x_m) \subseteq \mathcal{F}$, and then $SR^n x_m \in X_{SR^n}(\mathcal{F})$. By [9] (Lemma 2.3) $x_m \in X_{SR^n}(\mathcal{F})$, therefore $x \in X_{SR^n}(\mathcal{F})$. Hence, $\sigma_{SR^n}(x) \subseteq \mathcal{F}$. Since by (10) $\sigma_{R^{j-n}}(R^n y) \subseteq \sigma_{SR^n}(y)$ for all $y \in X$, then, if $y = R^{j-n}x$, we have $\sigma_{R^{j-n}}(R^J x) \subseteq \sigma_{SR^n}(R^{j-n}x)$. By (6), we have

$$\begin{aligned}\sigma_{R^{j-n}}(R^{j-n}x) &\subseteq \sigma_{R^{j-n}}(R^j x) \cup \{0\} \subseteq \sigma_{SR^n}(R^{j-n}x) \cup \{0\} \\ &\subseteq \sigma_{SR^n}(SR^j x) \cup \{0\} \subseteq \sigma_{SR^n}[(SR^n)^2 x] \cup \{0\} \\ &\subseteq \sigma_{SR^n}(SR^n x) \cup \{0\} \subseteq \sigma_{SR^n}(x) \cup \{0\} \subseteq \mathcal{F}\end{aligned}$$

i.e., $R^{j-n}x \in X_{R^{j-n}}(\mathcal{F})$. Hence, $\sigma_{R^{j-n}}(R^{j-n}x) \subseteq \mathcal{F}$ i.e., $R^{j-n}x \in X_{R^{j-n}}(\mathcal{F})$. By [9] (Lemma 2.3) $x \in X_{R^{j-n}}(\mathcal{F})$. □

The following result is inspired by [1] and ([11], Theorem 2.1).

Lemma 2. *Let $S, R \in L(X)$ be such that $R^n SR^n = R^j$ for same integers $j \geq n \geq 0$. If R^{j-n} has SVEP, then SR^n and $R^n S$ have SVEP.*

Proof. By ([12], Proposition 2.1), SR^n has SVEP if and only if $R^n S$ has Svep. Suppose that R^{j-n} has SVEP at λ_0 and let $f : \mathcal{U}_0 \to X$ be an analytic function for which $(\lambda I - SR^n)f(\lambda) = 0$ for all $\lambda \mathcal{U}_0$. Then, $SR^n f(\lambda) = \lambda f(\lambda)$.

$$\begin{aligned}R^n(\lambda I - SR^n)f(\lambda) &= \\ (\lambda R^n - R^j)f(\lambda) &= (\lambda I - R^{j-n})R^n f(\lambda) = 0.\end{aligned}$$

The SVEP of R^{j-n} at λ_0 implies that $R^n f(\lambda) = 0$ and hence $SR^n f(\lambda) = \lambda f(\lambda) = 0$. Thus, if $0 \notin \mathcal{U}_0$, then $f(\lambda) = 0$ for $\lambda \neq 0$ and by continuity $f(0) = 0$. Therefore, SR^n has SVEP at λ_0. □

We now consider the case where $0 \notin \mathcal{F}$

Theorem 4. *Let $\mathcal{F}$ be a closed subset of $\mathbf{C}$ such that $0 \notin \mathcal{F}$. Suppose that R, $S \in Ł(X)$ satisfy $R^n SR^n = R^j$ for some integers $j \geq n \geq 0$ and R^{j-n} has SVEP. If $X_{R^{j-n}}(\mathcal{F})$ is closed, then $X_{SR^n}(\mathcal{F})$ is closed.*

Proof. Let $\mathcal{F}_1 := \mathcal{F} \cup \{0\}$; by assumption, $X_{R^{j-n}}(\mathcal{F}_1)$ is closed. By (3), $X_{SR^n}(\mathcal{F}_1)$ is closed. By (2), SR^n has SVEP; therefore, by ([9], Lemma 1.4), $X_{SR}(\mathcal{F})$ is closed. □

Definition 5. *An operator $T \in L(X)$ is said to have Dunford's property (abbreviated property (C)) if $X_T(\mathcal{F})$ is closed for every closed set $\mathcal{F} \subseteq \mathbf{C}$*

It is known that Dunford property (C) entails SVEP for T.

Theorem 5. *Let $S, R \in L(X)$ be such that $R^n SR^n = R^j$ for some integers $j \geq n \geq 0$. If R^{j-n} has the property (C), then SR^n and $R^n S$ have the property (C).*

Proof. Suppose that $\mathcal{F}$ is a closed set and R^{j-n} has property (C); then, R^{j-n} has SVEP. If $0 \in \mathcal{F}$, by (3) and by assumptions $X_{R^{j-n}}(\mathcal{F})$ is closed, it follows that $X_{SR^n}(\mathcal{F})$ is closed. Similarly, if $0 \notin \mathcal{F}$, then by (4) we have that $X_{SR^n}(\mathcal{F} \cup \{0\})$ is closed. Therefore, SR^n has property (C). □

We prove that somehow there exists a bond, i.e., SR and RS share Dunford's property (C) when $R^n SR^n = R^j$ for same integers $j \geq n \geq 0$.

Definition 6. *An operator $T \in L(X)$ is said to have property (Q) if the* quasi-nilpotent part $H_0(\lambda I - T)$ *of $\lambda I - T$ defined by*

$$H_0(\lambda I - T) := \{x \in X : \limsup_{n\to\infty} \|(\lambda I - T)^n x\|^{1/n} = 0\}$$

is closed for every $\lambda \in \mathbf{C}$.

It is known that

$$\text{Property}(C) \Rightarrow \text{Property}(Q) \Rightarrow SVEP,$$

and moreover for operator T we have $\mathfrak{X}_T(\lambda) = H_0(\lambda I - T)$.

Then, if T has SVEP,

$$X_T(\lambda) = \mathfrak{X}_T(\lambda) = H_0(\lambda I - T). \tag{13}$$

Every multiplier of a semi-simple commutative Banach algebra has property (Q), see ([13], Theorem 1.8), in particular every convolution operator T_μ, $\mu \in M(G)$, on the group algebra $L_1(G)$ has property (Q), but there are convolution operators which do not enjoy property (C) (see [7], Chapter 4).

Observe that, if T has property (Q) and f is an injective analytic function defined on an open neighborhood U of $\sigma(T)$, then $f(T)$ also has property (Q). To see this, recall first that the equality

$$\mathcal{X}_{f(T)}(\mathcal{F}) = \mathcal{X}_T(f^{-1}(\mathcal{F})) \tag{14}$$

holds for every closed subset of $\mathbb{C}$ and every analytic function f on an open neighborhood U of $\sigma(T)$, see ([7], Theorem 3.3.6). Now, to show that $f(T)$ has property (Q) amd f is injective, we have to prove that $H_0(\lambda I - f(T))$ is closed for every $\lambda \in \mathbb{C}$. If $\lambda \notin \sigma(f(T))$, then $H_0(\lambda I - f(T)) = \{0\}$, while, if $\lambda \in \sigma(f(T)) = f(\sigma(T))$, then

$$H_0(\lambda I - f(T)) = X_{f(T)}(\{\lambda\}) = \mathcal{X}_T(f^{-1}\{\lambda\}) = H_0(\mu I - T),$$

where $f(\lambda) = \mu$, and, consequently, $H_0(\lambda I - f(T))$ is closed. In particular, considering the function $f(\lambda) := \frac{1}{\lambda}$, we see that, if T is invertible and has property (Q), then its inverse has property (Q). Furthermore, property (Q) for T implies property (Q) for T^n, for every $n \in \mathbb{N}$.

Theorem 6. *Let $S, R \in L(X)$ be such that $R^n S R^n = R^j$ for some integers $j \geq n \geq 0$. If R^{j-n} has the property (Q), then SR^n has the property (Q).*

Proof. Suppose that R^{j-n} has property (Q). Then, R^{j-n} has SVEP, hence by Lemma 2 SR^n has SVEP. Therefore, by (13) and by assumption, $H_0(\lambda I - R^{j-n}) = X_{R^{j-n}}(\{\lambda\})$ is closed for every $\lambda \in \mathbf{C}$. By (13) and (3), $H_0(SR^n) = X_{SR^n}(\{0\})$ is closed. Following the procedure of [1], let $0 \neq \lambda \in \mathbf{C}$; by ([7], Proposition 3.3.1, part (f)) we have

$$X_{R^{j-n}}(\{\lambda\} \cup \{0\}) = X_{R^{j-n}}(\{\lambda\}) + X_{R^{j-n}}(\{0\}) = H(\lambda_0 I - R^{j-n}) + H_0(R^{j-n}).$$

Since R^{j-n} is upper semi-Fredholm, the SVEP at 0 implies that $H_0(R^{j-n})$ is finite-dimensional (see [8], Theorem 3.18). Then, $X_{R^{j-n}}(\{\lambda\} \cup \{0\})$ is closed. By Theorem 5, we then have

$$H_0(\lambda I - SR^n) = X_{SR^n}\{\lambda\}$$

is closed, therefore SR^n has property Q. □

Following the procedure of [1] (Theorem 3), it is possible to prove the following:

Theorem 7. *Let $S, R \in L(X)$ be such that $R^n S R^n = R^j$ for same integers $j \geq n \geq 0$.*

1. *(i) If $0 \neq \lambda \in \mathbf{C}$, then $K(\lambda I - R^{j-n})$ is closed if and only $K(\lambda I - SR^n)$ is closed, or equivalently $K(\lambda I - R^n S)$ is closed.*
2. *(ii) If R^{j-n} is injective, then $K(\lambda I - R^{j-n})$ is closed if and only $K(\lambda I - SR^n)$ is closed, or equivalently $K(\lambda I - R^n S)$ is closed for all $\lambda \in \mathbf{C}$.*

Corollary 1. *Suppose $R^n S R^n = R^j$, $S^j R S^J = S^n$, for some integers $j \geq n \geq 0$ and $\lambda \neq 0$. Then, the following statements are equivalent:*

1. *$K(\lambda I - R^j)$ is closed.*
2. *$K(\lambda I - SR^n)$ is closed.*
3. *$K(\lambda I - R^n S)$ is closed.*
4. *$K(\lambda I - S^n)$ is closed.*

When R is injective, the equivalence also holds for $\lambda \neq 0$.

Proof. The equivalence of (3) and (4) follows from Theorem 3. Since, the injectivity of R is equivalent to the injectivity of S, the equivalence of (1) and (4) also holds for $\lambda = 0$. □

We show now that property (Q) is also transmitted between operators R into S. Let $S, R \in L(X)$ be such that $R^n S R^n = R^j$ for some integers $j \geq n \geq 0$. If R has the property (Q) and R^{j-n} has the property (Q), then SR^n has the property (Q), therefore S^n has the property (Q), thus S has the property (Q).

4. Example: Drazin Invertible Operators

In this section, we give an example that plays a crucial role for the theory, of operators R, $S \in Ł(X)$ that satisfy the equation $R^n C R^n = R^j$ for some integers $j \geq n \geq 0$.

In the literature, the concept of invertibility admits several generalizations. Another generalization of the notion of invertibility, which satisfies the relationships of "reciprocity" observed above, is provided by the concept of *Drazin invertibility*.

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory [14]. In the case of the Banach algebra $L(X)$, $R \in L(X)$ is said to be *Drazin invertible* (with a finite index) if there exists an operator $S \in L(X)$ and $n \in \mathbb{N}$ such that

$$RS = SR, \quad SRS = S, \quad R^n S R = R^n. \tag{15}$$

The smallest nonnegative integer ν such that (15) holds is called the index $i(R)$ of R. In this case, the operator S is called *Drazin inverse* of R.

Clearly, in this case,

$$R^n S R^n = R^n S R R^{n-1} = R^n R^{n-1} = R^j \text{ for same integers } j = 2n - 1 > n \geq 0. \tag{16}$$

Clearly, any invertible operator or a nilpotent operator R is Drazin invertible.

5. Conclusions

In this paper we give a proof that the operators S and R share property (Q) and in some modes Dunford's property (C); we prove further results concerning the local spectral theory of R, S, RS and SR, in particular we show several results concerning the quasi-nilpotent parts and the analytic cores of these operators. It should be noted that these results are established in a very general framework. Therefore, we hope to discuss some aspect in a further paper.

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