LE MATEMATICHE
Vol. LX (2005) – Fasc. II, pp. 363–369

SOME REMARKS ON THE EXTINCTION FOR THE MEANCURVATURE FLOW

B. BRANDOLINI - M. CICALESE - C. NITSCH - C. TROMBETTI

1. Introduction.

Let us consider a family of bounded open sets $(\Omega_t)_{t\geq 0}$ in $\mathbb{R}^n$ $(n \geq 2)$ and sets $\Gamma_t = \partial\Omega_t$. If Γ_t is a smooth $(n-1)$-dimensional hypersurface it is said to be moving by mean curvature if the following initial value problem is satisfied

$$\begin{cases} V = H & \text{on} \quad \Gamma_t \\ (\Gamma_t)_{t=0} = \Gamma_0 \end{cases} \tag{1.1}$$

where $V(x, t)$ and $H(x, t)$ denotes respectively the inward normal velocity and $(n-1)$ times the mean curvature of Γ_t at a point $x \in \Gamma_t$.

It is well known (see [9] for smooth convex, and [4] for general continuous hypersurfaces) that Γ_t shrinks to a point in a finite time t^* defined as

$$t^* = t^*(\Gamma_0) = \inf\{t : \Gamma_t \neq \emptyset\}$$

and called extinction time. The simplest upper bound estimate for t^* relies on a monotonicity property of the mean curvature equation according to which, given two sets Ω_0 and D_0 in $\mathbb{R}^n$ such that $\overline{\Omega_0} \subset D_0$, the inclusion remains true during the whole evolution of their boundaries: $\overline{\Omega_t} \subset D_t$. Therefore, denoting by d_0 the diameter of Ω_0, since Ω_0 lies in a ball of radius $R = \left(\frac{n}{2(n+1)}\right)^{1/2} d_0$, by the monotonicity it follows that t^* can be estimated with the extinction time of a ball of radius R, that is

$$0 \le t^* \le \frac{n}{4(n^2-1)} d_0^2. \tag{1.2}$$

This estimate is not sharp and it has been refined in [6], where the authors have proved that

$$0 \le t^* \le C \left((\mathcal{H})^{n-1}(\Gamma_0) \right)^{2/n-1}. \tag{1.3}$$

Here $(\mathcal{H})^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure and the constant $C = C(n)$ comes from a Sobolev type inequality on manifolds whose best constant is still unknown (see [10]). In this paper we will prove a sharp upper bound for t^* involving the $n-$dimensional measure of Ω_0 rather than the $(n-1)$-dimensional measure of its surface. More precisely we will show that the extinction time of Γ_0 can be estimated from above by the extinction time of the ball having the same volume as Ω_0. The sharpness of our estimate relies on an isoperimetric inequality involving the total mean curvature of mean convex sets (see Section 2 for definitions). For this reason our upper bound holds true in the case of general bounded convex sets and smooth mean convex sets.

2. Notation and Preliminaries.

We begin by recalling some definitions and properties of rearrangements of functions. Let Ω be a bounded open set of $\mathbb{R}^n$ and let $u : \Omega \to]-\infty, 0]$ be a measurable function. We denote by

$$\mu(\theta) = (\mathcal{L})^n(\{x \in \Omega : u(x) < \theta\}), \quad \theta \le 0,$$

the distribution function of u, where $(\mathcal{L})^n$ will denote here and in what follows the Lebesgue measure in $\mathbb{R}^n$, and by

$$u^*(s) = \sup\{\theta \le 0 : \mu(\theta) < s\}, \quad s \in (0, |\Omega|)$$

the increasing rearrangement of u. In the following we will denote by $\Omega^\#$ the ball centered at the origin having the same measure as Ω and by $u^\#$ the negative spherically symmetric increasing function whose level sets are balls having the same measure as the corresponding level sets of u. This means

$$u^\#(x) = u^*(\omega_n |x|^n) \quad x \in \Omega^\#,$$

where ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. If $u : \Omega \to]-\infty, 0]$ is a function whose level sets $\{x \in \Omega : u(x) = \theta\}$ have finite perimeter, then we denote by

$$\lambda(\theta) = \mathcal{H}^{n-1}(\{x \in \Omega : u(x) = \theta\}), \quad \theta \leq 0,$$

and we define the rearrangement of u with respect to the perimeter of its level sets as

$$u_1^*(s) = \sup\{\theta \leq 0 : \lambda(\theta) < s\}, \quad s \in (0, \mathcal{H}^{n-1}(\partial\Omega)).$$

We will denote by $\Omega^\bigstar$ the ball centered at the origin having the same perimeter as Ω and set

$$u^\bigstar(x) = u_1^*(n\omega_n|x|^{n-1}), \quad x \in \Omega^\bigstar.$$

We explicitly remark that $u^\bigstar$ is a negative spherically symmetric increasing function whose level sets are balls having the same perimeter as the corresponding level sets of u.

If Ω has a C^2 boundary, then the principal curvature (oriented so that convex sets have non-negative curvatures) will be denoted by $k_1, ..., k_{n-1}$ and $(n-1)$ times the mean curvature will be denoted by $H[\partial\Omega]$, that is

$$H[\partial\Omega] = k_1 + k_2 + ... + k_{n-1}.$$

It easily follows from the above definitions and the classical isoperimetric inequality that

$$H[\{u^\bigstar = \theta\}] \geq H[\{u^\# = \theta\}]. \tag{2.1}$$

According to [13] we will say that a domain Ω is mean convex or $1-$convex if and only if $H[\partial\Omega] \geq 0$.

Finally we recall the following Alexandrov-Fenchel inequality involving the total mean curvature of level sets of a function u (see [2], [12]).

Theorem 2.1. *Let u be a nonpositive measurable function having mean convex level sets; then*

$$\int_{u=\theta} H[\{u = \theta\}]\, d\mathcal{H}^{n-1} \geq \int_{u^\bigstar=\theta} H[\{u^\bigstar = \theta\}]\, d\mathcal{H}^{n-1}, \quad \theta < 0. \tag{2.2}$$

3. A sharp estimate of the extinction time.

We first recall a different approach to motion by mean curvature first proposed by Osher and Sethian in a numerical framework (see [11]) and then studied by Evans and Spruck in [4].
Let Ω be a bounded open set in $\mathbb{R}^n$ and let us choose a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Gamma_0 = \partial\Omega = \{x \in \mathbb{R}^n : \ f(x) = 0\}.$$

In the following parabolic problem

$$\begin{cases} w_t = |Du| \mathrm{div}\left(\dfrac{Dw}{|Dw|}\right) & \text{in } \mathbb{R}^n \times (0, T) \\ w(x, 0) = f(x); \end{cases} \tag{3.1}$$

the equation states that each level set of w evolves according to its mean curvature. Consequently, the evolution of Γ_0 is given by $\Gamma_t = \{x \in \mathbb{R}^n : w(x, t) = 0\}$, for each time $t > 0$. In particular, if Ω is a mean convex open set, we can set

$$w(x, t) = u(x) + t$$

and problem (3.1) becomes

$$\begin{cases} |Du| \mathrm{div}\left(\dfrac{Du}{|Du|}\right) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Note that, in general, neither a smooth solution to (3.1) nor a smooth solution to (3.2) exists, but it has been proved in [4] (see also [3], [5], [6] and [7]), that problems (3.1) and (3.2) admit a unique viscosity solution which provides a possible generalization of the classical mean curvature motion (1.1).

Proposition 3.1. *Let u be a smooth solution to problem* (3.2) *and let v be the solution of the following symmetrized problem*

$$\begin{cases} |Dv| \, \mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) = 1 & \textit{in } \Omega^\star \\ v = 0 & \textit{on } \partial\Omega^\star. \end{cases} \tag{3.3}$$

Then

$$0 \geq u^\#(x) \geq v(x), \quad x \in \Omega^\star. \tag{3.4}$$

Proof. Let $\theta \leq 0$; by integrating on the set $\{u < \theta\}$ the equation in (3.2), using the coarea-formula, and the fact that $\operatorname{div}\left(\frac{Du}{|Du|}\right)\Big|_{\{u=\sigma\}} = H[\{u = \sigma\}]$ we get

$$\mu(\theta) = \int_{-\infty}^{\theta} \left(\int_{u=\sigma} H[\{u = \sigma\}]\, d\mathcal{H}^{n-1} \right) d\sigma.$$

Differentiating with respect to θ, using the Alexandrov-Fenchel inequality (2.2) and (2.1) we have

$$\begin{aligned} \mu'(\theta) &= \int_{u=\theta} H[\{u = \theta\}] d\mathcal{H}^{n-1} \geq \int_{u^\star=\theta} H[\{u^\star = \theta\}] d\mathcal{H}^{n-1} \\ &\geq \int_{u^\#=\theta} H[\{u^\# = \theta\}] d\mathcal{H}^{n-1} = C_n \mu(\theta)^{(n-2)/n}, \end{aligned}$$

where $C_n = n(n-1)\omega_n^{2/n}$. Thus μ solves the following problem

$$\begin{cases} \mu'(\theta) \geq C_n \mu(\theta)^{(n-2)/n}, & \theta \leq 0 \\ \mu(0) = |\Omega|. \end{cases}$$

Arguing for v in an analogous way, all the inequalities become equalities and then the distribution function ν of v solves the problem

$$\begin{cases} \nu'(\theta) = C_n \nu(\theta)^{(n-2)/n}, & \theta \leq 0 \\ \nu(0) = |\Omega|. \end{cases}$$

Then $\mu(\theta) \leq \nu(\theta)$ and the claim immediately follows. □

From (3.4) straightly follows our main theorem.

Theorem 3.1. *Let Ω be a smooth mean convex bounded open set in $\mathbb{R}^n$ and let $\Gamma_0 = \partial\Omega$. If Γ_t denotes the evolution of Γ_0 by mean curvature and $\Gamma_t = \partial\Omega_t$, where Ω_t is a smooth mean convex bounded open set in $\mathbb{R}^n$, then the following estimate holds*

$$0 \leq t^* \leq \frac{1}{2(n-1)} \left(\frac{|\Omega|}{\omega_n} \right)^{\frac{2}{n}}. \tag{3.5}$$

Actually, we can prove a more precise pointwise comparison result as stated in the following

Proposition 3.2. *Under the assumptions of Proposition 3.1 we get*

$$0 \geq u^{\bigstar}(x) \geq v(x), \quad x \in \Omega^{\bigstar}. \tag{3.6}$$

Proof. Let $\theta \leq 0$. It is well known (see [9]) that

$$\lambda'(\theta) = \int_{u=\theta} H^2[\{u=\theta\}]\, d\mathcal{H}^{n-1}.$$

By Hölder inequality we get

$$\begin{aligned}\mu'(\theta) = \int_{u=\theta} H[\{u=\theta\}] d\mathcal{H}^{n-1} &\leq (\lambda(\theta))^{1/2} \left(\int_{u=\theta} H^2[\{u=\theta\}]\, d\mathcal{H}^{n-1} \right)^{1/2} \\ &= \lambda(\theta)^{1/2} \left(\lambda'(\theta)\right)^{1/2}.\end{aligned}$$

On the other hand

$$\mu'(\theta) \geq \int_{u_1^{\bigstar}=\theta} H[\{u^{\bigstar}=\theta\}]\, d\mathcal{H}^{n-1} = \tilde{c}(n)\lambda(\theta)^{(n-2)/(n-1)},$$

where $\tilde{c}(n) = (n-1)(n\omega_n)^{1/n-1}$. Hence we can say that λ satisfies

$$\begin{cases} \lambda'(\theta) \geq \tilde{c}(n)(\lambda(\theta))^{(n-2)/(n-1)} \\ \lambda(0) = \mathcal{H}^{n-1}(\partial\Omega). \end{cases}$$

In a similar way we find that the function σ, which denotes the perimeter of the level sets of v, is the solution of the following problem

$$\begin{cases} \sigma'(\theta) = \tilde{c}(n)(\sigma(\theta))^{(n-2)/(n-1)} \\ \sigma(0) = \mathcal{H}^{n-1}(\partial\Omega). \end{cases}$$

Then $\lambda(\theta) \leq \sigma(\theta)$,for all $\theta < 0$ and the claim follows. □

For the non-smooth case and for some numerical example we refer to [1].

REFERENCES

[1] B. Brandolini - M. Cicalese - C. Nitsch - C. Trombetti, *A sharp estimate of the extinction time for the mean curvature flow,* in preparation.

[2] Y.D. Burago - V.A. Zalgaller, *Geometric inequalities,* New York, Springer Verlag (1988).

[3] Y.G. Chen - Y. Giga - S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations,* J. Diff. Geom., 3 (1991).

[4] L.C. Evans - J. Spruck, *Motion of level sets by mean curvature I,* J. Diff. Geom., 33 (1991), pp. 635–681.

[5] L.C. Evans - J. Spruck, *Motion of level sets by mean curvature II,* Trans. AMS , 330 (1992), pp. 321–332.

[6] L.C. Evans - J. Spruck, *Motion of level sets by mean curvature III,* J. Geom. Anal., 2 (1992), pp. 121–150.

[7] L.C. Evans - J. Spruck, *Motion of level sets by mean curvature IV ,* J. Geom. Anal., 5 (1) (1995), pp. 77-.114.

[8] Y. Giga - K. Yama-uchi, *On a lower bound for the extinction time of surfaces moved by mean curvature,* Calc. Var., 1 (1993), pp. 417–428.

[9] G. Huisken, *Flow by mean curvature of convex surfaces into spheres,* J. Diff. Geom., 20 (1984), pp. 237-266.

[10] J.H. Michael - L.M. Simon, *Sobolev and meanvalue inequalities on generalized submanifolds of* $\mathbb{R}^n$*,* Comm. Pure Appl. Math., 26 (1973), pp. 361–379.

[11] S. Osher - J.A. Sethian, *Fronts propagating with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations,* J. Computational Phys., 79 (1988), pp. 12–49.

[12] N.S. Trudinger, *Isoperimetric inequalities for quermassintegrals,* Ann. Inst. H. Poicaré, 11 (4) (1994), pp. 411–425.

[13] N.S. Trudinger, *On new isoperimetric inequalities and symmetrization,* J. Reine Angew. Math., 488 (1997), pp. 203–220.

Dipartimento di Matematica e Applicazioni "R. Caccioppoli",
Università degli Studi di Napoli "Federico II",
Complesso Monte S. Angelo, via Cintia - 80126 Napoli (ITALY)
e-mail: brandolini@unina.it
cicalese@unina.it
carlo.nitsch@unina.it
cristina@unina.it