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Continuous frames for unbounded operators

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Abstract

Few years ago Găvruţa gave the notions of K-frame and atomic system for a linear bounded operator K in a Hilbert space $\mathcal{H}$ in order to decompose $\mathcal{R}(K)$, the range of K, with a frame-like expansion. These notions are here generalized to the case of a densely defined and possibly unbounded operator A on a Hilbert space in a continuous setting, thus extending what have been done in a previous paper in a discrete framework.

1 Introduction

The notion of discrete frame was introduced by Duffin and Schaefer in 1952 [19] even though it raised on the mathematical and physical scene in 1986 with the paper of Daubechies, Grossmann, Meyer because of their use in wavelet analysis. In the early '90s Ali, Antoine and Gazeau [1] and, independently, Kaiser [25] extended this notion to the continuous case. Continuous frames have been deeply investigated also in [27]. Over the years many extensions of frames have been introduced and studied. Most of them have been considered in the discrete case because of their wide use in applications e.g. in signal processing [19]. Frames have been studied for the whole Hilbert space or for a closed subspace until 2012, when Găvruţa [22] gave

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the notions of *K*-frame and of atomic system for a bounded operator *K* everywhere defined on $\mathcal{H}$, thus generalizing the notion of frame and that of atomic system for a subspace due to Feichtinger and Werther [21]. *K*-frames allow to write each element of $\mathcal{R}(K)$, the range of *K*, which is not a closed subspace in general, as a combination of the elements of the *K*-frame, which do not necessarily belong to $\mathcal{R}(K)$ with $K \in \mathcal{B}(\mathcal{H})$. *K*-frames have been generalized in [4] and [23] where the notion of *K*-g-frames was investigated and have been further generalized in 2018 to the continuous case in [2].

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and norm $\| \cdot \|$, (X, μ) a measure space where μ is a positive measure and *A* a densely defined operator on $\mathcal{H}$. Let $\phi : x \in X \rightarrow \phi_x \in \mathcal{H}$ be a Bessel function, i.e. ϕ be such that for all $f \in \mathcal{H}$, the map $x \rightarrow \langle f | \phi_x \rangle$ is a measurable function on *X* and there exists a constant $\beta > 0$ such that $\int_X |\langle f | \phi_x \rangle|^2 d\mu(x) \leq \beta \|f\|^2$, $\forall f \in \mathcal{H}$. Assume that for $f \in \mathcal{D}(A)$ (the domain of *A*) we have the decomposition

$$\langle Af | u \rangle = \int_X a_f(x) \langle \phi_x | u \rangle d\mu(x), \quad \forall u \in \mathcal{D}(A^*)$$

for some $a_f \in L^2(X, \mu)$. If *A* is unbounded, the function a_f can not depend continuously on *f*, differently to what occurs when *A* is bounded. In order to decompose the range of a densely defined unbounded operator *A* as a combination of vectors in $\mathcal{H}$, we need somewhat which takes on its unboundedness. In literature there are some generalizations to the continuous case of the notion of *K*-frame, as e.g. c-*K*-g-frames in [2]); however, as far as the author knows, the case of an unbounded operator *K* in $\mathcal{H}$ has been little considered.

In [10] this problem has been addressed in the discrete case. In the present paper both the approaches introduced in [10] are extended to the *continuous setting*. One of the approaches involves a Bessel function ϕ and the coefficient function a_f depends continuously on $f \in \mathcal{D}(A)$ only in the graph topology of *A*, which is stronger than the norm of $\mathcal{H}$; the other one involves a non-Bessel function ϕ but the coefficient function a_f depends continuously on $f \in \mathcal{D}(A)$. In the latter approach, the notions of continuous weak *A*-frame and continuous weak atomic system for an unbounded operator *A* are introduced and studied.

If $\phi : X \rightarrow \mathcal{H}$ is a continuous frame for $\mathcal{H}$ then of course

$$\langle Af | h \rangle = \int_X \langle Af | \zeta_x \rangle \langle \phi_x | h \rangle d\mu(x), \quad \forall f \in \mathcal{D}(A), h \in \mathcal{H}$$

where $\zeta : X \rightarrow \mathcal{H}$ is a dual frame of ϕ. In contrast, if ϕ is a continuous weak *A*-frame, then there exists a Bessel function $\psi : X \rightarrow \mathcal{H}$ such that

$$\langle Ah | u \rangle = \int_X \langle h | \psi_x \rangle \langle \phi_x | u \rangle d\mu(x), \quad \forall h \in \mathcal{D}(A), u \in \mathcal{D}(A^*)$$

and the action of the operator *A* does not appear in the weak decomposition of the range of *A*, see Theorem 3.20. Still, continuous weak *A*-frames clearly call to mind continuous multipliers which are the object of interest of a recent literature even though unbounded multipliers, as far the author knows, have been little looked over.

For example, some initial steps toward this direction have been done, in the discrete case, in [5–8, 24] where some unbounded multipliers have been defined. Therefore this paper can spur investigation in the direction of unbounded multipliers in the continuous case.

The paper is organized as follows. In Sect. 2 we recall some well known definitions and introduce the generalized frame operator T_ϕ which is the operator associated to a sesquilinear form defined by means of a function $\phi : x \in X \to \phi_x \in \mathcal{H}$. In Sect. 3 we introduce, prove the existence under opportune hypotheses, and study the notions of continuous weak A-frame and continuous weak atomic system for a densely defined operator A in a Hilbert space $\mathcal{H}$. To go into more detail, after having introduced and studied the notion of continuous weak A-frame, Sect. 3.1 is devoted to the study of frame-related operators as the analysis, synthesis and (generalized) frame operators of a continuous weak A-frame. In Sect. 3.2 the notion of continuous weak atomic system for an unbounded operator A in Hilbert space $\mathcal{H}$ is given.

Under some hypotheses, this notion is equivalent to that of continuous weak A-frame. Moreover, given a suitable function $\phi : x \in X \to \phi_x \in \mathcal{H}$, for every bounded operator $M \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$, an operator A_M can be constructed in order ϕ to be a continuous weak atomic system for A_M. Section 4 is devoted to the second approach to the problem of decomposing the range of an unbounded operator in Hilbert space: we consider a bounded operator K from a Hilbert space $\mathcal{J}$ into another one $\mathcal{H}$ and give some results about both continuous K-frames and continuous atomic systems for K and about their frame-related operators, then in Sect. 4.1, we use them to study the case of an unbounded closed and densely defined operator $A : \mathcal{D}(A) \to \mathcal{H}$ viewing it as a bounded one $A : \mathcal{H}_A \to \mathcal{H}$, where $\mathcal{H}_A$ is the Hilbert space obtained by giving $\mathcal{D}(A)$ the graph norm.

2 Definitions and preliminary results

Throughout the paper we denote by $\mathcal{H}$ an infinite dimensional complex Hilbert space with inner product $\langle\cdot|\cdot\rangle$ linear in the first entry and conjugate linear in the second entry, and induced norm $\|\cdot\|$. The term operator is used for a linear mapping. Given an operator A, we denote its domain by $\mathcal{D}(A)$, its range by $\mathcal{R}(A)$ and its adjoint by A^*, if A is densely defined. By $\mathcal{B}(\mathcal{H})$ we denote the Banach space of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$ and by $\|A\|$ the usual norm of the operator $A \in \mathcal{B}(\mathcal{H})$. We indicate by (X, μ) a measure space X with positive measure μ, sometimes we will require it is σ-finite, i.e. X can be covered with at most countably many measurable, possibly disjoint, sets $\{X_n\}_{n \in \mathbb{N}}$ of finite measure. For brevity we indicate by $L^2(X, \mu)$ the class of all μ-measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_2^2 = \int_X |f(x)|^2 \mathrm{d}\mu(x) < \infty,$$

by identifying functions which differ only on a μ-null subset of X.

Let us briefly recall the notion of continuous frame, see e.g. [1, Definition 2.1], [13, Definition 5.6.1].

Definition 2.1   A continuous frame for $\mathcal{H}$ is a function $\phi : x \in X \to \phi_x \in \mathcal{H}$ for which

(i)   for all $h \in \mathcal{H}$, the map $x \to \langle h | \phi_x \rangle$ is a measurable function on X (i.e. the function ϕ is weakly measurable),

(ii)   there exist constants $\alpha, \beta > 0$ such that

$$\alpha \|h\|^2 \le \int_X |\langle h | \phi_x \rangle|^2 \mathrm{d}\mu(x) \le \beta \|h\|^2, \quad \forall h \in \mathcal{H}. \tag{2.1}$$

The function ϕ is called *a Bessel function* if at least the upper condition in (2.1) holds. If $\alpha = \beta = 1$ then the function ϕ is called a Parseval frame.

The main feature of a frame, hence of a continuous frame too, is the possibility of writing each vector of a Hilbert space as a sum of a infinite linear combination of vectors in the space getting rid of rigidness of orthonormality of the vectors of a basis and of the uniqueness of the decomposition, but still maintaining numerical stability of the reconstruction and fast convergence. By a continuous frame it is possible to represent every element of the Hilbert space by a reconstruction formula: if $\phi : x \in X \to \phi_x \in \mathcal{H}$ is a continuous frame for the Hilbert space $\mathcal{H}$, then any $h \in \mathcal{H}$ can be expressed as

$$h = \int_X \langle h | \psi_x \rangle \phi_x \mathrm{d}\mu(x),$$

where $\psi : x \in X \to \psi_x \in \mathcal{H}$ is a function called *dual of* ϕ and the integrals have to be understood in the weak sense, as usual.

2.1 Frame-related operators and sesquilinear forms

In this section we recall the definitions of the main operators linked to a $\phi : x \in X \to \phi_x \in \mathcal{H}$ and prove some results about them. We want to drive the attention of the reader on the fact that, in contrast with the discrete case where some results involve strong convergence [10], in the continuous case we can prove our results just in weak sense.

In the sequel we will briefly indicate the range $\{\phi_x\}_{x \in X}$ of a function $\phi : x \in X \to \phi_x \in \mathcal{H}$ by $\{\phi_x\}$. Consider the function $\phi : x \in X \to \phi_x \in \mathcal{H}$ and the set

$$\mathcal{D}(C_\phi) = \left\{ h \in \mathcal{H} : \int_X |\langle h | \phi_x \rangle|^2 \mathrm{d}\mu(x) < \infty \right\}.$$

The operator $C_\phi : h \in \mathcal{D}(C_\phi) \subset \mathcal{H} \to \langle h | \phi_x \rangle \in L^2(X, \mu)$ strongly defined, for every $h \in \mathcal{D}(C_\phi)$ and for every $x \in X$, by

$$(C_\phi h)(x) = \langle h | \phi_x \rangle \tag{2.2}$$

is called *the analysis operator of the function* ϕ, borrowing the terminology from frame theory.

Remark 2.2 In general the domain of C_ϕ is not dense, hence C_ϕ^* is not well-defined. An example of function whose analysis operator is densely defined can be found in Example 2.8, where $\mathcal{D}(C_\phi) = \mathcal{D}(\Omega_\phi)$. Moreover, a sufficient condition for $\mathcal{D}(C_\phi)$ to be dense in $\mathcal{H}$ is that $\phi_x \in \mathcal{D}(C_\phi)$ for every $x \in X$, see [3, Lemma 2.3].

The next result will be often needed in Sect. 3. It is a part of Lemma 2.1 in [3]; there, $\{\phi_x\}$ needs not to be total.

Proposition 2.3 *Let* $\phi : x \in X \to \phi_x \in \mathcal{H}$. *The analysis operator* C_ϕ *is closed.*

If C_ϕ is densely defined, let us calculate its adjoint operator: let $a \in \mathcal{D}(C_\phi^*)$ with

$$\mathcal{D}(C_\phi^*) = \{a \in L^2(X, \mu) : \exists g \in \mathcal{H} \text{ such that } \langle C_\phi h | a \rangle_2 = \langle h | g \rangle, \forall h \in \mathcal{D}(C_\phi)\}$$

$$\left\langle C_\phi^* a | h \right\rangle = \langle a | C_\phi h \rangle_2 = \int_X a(x) \langle \phi_x | h \rangle \mathrm{d}\mu(x), \quad h \in \mathcal{D}(C_\phi)$$

hence $C_\phi^* : \mathcal{D}(C_\phi^*) \subset L^2(X, \mu) \to \mathcal{H}$ is weakly defined by:

$$\left\langle C_\phi^* a | h \right\rangle = \int_X \langle a(x) \phi_x | h \rangle \mathrm{d}\mu(x), \quad a \in \mathcal{D}(C_\phi^*), h \in \mathcal{D}(C_\phi)$$

and is called the *synthesis operator of the function* ϕ where

$$\mathcal{D}(C_\phi^*) := \left\{ a \in L^2(X, \mu) : \int_X \langle a(x) \phi_x | h \rangle \mathrm{d}\mu(x) \text{ exists } \forall h \in \mathcal{D}(C_\phi) \right\}.$$

Remark 2.4 Thus, if C_ϕ is densely defined, then the synthesis operator C_ϕ^* is a densely defined closed operator.

Proposition 2.5 *[20] The function* $\phi : x \in X \to \phi_x \in \mathcal{H}$ *is Bessel with bound* $\beta > 0$ *if and only if the synthesis operator* C_ϕ^* *is linear and bounded on* $L^2(X, \mu)$ *with* $\|C_\phi^*\|_{L^2, \mathcal{H}} \le \sqrt{\beta}$. *Moreover, the analysis operator* C_ϕ *is linear and bounded on* $\mathcal{H}$ *with* $\|C_\phi\|_{\mathcal{H}, L^2} \le \sqrt{\beta}$. *More precisely*

$$\|C_\phi^*\|_{L^2, \mathcal{H}} = \|C_\phi\|_{\mathcal{H}, L^2} = \sup_{f \in \mathcal{H}, \|f\|=1} \left(\int_X |\langle f | \phi_x \rangle|^2 \mathrm{d}\mu(x) \right)^{1/2} \le \sqrt{\beta}.$$

Extending to the continuous case [15], consider the set

$$\mathcal{D}(\Omega_\phi) = \left\{ f \in \mathcal{H} : \int_X |\langle f | \phi_x \rangle|^2 \mathrm{d}\mu(x) < \infty \right\} = \mathcal{D}(C_\phi)$$

and the mapping $\Omega_\phi : \mathcal{D}(\Omega_\phi) \times \mathcal{D}(\Omega_\phi) \to \mathbb{C}$ defined by

$$\Omega_\phi(f, g) := \int_X \langle f|\phi_x\rangle\langle\phi_x|g\rangle \mathrm{d}\mu(x). \tag{2.3}$$

Ω_ϕ is clearly a non-negative symmetric sesquilinear form which is well defined for every $f, g \in \mathcal{D}(\Omega_\phi)$ because of the Cauchy–Schwarz inequality. It is unbounded in general. Moreover, since $\mathcal{D}(\Omega_\phi)$ is the largest domain such that Ω_ϕ is defined on $\mathcal{D}(\Omega_\phi) \times \mathcal{D}(\Omega_\phi)$, it results that

$$\Omega_\phi(f, g) = \langle C_\phi f | C_\phi g \rangle_2, \quad \forall f, g \in \mathcal{D}(C_\phi) = \mathcal{D}(\Omega_\phi) \tag{2.4}$$

where C_ϕ is the analysis operator defined in (2.2). Since C_ϕ is a closed operator, Ω_ϕ is a closed non-negative symmetric sesquilinear form in $\mathcal{H}$, see e.g. [26, Example VI.1.13]. Let us assume that $\mathcal{D}(\Omega_\phi)$ is dense in $\mathcal{H}$, then by Kato's first representation theorem [26, Theorem VI.2.1] there exists a positive self-adjoint operator T_ϕ *associated to the sesquilinear form* Ω_ϕ on

$$\mathcal{D}(T_\phi) = \{f \in \mathcal{D}(\Omega_\phi) : h \to \int_X \langle f|\phi_x\rangle\langle\phi_x|h\rangle d\mu(x) \tag{2.5}$$
$$\text{is bounded on } \mathcal{D}(\Omega_\phi) \text{ w.r. to } \|\cdot\|\}$$

defined by

$$T_\phi f := h \tag{2.6}$$

with h as in (2.5), h is uniquely determined because of the density of $\mathcal{D}(\Omega_\phi)$. The operator T_ϕ is the greatest one whose domain is contained in $\mathcal{D}(\Omega_\phi)$ and for which the following representation holds

$$\Omega_\phi(f, g) = \langle T_\phi f | g \rangle, \qquad f \in \mathcal{D}(T_\phi), \, g \in \mathcal{D}(\Omega_\phi).$$

The set $\mathcal{D}(T_\phi)$ is dense in $\mathcal{D}(\Omega_\phi)$, see [26, p. 279]. Furthermore, by Kato's second representation theorem [26, Theorem VI.2.23], $\mathcal{D}(\Omega_\phi) = \mathcal{D}(T_\phi^{1/2})$ and

$$\Omega_\phi(f, g) = \langle T_\phi^{1/2} f | T_\phi^{1/2} g \rangle, \quad \forall f, g \in \mathcal{D}(\Omega_\phi)$$

and comparing with (2.4), we obtain $T_\phi = C_\phi^* C_\phi = |C_\phi|^2$ on $\mathcal{D}(T_\phi)$.

Definition 2.6 The operator $T_\phi : \mathcal{D}(T_\phi) \subset \mathcal{H} \to \mathcal{H}$ defined by (2.6) will be said the *generalized frame operator* of the function $\phi : x \in X \to \phi_x \in \mathcal{H}$.

Given $\phi : x \in X \to \phi_x \in \mathcal{H}$, coherently with [3], the operator $S_\phi : \mathcal{D}(S_\phi) \subset \mathcal{H} \to \mathcal{H}$ weakly defined by

$$\langle S_\phi f | g \rangle = \int_X \langle f|\phi_x\rangle\langle\phi_x|g\rangle \mathrm{d}\mu(x), \quad f \in \mathcal{D}(S_\phi), g \in \mathcal{H}$$

where

$$\mathcal{D}(S_\phi) = \{f \in \mathcal{H} : \int_X \langle f|\phi_x\rangle \phi_x \, d\mu(x) \text{ converges weakly in } \mathcal{H}\}$$

is called the *frame operator* of ϕ. It is a positive operator on its domain and symmetric indeed for every $f, g \in \mathcal{D}(S_\phi)$

$$\langle S_\phi f|g\rangle = \int_X \langle f|\phi_x\rangle \langle \phi_x|g\rangle d\mu(x) = \int_X \overline{\langle \phi_x|f\rangle \langle g|\phi_x\rangle} \, d\mu(x)$$
$$= \int_X \langle g|\phi_x\rangle \langle \phi_x|f\rangle d\mu(x) = \langle f|S_\phi g\rangle,$$

but non densely defined in general. If ϕ is a continuous frame for $\mathcal{H}$, then the frame operator S_ϕ is a bounded operator in $\mathcal{H}$, positive, invertible with bounded inverse, see e.g. [1].

Remark 2.7 The generalized frame operator T_ϕ and the frame operator S_ϕ coincide on $\mathcal{D}(S_\phi) \subset \mathcal{D}(T_\phi)$. If in particular ϕ is a continuous frame for $\mathcal{H}$, then C_ϕ, S_ϕ are defined on the whole $\mathcal{H}$ and C_ϕ^* on the whole $L^2(X, \mu)$, see also [3], and $T_\phi = C_\phi^* C_\phi = S_\phi$ on $\mathcal{H}$. However, in general, they are not the same operator, as the following example shows.

Example 2.8 Let X be such that $\mu(X) = \infty$ and having a covering made up of a countable collection $\{X_n\}_{n \in \mathbb{N}}$ of disjoint measurable subspaces of X each of measure $M > 0$, $\mathcal{H}$ a separable Hilbert space and $\{e_n\}_{n \in \mathbb{N}}$ an orthonormal basis of $\mathcal{H}$. Let $\alpha > 1, \beta > 0$ and define $\phi : x \in X \rightarrow \phi_x \in \mathcal{H}$ with

$$\phi_x = \begin{cases} \phi_{2n-1} := n^\beta e_n, & \text{if } x \in X_{2n-1} \\ \phi_{2n} := (n+1)^\alpha (e_{n+1} - e_n), & \text{if } x \in X_{2n}. \end{cases}$$

Then

$$\mathcal{D}(\Omega_\phi) = \left\{ f \in \mathcal{H} : \sum_{n=1}^\infty n^{2\beta}|\langle f|e_n\rangle|^2 + \sum_{n=1}^\infty (n+1)^{2\alpha}|\langle f|e_{n+1} - e_n\rangle|^2 < \infty \right\}$$

is dense. Indeed, consider the sequence $\{\phi_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$, then for every $m \in \mathbb{N}$

$$\int_X |\langle \phi_m|\phi_x\rangle|^2 d\mu(x) = \sum_{k=1}^\infty \int_{X_k} |\langle \phi_m|\phi_k\rangle|^2 d\mu(x) < \infty$$

because only two, three, or six terms in the series are different from zero, depending on the value of m. Then $span\{\phi_x\} = span\{\phi_n\} \subset \mathcal{D}(\Omega_\phi)$. On the other hand $(span\{\phi_n\})^\perp \subset \mathcal{D}(\Omega_\phi)$, hence

$$\mathcal{H} = \overline{span}\{\phi_n\} \oplus (\overline{span}\{\phi_n\})^\perp \subset \overline{\mathcal{D}(\Omega_\phi)}$$

hence $\mathcal{D}(\Omega_\phi)$ is dense in $\mathcal{H}$. We shall prove that there exists a $f \in \mathcal{D}(T_\phi)$ such that $f \notin \mathcal{D}(S_\phi)$. Let $f \in \mathcal{H}$ be such that $\langle f|e_n\rangle = \frac{1}{n^p}$ for every $n \in \mathbb{N}$, for a fixed

$p \in \{1, 2, \ldots\}$. We want to calculate for which values of α and β such an $f \in \mathcal{H}$ is in $\mathcal{D}(T_\phi) \setminus \mathcal{D}(S_\phi)$. For $f \in \mathcal{D}(\Omega_\phi)$ it has to be

$$\sum_{n=1}^{\infty} \frac{n^{2\beta}}{n^{2p}} + \sum_{n=1}^{\infty} (n+1)^{2\alpha} \frac{|n^p - (n+1)^p|^2}{n^{2p}(n+1)^{2p}} < \infty. \tag{2.7}$$

For $p > \beta + \frac{1}{2}$ the first series in (2.7) converges, the second has general term that behaves like $\frac{1}{n^{2(p-\alpha+1)}}$ hence if $p > \alpha - \frac{1}{2}$ too, then the series converges. To be $f \in \mathcal{D}(T_\phi)$ the functional $g \in \mathcal{D}(\Omega_\phi) \to \int_X \langle f|\phi_x\rangle\langle\phi_x|g\rangle \mathrm{d}\mu(x)$ has to be bounded. Take any $g \in \mathcal{D}(\Omega_\phi)$, then $\int_X \langle f|\phi_x\rangle\langle\phi_x|g\rangle \mathrm{d}\mu(x) = M\langle \sum_{n=1}^{\infty}\langle f|\phi_n\rangle\phi_n|g\rangle$. Let us consider the sequence of partial sums of the series $\sum_{n=1}^{\infty}\langle f|\phi_n\rangle\phi_n$:

$$s_{2m-1} = \sum_{n=1}^{m}\langle f|\phi_{2n-1}\rangle\phi_{2n-1} + \sum_{n=1}^{m-1}\langle f|\phi_{2n}\rangle\phi_{2n}$$

$$= ae_1 + \sum_{n=2}^{m-1} b_n(p)e_n + c_m(p)e_m$$

and

$$s_{2m} = \sum_{n=1}^{m}\langle f|\phi_{2n-1}\rangle\phi_{2n-1} + \sum_{n=1}^{m}\langle f|\phi_{2n}\rangle\phi_{2n}$$

$$= ae_1 + \sum_{n=2}^{m} b_n(p)e_n + d_{m+1}(p)e_{m+1}$$

with $a = \left[1 + 2^{2\alpha}\left(1 - \frac{1}{2^p}\right)\right] > 0$,

$$b_n(p) = \frac{n^{2\beta}}{n^p} + \frac{n^{2\alpha}[(n-1)^p - n^p]}{n^p(n-1)^p} - \frac{(n+1)^{2\alpha}[n^p - (n+1)^p]}{n^p(n+1)^p} = \frac{n^{2\beta}}{n^p} + b'_n(p)$$

and

$$c_m(p) = \frac{m^{2\beta}}{m^p} + d_m(p), \quad d_{m+1}(p) = \frac{(m+1)^{2\alpha}[m^p - (m+1)^p]}{m^p(m+1)^p}$$

where $b'_n(p) = \frac{p(p-1)}{n^{p-2\alpha+2}} + o\left(\frac{1}{n^{p-2\alpha+2}}\right)$ and $d_m(p) = \frac{-p}{n^{p-2\alpha+1}} + o\left(\frac{1}{n^{p-2\alpha+1}}\right)$. For $p > 2\beta + \frac{1}{2}$ and $p > 2\alpha - \frac{3}{2}$ the sequence $\{b_n(p)\}$ belongs to ℓ^2. Moreover, for every $g \in \mathcal{D}(\Omega_\phi)$ we have that $n^\beta|\langle e_n|g\rangle| \to 0$, hence, if also $p \geq 2\alpha - \beta - 1$, then $|c_m(p)\langle e_m|g\rangle| \leq (1 + p)m^\beta|\langle e_m|g\rangle| \to 0$ and $|d_m(p)\langle e_m|g\rangle| \leq pm^\beta|\langle e_m|g\rangle| \to 0$ as $m \to \infty$. Hence, the series $\langle \sum_{n=1}^{\infty}\langle f|\phi_n\rangle\phi_n|g\rangle$ converges. Now we want to calculate values of α and β in order $f \notin \mathcal{D}(S_\phi)$. A vector $h \in \mathcal{D}(S_\phi)$ if and only if for every $g \in \mathcal{H}$

$$\left| \int_X \langle h|\phi_x\rangle \langle \phi_x|g\rangle \mathrm{d}\mu(x) \right| = \left| M \left\langle \sum_{k=1}^{\infty} \langle h|\phi_k\rangle \phi_k |g \right\rangle \right| < \infty$$

i.e. if the series $\sum_{k=1}^{\infty} \langle h|\phi_k\rangle \phi_k$ weakly converges in $\mathcal{H}$, however, if $h = f$ and $0 < 2\alpha - 1 - p < \beta$ the norm of s_k goes to infinity as $k \to \infty$.

As an example, if $p = 3$ it can be $\alpha = \frac{17}{8}$ and $\beta = \frac{1}{3}$ or, as in [14], $p = 2$, $\alpha = \frac{8}{5}$ and $\beta = \frac{1}{2}$.

Proposition 2.9 *Let $\phi : x \in X \to \phi_x \in \mathcal{H}$ and $\mathcal{D}(\Omega_\phi)$ be dense. Then the frame operator S_ϕ is closable.*

Proof The sesquilinear form Ω_ϕ is non-negative closed and densely defined, hence the generalized frame operator T_ϕ is self-adjoint. We conclude the proof by recalling that $S_\phi \subset T_\phi$. □

In the following sections we will use the next two lemmas.

Lemma 2.10 [11] *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Let $W : \mathcal{D}(W) \subset \mathcal{K} \to \mathcal{H}$ a closed, densely defined operator with closed range $\mathcal{R}(W)$. Then, there exists a unique $W^\dagger \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that*

$$\mathcal{N}(W^\dagger) = \mathcal{R}(W)^\perp, \quad \overline{\mathcal{R}(W^\dagger)} = \mathcal{N}(W)^\perp, \quad WW^\dagger f = f, \quad f \in \mathcal{R}(W).$$

The operator $W^\dagger$ is called the *pseudo-inverse* of the operator W.

The following lemma is a partial variation of two Douglas majorization theorems [18, Theorem 1, Theorem 2], see also [10].

Lemma 2.11 *Let $(\mathcal{H}, \|\cdot\|), (\mathcal{H}_1, \|\cdot\|_1)$ and $(\mathcal{H}_2, \|\cdot\|_2)$ be Hilbert spaces and $T_1 : \mathcal{D}(T_1) \subseteq \mathcal{H}_1 \to \mathcal{H}, T_2 : \mathcal{D}(T_2) \subseteq \mathcal{H} \to \mathcal{H}_2$ densely defined operators. Assume that T_1 is closed and $\mathcal{D}(T_1^*) = \mathcal{D}(T_2)$. Consider the following statements*

(i) $\|T_1^* f\|_1 \le \lambda \|T_2 f\|_2$ for all $f \in \mathcal{D}(T_1^*)$ and some $\lambda > 0$,

(ii)   there exists a bounded operator $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $T_1 = T_2^* U$.

Then (i) $\Rightarrow$ (ii). If, in addition, T_2 is a bounded operator on $\mathcal{H}$, then (i) $\Leftrightarrow$ (ii) and both are equivalent to

(iii) $\mathcal{R}(T_1) \subset \mathcal{R}(T_2^*)$.

3 Continuous weak *A*-frame and continuous atomic systems for unbounded operators

In this section we introduce and study our extension to the continuous case of the notions of discrete weak *A*-frame and discrete weak atomic system for a densely defined operator *A* on a Hilbert space, given in [10].

Definition 3.1 Let A be a densely defined operator on $\mathcal{H}$. A *continuous weak A-frame* for $\mathcal{H}$ is a function $\phi : x \in X \to \phi_x \in \mathcal{H}$ such that for all $u \in \mathcal{D}(A^*)$, the map $x \to \langle u | \phi_x \rangle$ is a measurable function on X and

$$\alpha \| A^* u \|^2 \leq \int_X |\langle u | \phi_x \rangle|^2 \mathrm{d}\mu(x) < \infty,$$

for every $u \in \mathcal{D}(A^*)$ and some $\alpha > 0$.

Remark 3.2 If $X = \mathbb{N}$ and μ is the counting measure, a continuous weak A-frame clearly reduces to a discrete weak A-frame in the sense of [10].

Remark 3.3 Let (X, μ) be a σ-finite measure space. If $A \in \mathcal{B}(\mathcal{H})$, a continuous weak A-frame is a continuous A-g-frame in the sense of [2, Definition 2.1] with $\Lambda_x = \langle h | \phi_x \rangle$ for every $h \in \mathcal{H}$, with $x \in X$, since C_ϕ is a bounded operator in that case (see [2, Theorem 2.5]).

Remark 3.4 Let A be a densely defined operator on $\mathcal{H}$ and $\phi : x \in X \to \phi_x \in \mathcal{D}(A) \subset \mathcal{H}$ a continuous frame for $\mathcal{H}$. Then $A\phi$ is a continuous weak A-frame for $\mathcal{H}$. Indeed, there exist constants $\alpha, \beta > 0$ such that

$$\alpha \| A^* u \|^2 \leq \int_X |\langle A^* u | \phi_x \rangle|^2 \mathrm{d}\mu(x) \leq \beta \| A^* u \|^2, \quad \forall u \in \mathcal{D}(A^*).$$

Example 3.5 Let $X = \mathbb{R}^2$ and let μ be the Lebesgue measure on $\mathbb{R}^2$. Let $\mathcal{H} = L^2(\mathbb{R})$ and let $H^1(\mathbb{R})$ be the set of all functions f such that $f, f' \in L^2(\mathbb{R})$ and such that f is absolutely continuous on every closed bounded interval $[a, b] \subseteq \mathbb{R}$ (see [30, Appendix E]). Let us consider the differentiation operator $Af = -if'$ with domain $H^1(\mathbb{R})$ which is a self-adjoint operator of $L^2(\mathbb{R})$ (see [30, Example 1.7]). Fix $g \in H^1(\mathbb{R})$ with $\|g\|_2 = 1$, then $\phi_g : (s, t) \in \mathbb{R}^2 \to L^2(\mathbb{R})$ defined by $\phi_g(s, t) = -e^{2\pi i t \cdot}(2\pi t g(\cdot - s) + g'(\cdot - s))$ is a continuous weak A-frame for $L^2(\mathbb{R})$. Indeed, let $h \in H^1(\mathbb{R}) \setminus \{0\}$ and consider $\Theta_h(f)(t, s) = \int_{\mathbb{R}} f(x)\overline{h(x - s)}e^{-2\pi i t x}dx = \langle f | \phi_h(t, s) \rangle_2$, $t, s \in \mathbb{R}$, the short-time Fourier transform of $f \in L^2(\mathbb{R})$ with respect to the window h, with $\phi_h(f) : \mathbb{R}^2 \to H^1(\mathbb{R}) \subset L^2(\mathbb{R})$ defined by $\phi_h(t, s) = e^{2\pi i t \cdot}h(\cdot - s)$, $t, s \in \mathbb{R}$, we have the well-known identity for any $f \in L^2(\mathbb{R})$, see [13, Proposition 11.1.2]

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\langle f | \phi_h(s, t) \rangle|_2^2 \, ds dt = \|f\|_2^2 \|h\|_2^2,$$

hence, if $\|h\|_2 = 1$, then ϕ_h is a continuous Parseval frame in $L^2(\mathbb{R})$, see [12, Example 4.3]. Hence, $\phi_g = A\phi_g$ is a continuous weak A-frame.

Example 3.6 Let $X = \mathbb{R}$ and let μ be the Lebesgue measure on $\mathbb{R}$. Let $\mathcal{H} = L^2(0, 1)$, $H^1(0, 1)$ be the set of absolutely continuous functions f which are a.e. differentiable and such that $f' \in L^2(0, 1)$ and let $\mathcal{I}_{(0,1)}$ be the identity of $L^2(0, 1)$. Let us consider

the differentiation operator $Af = -if'$ with domain $H^1(0,1)$ which is a densely defined closed operator of $L^2(0,1)$, see [30, Section 1.3]. The function $\phi : t \in \mathbb{R} \to \phi_t \in L^2(0,1)$ with $\phi_t = 2\pi t e^{2\pi i t \cdot} \mathcal{I}_{(0,1)}$ is a continuous weak A-frame for $L^2(0,1)$. Indeed, as proved in [12, Example 4.2], the function $\psi : t \in \mathbb{R} \to \psi_t \in H^1(0,1) \subset L^2(0,1)$ such that $\psi_t := e^{2\pi i t \cdot} \mathcal{I}_{(0,1)}$ is a Parseval frame in $L^2(0,1)$. Hence $\phi = A\psi$ is a continuous weak A-frame for $L^2(0,1)$.

Proposition 3.7 *Let A be a densely defined operator on $\mathcal{H}$ and ϕ be a continuous weak A-frame for $\mathcal{H}$ with lower bound $\alpha > 0$. If $F \in \mathcal{B}(\mathcal{H})$ is such that the domain $\mathcal{D}(AF)$ is dense, then ϕ is a continuous weak AF-frame for $\mathcal{H}$ too, with lower bound $\alpha\|F^*\|^{-2}$.*

Proof By hypothesis there exists $\alpha > 0$ such that for every $u \in \mathcal{D}(A^*)$

$$\alpha\|A^*u\|^2 \le \int_X |\langle u|\phi_x\rangle|^2 \mathrm{d}\mu(x) < \infty.$$

The adjoint $(AF)^*$ is well defined and $F^*A^* = (AF)^*$ by [28, Theorem 13.2]. Hence, for every $u \in \mathcal{D}((AF)^*) = \mathcal{D}(F^*A^*)$

$$\begin{aligned}\|(AF)^*u\|^2 &= \|F^*A^*u\|^2 \le \|F^*\|^2\|A^*u\|^2 \\ &\le \frac{1}{\alpha}\|F^*\|^2 \int_X |\langle u|\phi_x\rangle|^2 \mathrm{d}\mu(x) < \infty\end{aligned}$$

since $u \in \mathcal{D}(F^*A^*) = \mathcal{D}(A^*)$. $\square$

If, in particular, F is also a unitary operator, then $\|F^*\| = 1$, hence ϕ is a continuous weak AF-frame for $\mathcal{H}$ with the same lower bound α.

Proposition 3.8 *Let A be a self-adjoint operator and $\phi : x \in X \to \phi_x \in \mathcal{D}(A) \subset \mathcal{H}$ a continuous weak A-frame for $\mathcal{H}$ with lower bound α, then $A\phi$ is a continuous weak A^2-frame for $\mathcal{H}$ with the same lower bound α. Moreover, if $\phi : x \in X \to \phi_x \in \bigcap_{k=1}^n \mathcal{D}(A^k) \subset \mathcal{H}$, then $A^n\phi$ is a continuous weak A^{n+1}-frame for $\mathcal{H}$, for every fixed $n \in \mathbb{N}$, with the same lower bound α. In particular, if $\phi : x \in X \to \phi_x \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n) \subset \mathcal{H}$ is a continuous weak A-frame for $\mathcal{H}$ with lower bound α, then $A^n\phi$ is a continuous weak A^{n+1}-frame for $\mathcal{H}$, for every $n \in \mathbb{N}$, with the same lower bound α.*

Proof By hypotheses A^2 is self-adjoint with dense domain $\mathcal{D}(A^2) \subset \mathcal{D}(A)$ and there exists $\alpha > 0$ such that for every $f \in \mathcal{D}(A)$

$$\alpha\|Af\|^2 \le \int_X |\langle f|\phi_x\rangle|^2 \mathrm{d}\mu(x) < \infty.$$

Hence, for every $h \in \mathcal{D}(A^2)$

$$\|A^2h\|^2 = \|A(Ah)\|^2 \leq \frac{1}{\alpha}\int_X |\langle Ah|\phi_x\rangle|^2 \mathrm{d}\mu(x)$$
$$= \frac{1}{\alpha}\int_X |\langle h|A\phi_x\rangle|^2 \mathrm{d}\mu(x) < \infty$$

since $Ah \in \mathcal{D}(A)$. Fix now an arbitrary $n \in \mathbb{N}$. If $\phi : x \in X \to \phi_x \in \mathcal{D}(A^n) \subset \mathcal{H}$, then, as before, by hypotheses both A^n and A^{n+1} are self-adjoint with dense domain $\mathcal{D}(A^{n+1}) \subset \mathcal{D}(A^n) \subset \mathcal{D}(A)$ and for every $h \in \mathcal{D}(A^{n+1})$

$$\|A^{n+1}h\|^2 = \|A(A^n h)\|^2 \leq \frac{1}{\alpha}\int_X |\langle A^n h|\phi_x\rangle|^2 \mathrm{d}\mu(x)$$
$$= \frac{1}{\alpha}\int_X |\langle h|A^n\phi_x\rangle|^2 \mathrm{d}\mu(x) < \infty$$

being $A^n h \in \mathcal{D}(A)$. The last sentence in the Proposition is now obvious. $\qquad\square$

The following definition sounds like [17, Definition 2.1] but here the operator is, in principle, unbounded.

Definition 3.9 Let A be a densely defined operator and $\phi : x \in X \to \phi_x \in \mathcal{H}$, then a function $\psi : x \in X \to \psi_x \in \mathcal{H}$ is called a *weak A-dual of* ϕ if

$$\langle Af|u\rangle = \int_X \langle f|\psi_x\rangle\langle\phi_x|u\rangle \mathrm{d}\mu(x), \quad \forall f \in \mathcal{D}(A), u \in \mathcal{D}(A^*). \tag{3.1}$$

The weak A-dual ψ of ϕ is not unique, in general.

Example 3.10 Let us see two examples. Let A be a densely defined operator on a separable Hilbert space $\mathcal{H}$.

(i) Let (X, μ) be a σ-finite measure space and let $\{X_n\}_{n\in\mathbb{N}}$ be a covering of X made up of countably many measurable disjoint sets of finite measure. Without loss of generality we suppose that $\mu(X_n) > 0$ for every $n \in \mathbb{N}$. Let $\{e_n\} \subset \mathcal{D}(A)$ be an orthonormal basis of $\mathcal{H}$ and consider ϕ, with $\phi_x = \frac{Ae_n}{\sqrt{\mu(X_n)}}$, $x \in X_n, \forall n \in \mathbb{N}$, then ϕ is a continuous weak A-frame, see the first part of the proof of Theorem 3.19. One can take ψ with $\psi_x = \frac{e_n}{\sqrt{\mu(X_n)}}$, $x \in X_n, \forall n \in \mathbb{N}$.

(ii) If $\phi := A\zeta$, where $\zeta : x \in X \to \zeta_x \in \mathcal{D}(A) \subset \mathcal{H}$ is a continuous frame for $\mathcal{H}$, then one can take as ψ any dual frame of $\{\zeta_x\}$.

3.1 Frame-related operators of continuous weak A-frames

In this subsection we will establish some properties of the analysis, synthesis and (generalized) frame operators of a continuous weak A-frame with A a densely

defined operator. A theorem of characterization for a continuous weak A-frame is also given.

Consider the sesquilinear form Ω_ϕ defined in (2.3), then we can prove the following

Proposition 3.11 *Let A be a densely defined operator and ϕ a continuous weak A-frame, then $\mathcal{D}(A^*) \subset \mathcal{D}(\Omega_\phi)$. Moreover, if A is closable, then Ω_ϕ is densely defined.*

Proof By hypotheses and definitions $\mathcal{D}(A^*) \subset \mathcal{D}(\Omega_\phi)$. If A is closable, then $\mathcal{D}(A^*)$ is dense and this concludes the proof. $\square$

However, in general $\mathcal{D}(A^*) \subsetneq \mathcal{D}(\Omega_\phi)$.

Corollary 3.12 *Let A be a closable and densely defined operator, ϕ a continuous weak A-frame, then the synthesis operator C_ϕ^* is closed.*

Proof By Proposition 3.11, the domain $\mathcal{D}(C_\phi) = \mathcal{D}(\Omega_\phi)$ of the closed operator C_ϕ is dense, hence C_ϕ^* is closed and densely defined. $\square$

Remark 3.13 For what has been established until now, if A is closable and densely defined and ϕ is a continuous weak A-frame, by (2.4) the sesquilinear form Ω_ϕ is a *densely defined*, non-negative closed form. Then there exists the generalized frame operator T_ϕ of ϕ defined as in (2.6) and the analysis operator C_ϕ is closed and densely defined. Moreover, one has

$$\alpha \|A^* u\|^2 \leq \int_X |\langle u | \phi_x \rangle|^2 \mathrm{d}\mu(x) = \|C_\phi u\|_2^2 = \left\| T_\phi^{\frac{1}{2}} u \right\|^2, \quad \forall u \in \mathcal{D}(A^*).$$

Corollary 3.14 *Let A be a closable, densely defined operator, ϕ a continuous weak A-frame for $\mathcal{H}$. Then the generalized frame operator T_ϕ of ϕ is self-adjoint and the frame operator S_ϕ is closable.*

Proof By Proposition 3.11, the domain $\mathcal{D}(\Omega_\phi)$ is dense, hence the thesis follows by Proposition 2.9. $\square$

Proposition 3.15 *Let A be densely defined and closable, A^* injective and ϕ a continuous weak A-frame for $\mathcal{H}$. Then C_ϕ is injective on $\mathcal{D}(A^*)$.*

Proof The proof is straightforward once observed that in our hypotheses $\alpha \|A^* f\|^2 \leq \|C_\phi f\|_2^2$ for every $f \in \mathcal{D}(A^*)$ and some $\alpha > 0$. $\square$

The following is a theorem of characterization for continuous weak A-frames.

Theorem 3.16 *Let A be a closed densely defined operator and $\phi : x \in X \to \phi_x \in \mathcal{H}$. Then the following statements are equivalent.*

(i)   *ϕ is a continuous weak A-frame for $\mathcal{H}$;*

(ii)   for every $u \in \mathcal{D}(A^*)$, the map $x \to \langle u | \phi_x \rangle$ is a measurable function on X and there exists a closed densely defined extension R of C_ϕ^*, with $\mathcal{D}(R^*) = \mathcal{D}(A^*)$, such that $A = RM$ for some $M \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$.

Proof $(i) \Rightarrow (ii)$ Consider $B : \mathcal{D}(A^*) \to L^2(X, \mu)$ given by $(Bu)(x) = \langle u | \phi_x \rangle$, $\forall u \in \mathcal{D}(A^*), x \in X$ which is a restriction of the analysis operator C_ϕ. Since C_ϕ is closed, B is closable. B is also densely defined since $\mathcal{D}(A^*)$ is dense.

We apply Lemma 2.11 to $T_1 := A$ and $T_2 := B$ noting that $\|Bu\|_2^2 = \int_X |\langle u | \phi_x \rangle|^2 d\mu(x)$. There exists $M \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$ such that $A = B^*M$. Then the statement is proved taking $R = B^*$, indeed $R = B^* \supseteq C_\phi^*$ and $\mathcal{D}(R) \supset \mathcal{D}(C_\phi^*)$ is dense because C_ϕ is closed and densely defined. Note that we have $\mathcal{D}(A^*) = \mathcal{D}(R^*)$ indeed $\mathcal{D}(R^*) = \mathcal{D}(\overline{B})$,

$$\mathcal{D}(A^*) \subset \mathcal{D}(\overline{B}) = \mathcal{D}(M^*\overline{B}) \subset \mathcal{D}((B^*M)^*) = \mathcal{D}(A^*),$$

hence in particular B is closed.

$(ii) \Rightarrow (i)$ We have $\mathcal{D}(A^*) = \mathcal{D}(R^*)$ indeed

$$\mathcal{D}(A^*) \subset \mathcal{D}(R^*) = \mathcal{D}(M^*R^*) \subset \mathcal{D}((RM)^*) = \mathcal{D}(A^*).$$

For every $u \in \mathcal{D}(A^*) = \mathcal{D}(R^*)$

$$\|A^*u\|^2 = \|M^*R^*u\|^2 \leq \|M^*\|^2 \|R^*u\|^2 = \|M^*\|^2 \int_X |\langle u | \phi_x \rangle|^2 d\mu(x) < \infty$$

being $R^* \subset C_\phi$. This proves that ϕ is a continuous weak A-frame. $\square$

3.2 Atomic systems for unbounded operators A and their relation with A-frames

Now we define our generalization to the continuous case and to unbounded operators of the notion of atomic system for K, with $K \in \mathcal{B}(\mathcal{H})$ [22].

Definition 3.17 Let A be a densely defined operator on $\mathcal{H}$. A *continuous weak atomic system for A* is a function $\phi : x \in X \to \phi_x \in \mathcal{H}$ such that for all $u \in \mathcal{D}(A^*)$, the map $x \to \langle u | \phi_x \rangle$ is a measurable function on X and

(i) $\int_X |\langle u | \phi_x \rangle|^2 d\mu(x) < \infty$, for every $u \in \mathcal{D}(A^*)$;

(ii)   there exists $\gamma > 0$ such that, for every $f \in \mathcal{D}(A)$, there exists $a_f \in L^2(X, \mu)$, with $\|a_f\|_2 = \left(\int_X |a_f(x)|^2 d\mu(x) \right)^{1/2} \leq \gamma \|f\|$ and

$$\langle Af | u \rangle = \int_X a_f(x) \langle \phi_x | u \rangle d\mu(x), \quad \forall u \in \mathcal{D}(A^*). \tag{3.2}$$

Remark 3.18 If ϕ is a continuous weak atomic system for a densely defined operator A then, for every $f \in \mathcal{D}(A)$ and for every $u \in \mathcal{D}(A^*)$ the function $g_f^u(x) =$

$a_f(x)\langle\phi_x|u\rangle$ in (3.2) is μ-integrable. Indeed it is *absolutely* integrable: fix any $f \in \mathcal{D}(A)$, $u \in \mathcal{D}(A^*)$, then by Schwarz inequality

$$\int_X |a_f(x)\langle\phi_x|u\rangle|\mathrm{d}\mu(x) \le \|a_f\|_2 \left(\int_X |\langle\phi_x|u\rangle|^2\mathrm{d}\mu(x)\right)^{1/2} < \infty,$$

where the last inequality follows from both conditions in Definition 3.17.

The next theorem guarantees the existence of continuous weak atomic systems for densely defined operators on $\mathcal{H}$.

Theorem 3.19 *Let (X, μ) be a σ-finite measure space. Let $\mathcal{H}$ be a separable Hilbert space and A a densely defined operator on $\mathcal{H}$. Then there exists a continuous weak atomic system for A.*

Proof Let $\{e_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ be an orthonormal basis for $\mathcal{H}$. Then, every $f \in \mathcal{H}$ can be written as $f = \sum_{n=1}^{\infty}\langle f|e_n\rangle e_n$. For all $n \in \mathbb{N}$ denote with $\phi_n = Ae_n$. Let $\{X_n\}_{x\in\mathbb{N}}$ be a covering of X made up of countably many measurable disjoint sets of finite measure. It is not restrictive supposing that $\mu(X_n) > 0$ for every $n \in \mathbb{N}$. Then we define

$$\phi_x := \frac{\phi_n}{\sqrt{\mu(X_n)}}, \quad x \in X_n, n \in \mathbb{N}.$$

For every $f \in \mathcal{H}$ the map $x \in X \to \langle f|\phi_x\rangle \in \mathbb{C}$ is measurable because it is a step function.

Moreover, for every $u \in \mathcal{D}(A^*)$

$$\|A^*u\|^2 = \sum_{n=1}^{\infty}|\langle A^*u|e_n\rangle|^2 = \sum_{n=1}^{\infty}|\langle u|Ae_n\rangle|^2$$
$$= \sum_{n=1}^{\infty}|\langle u|\phi_n\rangle|^2 = \sum_{n=1}^{\infty}\int_{X_n}|\langle u|\phi_x\rangle|^2\mathrm{d}\mu(x)$$
$$= \int_X |\langle u|\phi_x\rangle|^2\mathrm{d}\mu(x) < \infty.$$

Now, for all $f \in \mathcal{D}(A)$, take a_f as the step function defined as follows:

$$a_f(x) := \frac{\langle f|e_n\rangle}{\sqrt{\mu(X_n)}}, \quad x \in X_n, n \in \mathbb{N}.$$

Then, for all $f \in \mathcal{D}(A)$, $a_f \in L^2(X, \mu)$, with

$$\|a_f\|_2^2 = \int_X |a_f(x)|^2 d\mu(x) = \sum_{n=1}^{\infty}\int_{X_n}\frac{|\langle f|e_n\rangle|^2}{\mu(X_n)}\mathrm{d}\mu(x)$$
$$= \sum_{n=1}^{\infty}|\langle f|e_n\rangle|^2 = \|f\|^2,$$

and for every $f \in \mathcal{D}(A)$, $u \in \mathcal{D}(A^*)$

$$\langle Af|u\rangle = \left\langle \sum_{n=1}^{\infty} \langle f|e_n\rangle Ae_n|u \right\rangle$$

$$= \sum_{n=1}^{\infty} \langle f|e_n\rangle \langle Ae_n|u\rangle$$

$$= \sum_{n=1}^{\infty} \int_{X_n} \frac{\langle f|e_n\rangle}{\sqrt{\mu(X_n)}} \frac{\langle Ae_n|u\rangle}{\sqrt{\mu(X_n)}} \mathrm{d}\mu(x) = \int_X a_f(x)\langle \phi_x|u\rangle \mathrm{d}\mu(x)$$

Therefore ϕ is a continuous weak atomic system for A. $\square$

The following theorem gives a characterization of continuous weak atomic systems for A and continuous weak A-frames.

Theorem 3.20 *Let $\phi : x \in X \to \phi_x \in \mathcal{H}$ and A be a closable densely defined operator. Then the following statements are equivalent.*

  (i)   *ϕ is a continuous weak atomic system for A;*
  (ii) *ϕ is a continuous weak A-frame;*
  (iii) *$\int_X |\langle u|\phi_x\rangle|^2 \mathrm{d}\mu(x) < \infty$ for every $u \in \mathcal{D}(A^*)$ and there exists a Bessel weak A-dual ψ of ϕ.*

Proof $(i) \Rightarrow (ii)$ For every $u \in \mathcal{D}(A^*)$ by the density of $\mathcal{D}(A)$ we have

$$\|A^*u\| = \sup_{f \in \mathcal{H}, \|f\|=1} |\langle A^*u|f\rangle| = \sup_{f \in \mathcal{D}(A), \|f\|=1} |\langle A^*u|f\rangle|$$

$$= \sup_{f \in \mathcal{D}(A), \|f\|=1} |\langle u|Af\rangle|$$

$$= \sup_{f \in \mathcal{D}(A), \|f\|=1} \left| \int_X \overline{a_f(x)}\langle u|\phi_x\rangle \mathrm{d}\mu(x) \right|$$

$$\leq \sup_{f \in \mathcal{D}(A), \|f\|=1} \left(\int_X |a_f(x)|^2 \mathrm{d}\mu(x) \right)^{1/2} \left(\int_X |\langle u|\phi_x\rangle|^2 \mathrm{d}\mu(x) \right)^{1/2}$$

$$\leq \gamma \left(\int_X |\langle u|\phi_x\rangle|^2 \mathrm{d}\mu(x) \right)^{1/2} < \infty,$$

for some $\gamma > 0$, the last two inequalities are due to the fact that ϕ is a continuous weak atomic system for A.

$(ii) \Rightarrow (iii)$ Following the proof of Theorem 3.16, there exists $M \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$ such that $\overline{A} = B^*M$, with $B : \mathcal{D}(A^*) \to L^2(X, \mu)$ a closable, densely defined operator which is a restriction of the analysis operator C_ϕ.

By the Riesz representation theorem, for every $x \in X$ there exists a unique vector $\psi_x \in \mathcal{H}$ such that $(Mh)(x) = \langle h|\psi_x\rangle$ for every $h \in \mathcal{H}$. The function $\psi : x \in X \to \psi_x \in \mathcal{H}$ is Bessel. Indeed,

$$\int_X |\langle h|\psi_x\rangle|^2 \mathrm{d}\mu(x) = \int_X |(Mh)(x)|^2 \mathrm{d}\mu(x)$$
$$= \|Mh\|_2^2 \le \|M\|_{L^2}^2 \|h\|^2, \quad \forall h \in \mathcal{H}.$$

Moreover, for $f \in \mathcal{D}(A), u \in \mathcal{D}(A^*) = \mathcal{D}(B)$

$$\langle Af|u\rangle = \langle \overline{A}f|u\rangle = \langle B^*Mf|u\rangle = \langle Mf|B^{**}u\rangle_2$$
$$= \langle Mf|Bu\rangle_2 = \int_X \langle f|\psi_x\rangle \langle \phi_x|u\rangle \mathrm{d}\mu(x).$$

$(iii) \Rightarrow (i)$ It suffices to take $a_f : x \in X \to a_x(f) = \langle f|\psi_x\rangle \in \mathbb{C}$ for all $f \in \mathcal{D}(A)$. Indeed $a_f \in L^2(X,\mu)$ and, for some $\gamma > 0$, we have $\int_X |a_x(f)|^2 \mathrm{d}\mu(x) = \int_X |\langle f|\psi_x\rangle|^2 \mathrm{d}\mu(x) \le \gamma \|f\|^2$ since ψ is a Bessel function. Moreover, by definition of weak A-dual it is $\langle Af|u\rangle = \int_X a_f(x)\langle \phi_x|u\rangle \mathrm{d}\mu(x)$, for $f \in \mathcal{D}(A), u \in \mathcal{D}(A^*)$. □

The proof of Theorem 3.20 suggests the following

Proposition 3.21 *Let $\mathcal{D} \subset \mathcal{H}$ be dense, $\phi : x \in X \to \phi_x \in \mathcal{H}$ be such that*

(i)   *for every $u \in \mathcal{D}$, the map $x \to \langle u|\phi_x\rangle$ is a measurable function on X*
(ii) $\int_X |\langle u|\phi_x\rangle|^2 \mathrm{d}\mu(x) < \infty$ *for every $u \in \mathcal{D}$.*

If $M \in \mathcal{B}(\mathcal{H}, L^2(X,\mu))$ and $x \in X$ denote by ψ_x the unique vector of $\mathcal{H}$ such that $(Mh)(x) = \langle h|\psi_x\rangle$ for every $h \in \mathcal{H}$. Then, there exists a closed, densely defined operator A_M such that ϕ is a continuous weak atomic system for A_M and $\psi : x \in X \to \psi_x \in \mathcal{H}$ is a Bessel function which is a weak A_M-dual of ϕ.

Proof Let us consider the operator $B : \mathcal{D} \to L^2(X,\mu)$ defined for every $u \in \mathcal{D}$ by $(Bu)(x) = \langle u|\phi_x\rangle, \forall x \in X$ which is a restriction of the analysis operator C_ϕ. Since B is densely defined, then B^*, the adjoint of B, is well defined. Now fix any $M \in \mathcal{B}(\mathcal{H}, L^2(X,\mu))$, for every $h \in \mathcal{H}$ and any $x \in X$ by the Riesz representation theorem there exists a function $\psi : x \in X \to \psi_x \in \mathcal{H}$ such that $(Mh)(x) = \langle h|\psi_x\rangle$. By the same calculations than in Theorem 3.20, ψ is a Bessel function. Consider the closed operator $E = B^*M$, then $E^* \supset M^*B^{**} \supset M^*B$ and define $F = E^*_{\restriction \mathcal{D}} = M^*B$ which is closable and densely defined. Then $\mathcal{D}(F^*)$ is dense and $\forall u \in \mathcal{D} = \mathcal{D}(F)$ and $\forall h \in \mathcal{D}(F^*)$ we have

$$\langle F^*h|u\rangle = \langle h|Fu\rangle = \langle h|M^*Bu\rangle = \langle Mh|Bu\rangle_2$$
$$= \int_X \langle h|\psi_x\rangle \langle \phi_x|u\rangle \mathrm{d}\mu(x).$$

It suffices now to take $A_M = F^*$. □

If $\mathcal{R}(A)$ is weakly decomposable, then $\mathcal{R}(A^*)$ is weakly decomposable too, as shown in the next Proposition.

Proposition 3.22 *Let A be a densely defined operator on $\mathcal{H}$, ϕ a continuous weak atomic system for A and ψ a Bessel weak A-dual of ϕ. Then, the adjoint A^* of A admits a weak decomposition and*

$$\langle A^*u|f\rangle = \int_X \langle u|\phi_x\rangle\langle\psi_x|f\rangle\mathrm{d}\mu(x), \qquad \forall u \in \mathcal{D}(A^*), \forall f \in \mathcal{D}(A).$$

Proof   Fix any $u \in \mathcal{D}(A^*)$ then, for every $f \in \mathcal{D}(A)$

$$\begin{aligned}\langle A^*u|f\rangle &= \langle u|Af\rangle = \int_X \overline{\langle f|\psi_x\rangle\langle\phi_x|u\rangle}\mathrm{d}\mu(x) \\ &= \int_X \langle u|\phi_x\rangle\langle\psi_x|f\rangle\mathrm{d}\mu(x).\end{aligned}$$

$\square$

Remark 3.23   In the discrete case, i.e. for $X = \mathbb{N}$ and μ a counting measure, albeit a strong decomposition of A is still not guaranteed in general, the adjoint A^* admits a strong decomposition [10, Remark 3.13], in the sense that

$$A^*u = \sum_{n=1}^{\infty}\langle u|\phi_n\rangle\psi_n, \quad \forall u \in \mathcal{D}(A^*)$$

with $\{\psi_n\}$ a Bessel weak A-dual of the weak A-frame $\{\phi_n\}$.

Remark 3.24   Contrarily to the case in which the operator is in $\mathcal{B}(\mathcal{H})$, given a closed densely defined operator A on $\mathcal{H}$ and a continuous weak A-frame ϕ, a weak A-dual ψ of ϕ is not a continuous weak A^*-frame, in general. For example, if A is unbounded and ψ is also a Bessel function, from the inequality

$$\alpha\|Af\|^2 \leq \int_X |\langle f|\psi_x\rangle|^2\mathrm{d}\mu(x), \quad \forall f \in \mathcal{D}(A)$$

with $\alpha > 0$, we obtain that A is bounded, a contradiction.

We conclude this section by proving that, under suitable hypotheses, we can weakly decompose the domain of A^* by means of a continuous weak A-frame.

Theorem 3.25   *Let A be a closed densely defined operator with $\mathcal{R}(A) = \mathcal{H}$ and $A^\dagger$ the pseudo-inverse of A. Let ϕ be a continuous weak A-frame and ψ a Bessel weak A-dual of ϕ. Then, the function ϑ with $\vartheta_x := (A^\dagger)^*\psi_x \in \mathcal{H}$, for every $x \in X$, is Bessel and every $u \in \mathcal{D}(A^*)$ can be weakly decomposed as follows*

$$\langle h|u\rangle = \int_X \langle h|\vartheta_x\rangle\langle\phi_x|u\rangle\mathrm{d}\mu(x) \quad \forall h \in \mathcal{H}, u \in \mathcal{D}(A^*).$$

Proof By Lemma 2.10 there exists a unique pseudo-inverse $A^\dagger \in \mathcal{B}(\mathcal{H})$ of A such that $h = AA^\dagger h$, $h \in \mathcal{H}$. Then,

$$\langle h|u\rangle = \left\langle AA^\dagger h|u\right\rangle = \int_X \langle A^\dagger h|\psi_x\rangle\langle\phi_x|u\rangle\mathrm{d}\mu(x) \quad \forall h \in \mathcal{H}, u \in \mathcal{D}(A^*).$$

Consider the adjoint $(A^\dagger)^* \in \mathcal{B}(\mathcal{H})$ of $A^\dagger$ and define $\vartheta_x := (A^\dagger)^*\psi_x \in \mathcal{H}$, for every $x \in X$. Then, for any $h \in \mathcal{H}$, we have

$$\begin{aligned}
\langle h|u\rangle &= \int_X \langle h|(A^\dagger)^*\psi_x\rangle\langle\phi_x|u\rangle\mathrm{d}\mu(x) \\
&= \int_X \langle h|\vartheta_x\rangle\langle\phi_x|u\rangle\mathrm{d}\mu(x), \quad \forall u \in \mathcal{D}(A^*)
\end{aligned}$$

and

$$\int_X |\langle h|\vartheta_x\rangle|^2 \mathrm{d}\mu(x) = \int_X \left|\left\langle h\left|(A^\dagger)^*\psi_x\right\rangle\right|^2 \mathrm{d}\mu(x) = \int_X \left|\left\langle A^\dagger h|\psi_x\right\rangle\right|^2 \mathrm{d}\mu(x)$$
$$\leq \gamma\|A^\dagger h\|^2 \leq \gamma\|A^\dagger\|^2\|h\|^2$$

for some $\gamma > 0$ since ψ is Bessel and $A^\dagger$ is bounded. Hence, $\vartheta : x \in X \to \vartheta_x \in \mathcal{H}$ is a Bessel function. $\qquad\square$

Remark 3.26 In the discrete case the decomposition of the domain of $\mathcal{D}(A^*)$ is strong [10].

4 Continuous atomic systems for bounded operators between different Hilbert spaces

In this section we introduce our second approach to the generalization of the notion of (discrete) atomic system for $K \in \mathcal{B}(\mathcal{H})$ and of K-frame in [22], to unbounded operators in a Hilbert space in the continuous framework. A closed densely defined operator in a Hilbert space $A : \mathcal{D}(A) \to \mathcal{H}$ can be seen as a *bounded* operator $A : \mathcal{H}_A \to \mathcal{H}$ between two different Hilbert spaces, with $\mathcal{H}_A$ the Hilbert space $\mathcal{D}(A)[\|\cdot\|_A]$ where $\|\cdot\|_A$ is the graph norm. Hence, before introducing new notions, we put the main definitions and results in [2, 22] for $K \in \mathcal{B}(\mathcal{H})$ in terms of bounded operators from a Hilbert space into another. Later, in Sect. 4.1, we return to the operator $A : \mathcal{H}_A \to \mathcal{H}$.

Let $\mathcal{H}, \mathcal{J}$ be two Hilbert spaces with inner products $\langle\cdot|\cdot\rangle_\mathcal{H}, \langle\cdot|\cdot\rangle_\mathcal{J}$ and induced norms $\|\cdot\|_\mathcal{H}, \|\cdot\|_\mathcal{J}$, respectively. We denote by $\mathcal{B}(\mathcal{J}, \mathcal{H})$ the set of bounded linear operators from $\mathcal{J}$ into $\mathcal{H}$. For any $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ we denote by $K^* \in \mathcal{B}(\mathcal{H}, \mathcal{J})$ its adjoint.

Definition 4.1 Let $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$. The function $\phi : x \in X \to \phi_x \in \mathcal{H}$ is a continuous atomic system for K if for all $h \in \mathcal{H}$, the map $x \to \langle h|\phi_x\rangle_\mathcal{H}$ is a measurable function on X and

(i) ϕ is Bessel function
(ii)   there exists $\gamma > 0$ such that for all $f \in \mathcal{J}$ there exists $a_f \in L^2(X, \mu)$, with
$\|a_f\|_2 = \left(\int_X |a_f(x)|^2 \mathrm{d}\mu(x) \right)^{1/2} \leq \gamma \|f\|_{\mathcal{J}}$ and for every $g \in \mathcal{H}$

$$\langle Kf|g \rangle_{\mathcal{H}} = \int_X a_f(x) \langle \phi_x|g \rangle_{\mathcal{H}} \mathrm{d}\mu(x).$$

If $\mathcal{J} = \mathcal{H}$ and μ is the counting measure, then the previous notion reduces to the notion of atomic system for $K \in \mathcal{B}(\mathcal{H})$ in [22].

Example 4.2 Let $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$. Every continuous frame ϕ for $\mathcal{H}$ is a continuous atomic system for K. Indeed, if ψ is a dual frame of ϕ, then for every $h \in \mathcal{H}$

$$\langle Kf|h \rangle_{\mathcal{H}} = \int_X \langle Kf|\psi_x \rangle_{\mathcal{H}} \langle \phi_x|h \rangle_{\mathcal{H}} \mathrm{d}\mu(x), \quad \forall f \in \mathcal{J}$$

and Definition 4.1 is satisfied by taking $a_f(x) = \langle Kf|\psi_x \rangle_{\mathcal{H}}$ for $f \in \mathcal{J}$.

Example 4.3 Let $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ and $\xi : x \in X \to \xi_x \in \mathcal{J}$ a continuous frame for $\mathcal{J}$ with dual frame $\vartheta : x \in X \to \vartheta_x \in \mathcal{J}$, then for all $f, g \in \mathcal{J}$

$$\langle f|g \rangle_{\mathcal{J}} = \int_X \langle f|\vartheta_x \rangle_{\mathcal{J}} \langle \xi_x|g \rangle_{\mathcal{J}} \mathrm{d}\mu(x),$$

hence, for every $h \in \mathcal{H}$

$$\langle Kf|h \rangle_{\mathcal{H}} = \langle f|K^*h \rangle_{\mathcal{J}} = \int_X \langle f|\vartheta_x \rangle_{\mathcal{J}} \langle K\xi_x|h \rangle_{\mathcal{H}} \mathrm{d}\mu(x).$$

Thus the function $\phi = K\xi$ is a continuous atomic system for K, taking $a_f(x) := \langle f|\vartheta_x \rangle_{\mathcal{J}}$.

In the discrete case, the decomposition of $\mathcal{R}(K)$, the range of K, is strong [10].

We give a result of existence of a continuous atomic system for a bounded operator.

Theorem 4.4 Let (X, μ) be a σ-finite measure space, $\mathcal{J}$ a separable Hilbert space and $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$. Then there exists a continuous atomic system for K.

Proof With the same notation than in Theorem 3.19 we have that

$$\int_X |\langle h|\phi_x\rangle_{\mathcal{H}}|^2 \mathrm{d}\mu(x) = \sum_{n=1}^{\infty} \int_{X_n} |\langle h|\phi_x\rangle_{\mathcal{H}}|^2 \mathrm{d}\mu(x) = \sum_{n=1}^{\infty} |\langle h|\phi_n\rangle_{\mathcal{H}}|^2$$

$$= \sum_{n=1}^{\infty} |\langle h|Ke_n\rangle_{\mathcal{H}}|^2 = \sum_{n=1}^{\infty} |\langle K^*h|e_n\rangle_{\mathcal{J}}|^2$$

$$= \|K^*h\|_{\mathcal{J}}^2 \le \|K^*\|_{\mathcal{H},\mathcal{J}}^2 \|h\|_{\mathcal{H}}^2,$$

where the last equality is due to the Parseval identity. The thesis follows from Theorem 3.19, with slight modifications due to the fact that $K \in \mathcal{B}(\mathcal{J},\mathcal{H})$. $\quad\square$

Definition 4.5 Let $K \in \mathcal{B}(\mathcal{J},\mathcal{H})$. A function $\phi : x \in X \to \phi_x \in \mathcal{H}$ is called a *continuous K-frame for* $\mathcal{H}$ if for all $h \in \mathcal{H}$, the map $x \to \langle h|\phi_x\rangle_{\mathcal{H}}$ is a measurable function on X and there exist $\alpha, \beta > 0$ such that for every $h \in \mathcal{H}$

$$\alpha\|K^*h\|_{\mathcal{J}}^2 \le \int_X |\langle h|\phi_x\rangle_{\mathcal{H}}|^2 \mathrm{d}\mu(x) \le \beta\|h\|_{\mathcal{H}}^2. \tag{4.1}$$

The constants α, β will be called frame bounds.

It is easy to see that if $K \in \mathcal{B}(\mathcal{J},\mathcal{H})$ and ϕ is a continuous frame for $\mathcal{J}$, then $K\phi$ is a continuous K-frame for $\mathcal{H}$. Then we give the following two examples.

Example 4.6 Let $X = \mathbb{R}$ and let μ be the Lebesgue measure. Let us identify $\mathcal{J} = \mathcal{H} = L^2(0,1)$ and let $\mathcal{I}_{(0,1)}$ be the identity of $L^2(0,1)$. Fix any $g \in C(0,1)$, the space of continuous functions on the open interval $(0,1)$ (or also $g \in L^\infty(0,1)$ the space of essentially bounded functions on $(0,1)$), and consider the self-adjoint operator $M_g \in \mathcal{B}(L^2(0,1))$ defined by $M_g f = gf$ for every $f \in L^2(0,1)$. Then, $\phi_t := ge^{2\pi i t\cdot}\mathcal{I}_{(0,1)}$ is a continuous M_g-frame. Indeed, as proved in [12, Example 4.2], the function $\phi : t \in \mathbb{R} \to \phi_t \in L^2(0,1)$ such that $\phi_t := e^{2\pi i t\cdot}\mathcal{I}_{(0,1)}$ is a Parseval frame in $L^2(0,1)$, hence $\phi = M_g\phi$ is a continuous M_g-frame.

Remark 4.7 If $\mathcal{J} = \mathcal{H}$ a continuous K-frame ϕ is a continuous K-g-frame in the sense of [2, Definition 2.1] with $\Lambda_x = \langle f|\phi_x\rangle$ for every $f \in \mathcal{H}$, with $x \in X$. If $K \in \mathcal{B}(\mathcal{J},\mathcal{H})$, $X = \mathbb{N}$ and μ is the counting measure, a continuous K-frame clearly reduces to a discrete K-frame in the sense of [10] and, if in addition $\mathcal{J} = \mathcal{H}$, coincides with that of K-frame in [22].

Proposition 4.8 *Let* $\mathcal{H}$, $\mathcal{J}$ *and* $\mathcal{F}$ *be Hilbert spaces*, $K \in \mathcal{B}(\mathcal{J},\mathcal{H})$, $E \in \mathcal{B}(\mathcal{H},\mathcal{F})$, $G \in \mathcal{B}(\mathcal{H},\mathcal{J})$ *and* ϕ *be a continuous K-frame for* $\mathcal{H}$, *then*

(i)   *$E\phi$ is a continuous EK-frame for $\mathcal{F}$;*
(ii)   *ϕ is a continuous KG-frame for $\mathcal{H}$ too.*

Proof (*i*) It is a slight modification of the proof in [2, Theorem 3.4].
(*ii*) It descends from Proposition 3.7 with obvious adaptations. $\quad\square$

A natural consequence is the following corollary, see also [2, Corollary 3.5].

Corollary 4.9 *Let $K \in \mathcal{B}(\mathcal{H})$ and ϕ be a continuous K-frame for $\mathcal{H}$, then ϕ and $K^n\phi$ are continuous K^{n+1}-frames for $\mathcal{H}$, for every integer $n \geq 0$.*

Let us give a characterization of continuous atomic systems for operators in $\mathcal{B}(\mathcal{J}, \mathcal{H})$.

Theorem 4.10 *Let $\phi : x \in X \rightarrow \phi_x \in \mathcal{H}$ and $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$. Then the following are equivalent.*

   (i)   *ϕ is a continuous atomic system for K;*
   (ii)   *ϕ is a continuous K-frame for $\mathcal{H}$;*
  (iii)   *ϕ is a Bessel function and there exists a Bessel function $\psi : X \rightarrow \mathcal{J}$ such that*

$$\langle Kf|h\rangle_{\mathcal{H}} = \int_X \langle f|\psi_x\rangle_{\mathcal{J}}\langle\phi_x|h\rangle_{\mathcal{H}}\mathrm{d}\mu(x) \quad \forall f \in \mathcal{J}, \forall h \in \mathcal{H}. \tag{4.2}$$

Proof The proof follows from Theorem 3.20, with suitable adjustments, recalling that if ϕ is a continuous K-frame for $\mathcal{H}$, then it is a Bessel function. □

As in the discrete case,

Definition 4.11 Let $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ and $\phi : x \in X \rightarrow \phi_x \in \mathcal{H}$ a continuous K-frame for $\mathcal{H}$. A function $\psi : X \rightarrow \mathcal{H}$ as in (4.2) is called a *K-dual* of ϕ.

Example 4.12 In general, a K-dual $\psi : x \in X \rightarrow \psi_x \in \mathcal{J}$ of a continuous K-frame $\phi : x \in X \rightarrow \phi_x \in \mathcal{H}$ is not unique. Let us see some examples.

   (i)   If $\phi = \zeta$, where $\zeta : X \rightarrow \mathcal{H}$ is a continuous frame for $\mathcal{H}$, then one can take $\psi = K^*\xi : X \rightarrow \mathcal{J}$ where $\xi : x \in X \rightarrow \xi_x \in \mathcal{H}$ is any dual frame of ζ.
   (ii)   If $\phi = K\zeta$, where $\zeta : x \in X \rightarrow \zeta_x \in \mathcal{J}$ is a continuous frame for $\mathcal{J}$, then one can take as ψ any dual frame of ζ.

Remark 4.13 Once at hand a continuous atomic system ϕ for K, a Bessel K-dual $\psi : X \rightarrow \mathcal{J}$ as in Theorem 4.10 is a continuous atomic system for K^*. Indeed,

$$\langle K^*h|f\rangle_{\mathcal{J}} = \langle h|Kf\rangle_{\mathcal{H}} = \int_X \overline{\langle f|\psi_x\rangle_{\mathcal{J}}\langle\phi_x|h\rangle_{\mathcal{H}}}\mathrm{d}\mu(x)$$
$$= \int_X \langle h|\phi_x\rangle_{\mathcal{H}}\langle\psi_x|f\rangle_{\mathcal{J}}\mathrm{d}\mu(x), \quad f \in \mathcal{J}, h \in \mathcal{H}.$$

We apply Theorem 4.10 to K^* and ψ to conclude that ψ is a continuous atomic system for K^*.

Following H.G. Feichtinger and T. Werther [21],

Definition 4.14 Let $\phi : x \in X \rightarrow \phi_x \in \mathcal{H}$ be a Bessel function and $\mathcal{H}_0$ a closed subspace of $\mathcal{H}$. The function ϕ is called a continuous family of local atoms for $\mathcal{H}_0$ if there exists a family of linear functionals $\{c_x\}$ with $c_x : \mathcal{H} \rightarrow \mathbb{C}$ for every $x \in X$, such that

(i)   exists $\gamma > 0$ with $\int_X |c_x(f)|^2 \mathrm{d}\mu(x) \leq \gamma \|f\|^2$, $\forall f \in \mathcal{H}_0$;

(ii) $\langle f|h \rangle = \int_X c_x(f)\langle \phi_x|h \rangle \mathrm{d}\mu(x)$, $\forall f \in \mathcal{H}_0, h \in \mathcal{H}$.

We will say that the pair $\{\phi_x, c_x\}$ provides an atomic decomposition for $\mathcal{H}_0$ and γ will be called an atomic bound of $\{\phi_x\}$.

If now $K = P_{\mathcal{H}_0} \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection on $\mathcal{H}_0$, i.e. $P_{\mathcal{H}_0} = P_{\mathcal{H}_0}^2 = P_{\mathcal{H}_0}^*$, a continuous $P_{\mathcal{H}_0}$-frame is a family of continuous local atoms for $\mathcal{H}_0$, similarly to [22, Theorem 5].

Corollary 4.15 *Let $\phi : x \in X \to \phi_x \in \mathcal{H}$ be a Bessel function and $\mathcal{H}_0$ a closed subspace of the Hilbert space $\mathcal{H}$. Then the following statements are equivalent.*

(i)   *$\{\phi_x\}$ is a family of continuous local atoms for $\mathcal{H}_0$;*

(ii)   *ϕ is a continuous atomic system for $P_{\mathcal{H}_0}$;*

(iii)   *there exists $\alpha > 0$ such that $\alpha \|P_{\mathcal{H}_0}f\|^2 \leq \int_X |\langle f|\phi_x \rangle|^2 \mathrm{d}\mu(x), f \in \mathcal{H}$;*

(iv)   *there exists a Bessel function $\psi : x \in X \to \psi_x \in \mathcal{H}$ such that*

$$\langle P_{\mathcal{H}_0}f|h \rangle = \int_X \langle f|\psi_x \rangle \langle \phi_x|h \rangle \mathrm{d}\mu(x),$$

for any $f, h \in \mathcal{H}$.

Not even if $\mathcal{J} = \mathcal{H}$ a Bessel function $\phi : X \to \mathcal{H}$ and a K-dual $\psi : X \to \mathcal{H}$ of its are interchangeable, in general. However, if we strengthen hypotheses on K, it can be proved the existence of a function with range in $\mathcal{H}$ which is interchangeable with ϕ in the weak decomposition of $\mathcal{R}(K) \subset \mathcal{H}$, see also [2, Theorem 3.2].

Theorem 4.16 *Let $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ with closed range $\mathcal{R}(K)$. Let ϕ be a continuous K-frame and ψ a Bessel K-dual of its. Then,*

(i)   *the function $\vartheta : x \in X \to \vartheta_x \in \mathcal{H}$ with $\vartheta_x := (K_{\restriction \mathcal{R}(K)}^{\dagger})^* \psi_x \in \mathcal{H}$, for every $x \in X$, is Bessel for $\mathcal{R}(K)$ and interchangeable with ϕ for any $h \in \mathcal{R}(K)$, i.e.*

$$\langle h|f \rangle_{\mathcal{H}} = \int_X \langle h|\vartheta_x \rangle_{\mathcal{H}} \langle \phi_x|f \rangle_{\mathcal{H}} \mathrm{d}\mu(x) = \int_X \langle h|\phi_x \rangle_{\mathcal{H}} \langle \vartheta_x|f \rangle_{\mathcal{H}} \mathrm{d}\mu(x), f \in \mathcal{H};$$

ii)   *ϑ is a continuous K-frame for $\mathcal{H}$ and $K^*\vartheta$ and $K^*\phi$ are Bessel K-duals of ϕ and of ϑ respectively. In particular, for every $h \in \mathcal{H}$*

$$\begin{aligned} \langle Kf|h \rangle_{\mathcal{H}} &= \int_X \langle f|K^*\vartheta_x \rangle_{\mathcal{J}} \langle \phi_x|h \rangle_{\mathcal{H}} \mathrm{d}\mu(x) \\ &= \int_X \langle f|K^*\phi_x \rangle_{\mathcal{J}} \langle \vartheta_x|h \rangle_{\mathcal{H}} \mathrm{d}\mu(x), \quad \forall f \in \mathcal{J}. \end{aligned} \tag{4.3}$$

Proof *(i)* See [2, Theorem 3.2] with obvious adjustments.

(ii) Clearly (4.3) follows from *(i)*. The function ϑ is a continuous K-frame for $\mathcal{H}$

by (i) and (4.3), taking for all $f \in \mathcal{J}$, $a_f(x) = \langle f | K^* \phi_x \rangle_{\mathcal{J}}$, for every $x \in X$. The functions $K^* \vartheta$ and $K^* \phi$ are Bessel for $\mathcal{J}$, indeed for all $f \in \mathcal{J}$, the maps $x \to \left\langle K^\dagger_{|\mathcal{R}(K)} K f | \psi_x \right\rangle_{\mathcal{J}} = \langle f | K^* \vartheta_x \rangle_{\mathcal{J}}$ and $x \to \langle K f | \phi_x \rangle_{\mathcal{H}} = \langle f | K^* \phi_x \rangle_{\mathcal{J}}$ are measurable functions on X and

$$\int_X |\langle f | K^* \vartheta_x \rangle_{\mathcal{J}}|^2 \mathrm{d}\mu(x) = \int_X |\langle K f | \vartheta_x \rangle_{\mathcal{H}}|^2 \mathrm{d}\mu(x)$$
$$\leq \beta \| K f \|_{\mathcal{H}}^2 \leq \beta \| K \|_{\mathcal{J}, \mathcal{H}}^2 \| f \|_{\mathcal{J}}^2, \quad \forall f \in \mathcal{J}$$

for some $\beta > 0$. Similarly, $K^* \phi$ is Bessel. The proof is concluded by using Theorem 4.10.   □

Remark 4.17 Consider the function $\phi : x \in X \to \phi_x \in \mathcal{H}$. In this section the frame operator S_ϕ of ϕ will be denoted by

$$\langle S_\phi f | g \rangle_{\mathcal{H}} = \int_X \langle f | \phi_x \rangle_{\mathcal{H}} \langle \phi_x | g \rangle_{\mathcal{H}} \mathrm{d}\mu(x), \qquad f \in \mathcal{D}(S_\phi), g \in \mathcal{H}$$

where

$$\mathcal{D}(S_\phi) = \{ f \in \mathcal{H} : \int_X \langle f | \phi_x \rangle_{\mathcal{H}} \phi_x \mathrm{d}\mu(x) \text{ converges weakly in } \mathcal{H} \}$$

Later on, in Remark 4.20, we will see that, as for continuous K-frames with $K \in \mathcal{B}(\mathcal{H})$, the domain $\mathcal{D}(S_\phi)$ of the frame operator of a continuous K-frame with $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ coincides with the whole $\mathcal{H}$.

The analysis operator of the function ϕ will be indicated by $C_\phi : h \in \mathcal{D}(C_\phi) \subset \mathcal{H} \to \langle h | \phi_x \rangle_{\mathcal{H}} \in L^2(X, \mu)$ strongly defined, for every $h \in \mathcal{D}(C_\phi)$ and for every $x \in X$, by

$$(C_\phi h)(x) = \langle h | \phi_x \rangle_{\mathcal{H}}$$

and the synthesis operator of ϕ by $C_\phi^* : \mathcal{D}(C_\phi^*) \subset L^2(X, \mu) \to \mathcal{H}$ will be denoted by:

$$\left\langle C_\phi^* a | h \right\rangle_{\mathcal{H}} = \int_X a(x) \langle \phi_x | h \rangle_{\mathcal{H}} \mathrm{d}\mu(x), \quad a \in \mathcal{D}(C_\phi^*), h \in \mathcal{H}$$

where

$$\mathcal{D}(C_\phi^*) := \left\{ a \in L^2(X, \mu) : \int_X a(x) \langle \phi_x | h \rangle_{\mathcal{H}} \mathrm{d}\mu(x) \text{ exists } \forall h \in \mathcal{H} \right\}.$$

We can characterize continuous K-frames for $\mathcal{H}$ by means of both their frame and synthesis operators.

Theorem 4.18 *Let $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ and $\phi : x \in X \to \phi_x \in \mathcal{H}$ such that for all $f \in \mathcal{H}$, the map $x \to \langle f | \phi_x \rangle$ is a measurable function on X. Then the following statements are equivalent.*

(i)   *ϕ is a continuous K-frame for $\mathcal{H}$;*

(ii)   *C_ϕ^* is bounded and $\mathcal{R}(K) \subset \mathcal{R}(C_\phi^*)$;*

(iii) *C_ϕ^* is bounded and there exists $M \in \mathcal{B}(\mathcal{J}, L^2(X, \mu))$ such that $K = C_\phi^* M$;*

(iv) *$S_\phi = C_\phi^* C_\phi \geq \alpha K K^*$ on $\mathcal{H}$ (i.e. $\langle S_\phi f | f \rangle_\mathcal{H} \geq \alpha \langle K K^* f | f \rangle_\mathcal{H}$ for every $f \in \mathcal{H}$) for some $\alpha > 0$ and ϕ is a Bessel function for $\mathcal{H}$;*

(v)   *$K = \left(S_\phi^{1/2} \right) U$, for some $U \in \mathcal{B}(\mathcal{J}, \mathcal{H})$.*

Proof $(i) \Rightarrow (ii)$ The operator C_ϕ^* is bounded by Proposition 2.5. Moreover, for every $h \in \mathcal{H}$

$$\alpha \|K^* h\|_\mathcal{J}^2 \leq \int_X |\langle h | \phi_x \rangle_\mathcal{H}|^2 \mathrm{d}\mu(x) = \|C_\phi h\|_2^2.$$

By Lemma 2.11, it follows that $\mathcal{R}(K) \subset \mathcal{R}(C_\phi^*)$.

$(ii) \Rightarrow (iii)$ By Lemma 2.11 there exists a bounded operator $M : \mathcal{J} \to L^2(X, \mu)$ such that $K = C_\phi^* M$.

$(iii) \Rightarrow (i)$ ϕ is a continuous K-frame for $\mathcal{H}$ since

$$\begin{aligned}
\|K^* h\|_\mathcal{J}^2 &= \|M^* C_\phi h\|_\mathcal{J}^2 \leq \|M^*\|_{L^2, \mathcal{J}}^2 \|C_\phi h\|_\mathcal{J}^2 \\
&= \|M^*\|_{L^2, \mathcal{J}}^2 \int_X |\langle h | \phi_x \rangle_\mathcal{H}|^2 \mathrm{d}\mu(x) \leq \beta \|M^*\|_{L^2, \mathcal{J}}^2 \|h\|_\mathcal{H}^2
\end{aligned}$$

by the boundedness of C_ϕ.

$(i) \Leftrightarrow (iv)$ See [2, Lemma 2.4] with $\Lambda_x = \langle f | \phi_x \rangle$ for every $f \in \mathcal{H}$, with $x \in X$.

$(i) \Rightarrow (v)$ The operator S_ϕ is positive, bounded and everywhere defined in $\mathcal{H}$ because, by definition of continuous K-frame for $\mathcal{H}$, there exists $\beta > 0$ such that

$$0 \leq \langle S_\phi f | f \rangle_\mathcal{H} = \int_X |\langle f | \phi_x \rangle_\mathcal{H}|^2 \mathrm{d}\mu(x) \leq \beta \|f\|_\mathcal{H}^2, \quad \forall f \in \mathcal{H}.$$

Hence $S_\phi = S_\phi^{1/2} S_\phi^{1/2}$, with $S_\phi^{1/2}$ positive self-adjoint operator and, by hypothesis, there exists $\alpha > 0$ such that

$$\alpha \|K^* f\|_\mathcal{H}^2 \leq \left\| S_\phi^{1/2} f \right\|_\mathcal{H}^2, \quad \forall f \in \mathcal{H}.$$

By Lemma 2.11, there exists $U \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ such that $K = \left(S_\phi^{1/2} \right) U$.

$(v) \Rightarrow (i)$ By hypothesis there exists $U \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ such that $K^* = \left(\left(S_\phi^{1/2} \right) U \right)^* = U^* S_\phi^{1/2}$, then, for every $f \in \mathcal{H}$

$$\|K^*f\|_{\mathcal{J}}^2 = \left\| U^* S_\phi^{1/2} f \right\|_{\mathcal{J}}^2 \le \|U^*\|_{\mathcal{H},\mathcal{J}}^2 \left\| S_\phi^{1/2} f \right\|_{\mathcal{H}}^2 \le \|U^*\|_{\mathcal{H},\mathcal{J}}^2 \left\| S_\phi^{1/2} \right\|_{\mathcal{H},\mathcal{H}}^2 \|f\|_{\mathcal{H}}^2,$$

hence ϕ is a continuous K-frame for $\mathcal{H}$. $\hspace{2cm}$ □

Remark 4.19 Nothing guarantees the closedness of $\mathcal{R}(C_\phi^*)$, then by Theorem 4.18 (*iii*) it follows that a continuous K-frame is not automatically a continuous frame for the subspace $\overline{\text{span}}\{\phi_x\}$, the closed linear span of $\{\phi_x\}$, which is in turn a Hilbert space, see [13, Corollary 5.5.2] for the discrete case.

Remark 4.20 As usual, the frame operator S_ϕ of a continuous K-frame for $\mathcal{H}$, with $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$, is a linear positive bounded operator in $\mathcal{H}$, indeed $S_\phi = C_\phi^* C_\phi$ with $C_\phi \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$, however, it is not invertible in general. Nevertheless, if we strenghten the hypotheses on K and X, S_ϕ can be invertible on its range. This has been shown in the discrete case in [31, p. 1245]. The proof of the following proposition is analogous to that given therein.

Proposition 4.21 *Let $\phi : x \in X \to \phi_x \in \mathcal{H}$ be a continuous K-frame for $\mathcal{H}$ with $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ having closed range. Then S_ϕ is linear, bounded, self-adjoint, positive and invertible on $\mathcal{R}(K)$.*

4.1 Continuous atomic systems for unbounded operators A and continuous A-frames

The results of Sect. 4 can be used to generalize continuous frames for bounded operators to the case of an unbounded closed and densely defined operator $A : \mathcal{D}(A) \to \mathcal{H}$ viewing it as a bounded operator between two different Hilbert spaces, more precisely, from the Hilbert space $\mathcal{H}_A = \mathcal{D}(A)[\|\cdot\|_A]$, where $\|\cdot\|_A$ is the graph norm induced by the graph inner product $\langle\cdot|\cdot\rangle_A$, into $\mathcal{H}$.

In order to simplify notations, we come back to denote again by $\langle\cdot|\cdot\rangle$ and $\|\cdot\|$ the inner product and the norm of $\mathcal{H}$, respectively.

We will indicate by $A^\sharp : \mathcal{H} \to \mathcal{H}_A$ the adjoint of the bounded operator $A : \mathcal{H}_A \to \mathcal{H}$. With this convention, if $A \in \mathcal{B}(\mathcal{H}_A, \mathcal{H})$, a function $\phi : x \in X \to \phi_x \in \mathcal{H}$ such that for all $f \in \mathcal{H}$, the map $x \to \langle f|\phi_x\rangle$ is a measurable function on X is said to be

(i) a *continuous atomic system* for A if ϕ is a Bessel function and there exists

$\gamma > 0$ such that for all $f \in \mathcal{D}(A)$ there exists $a_f \in L^2(X, \mu)$, with $\|a_f\|_2 = \left(\int_X |a_f(x)|^2 \mathrm{d}\mu(x)\right)^{1/2} \le \gamma\|f\|_A$ and for every $g \in \mathcal{H}$

$$\langle Af|g\rangle = \int_X a_f(x)\langle\phi_x|g\rangle\mathrm{d}\mu(x);$$

(ii) a *continuous A-frame* if there exist $\alpha, \beta > 0$ such that for every $h \in \mathcal{H}$

$$\alpha \|A^\sharp h\|_A^2 \leq \int_X |\langle h|\phi_x\rangle|^2 \mathrm{d}\mu(x) \leq \beta \|h\|^2.$$

Theorems 4.10 and 4.18 can be summarized and rewritten as follows.

Corollary 4.22 *Let $\phi : x \in X \to \phi_x \in \mathcal{H}$ and suppose that for all $h \in \mathcal{H}$, the map $x \to \langle h|\phi_x\rangle$ is a measurable function on X. Let A be a closed densely defined operator on $\mathcal{H}$. Then the following are equivalent.*

(i) *ϕ is a continuous atomic system for A;*

(ii) *ϕ is a continuous A-frame;*

(iii) *ϕ is a Bessel function and there exists ψ a Bessel function of $\mathcal{H}_A$ such that*

$$\langle Af|h\rangle = \int_X \langle f|\psi_x\rangle_A \langle \phi_x|h\rangle \mathrm{d}\mu(x), \quad \forall f \in \mathcal{D}(A), \forall h \in \mathcal{H};$$

(iv) *C_ϕ^* is bounded and $\mathcal{R}(A) \subset \mathcal{R}(C_\phi^*)$;*

(v) *C_ϕ^* is bounded and there exists $M \in \mathcal{B}(\mathcal{H}_A, L^2(X, \mu))$ such that $A = C_\phi^* M$;*

(vi) *$S_\phi = C_\phi^* C_\phi \geq \alpha A A^\sharp$ on $\mathcal{H}$, for some $\alpha > 0$ and ϕ is a Bessel function for $\mathcal{H}$;*

(vii) *$A = \left(S_\phi^{1/2}\right) U$, for some $U \in \mathcal{B}(\mathcal{H}_A, \mathcal{H})$.*

Note also that if $A \in \mathcal{B}(\mathcal{H})$, then the graph norm of A is defined on $\mathcal{H}$ and it is equivalent to $\|\cdot\|$, thus our notion of continuous A-frame reduces to that of literature, see e.g. [2].

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