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A sharp estimate for Neumann eigenvalues of the Laplace–Beltrami operator for domains in a hemisphere

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Here, we prove an isoperimetric inequality for the harmonic mean of the first $N-1$ non-trivial Neumann eigenvalues of the Laplace–Beltrami operator for domains contained in a hemisphere of $\mathbb{S}^N$.

Keywords: Neumann eigenvalues; Laplace–Beltrami operator; sphere; isoperimetric inequalities.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ and let us consider the eigenvalues of the classical Neumann–Laplacian in Ω,

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots$$

arranged in a non-decreasing sequence where each eigenvalue is repeated according to its multiplicity. Isoperimetric inequalities for the μ_i's go back to the classical theorem of Szegő [17] and Weinberger [19]: *the ball maximizes* $\mu_1(\Omega)$ *among all bounded smooth domains* Ω *in* $\mathbb{R}^N$ *having the same measure.* Szegő, using conformal maps, proved it for simply connected domains in $\mathbb{R}^2$, while Weinberger introduced a method that allowed him to get this result in full generality in $\mathbb{R}^N$. His technique has been adapted in different contexts to establish isoperimetric results for combination of eigenvalues of the Laplacian with Dirichlet or Neumann boundary conditions (see e.g. [2, 5, 6, 8, 9, 11, 12, 16]). For further references see, e.g. the monographs [10, 14, 15] and the survey paper [1]. Actually, as well-known, the conformal map technique used by Szegő allows to prove the stronger inequality

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} \geq \frac{2}{\mu_1(\Omega^\star)}, \tag{1}$$

again for simply connected domains in $\mathbb{R}^2$. Here and in the sequel, $\Omega^\star$ will denote the disk, or, more in general, the ball in $\mathbb{R}^N$ having the same measure as Ω. Inequality (1) is sharp since equality is achieved if and only if Ω is a disk. Later, in [3], the assumption of simply connectedness was removed. In the same paper, the authors conjectured that an inequality analogous to (1) holds true in $\mathbb{R}^N$ $(N \geq 1)$, namely

$$\frac{1}{\mu_1(\Omega)} + \cdots + \frac{1}{\mu_N(\Omega)} \geq \frac{N}{\mu_1(\Omega^\star)}.$$

Very recently, in [18] the authors made an important step toward the proof of this conjecture, by showing the following inequality

$$\frac{1}{\mu_1(\Omega)} + \cdots + \frac{1}{\mu_{N-1}(\Omega)} \geq \frac{N-1}{\mu_1(\Omega^\star)},$$

for $N \geq 2$.

The aim of this paper is to prove an analogous result for the Laplace–Beltrami operator with Neumann boundary conditions. Precisely, we deal with nontrivial Neumann eigenvalues of an arbitrary domain Ω contained in a hemisphere of $\mathbb{S}^N$, defined by the following boundary value problem:

$$\begin{cases} -\Delta_{\mathbb{S}^N} u = \mu u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where ν is the unit normal to $\partial\Omega$. We still denote with $\{\mu_i(\Omega)\}_i$ the non-decreasing sequence of eigenvalues of (2), where each eigenvalue is repeated according to its multiplicity, that is

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots .$$

Let us denote by u_i an eigenfunction corresponding to $\mu_i(\Omega)$, with $i \in \mathbb{N}_0$. The following variational characterization holds true

$$\mu_i(\Omega) = \min\left\{ \frac{\displaystyle\int_\Omega |\nabla\phi|^2 \, d\omega}{\displaystyle\int_\Omega \phi^2 d\omega} : \phi \in H^1(\Omega)\backslash\{0\},\ \phi \in \operatorname{span}\{u_0, u_1, \ldots, u_{i-1}\}^\perp \right\}. \tag{3}$$

The analogous of the Szegő–Weinberger result is already known and was proved in [4]. Our main result is the following.

Theorem 1.1. *With the notation as above,*

$$\sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \geq \sum_{i=1}^{N-1} \frac{1}{\mu_i(D_\gamma)} = \frac{N-1}{\mu_1(D_\gamma)}, \tag{4}$$

where D_γ is a geodesic ball contained in a hemisphere of $\mathbb{S}^N$ having the same N-volume as Ω, and γ is its radius. More precisely, γ is determined by

$$|\Omega| = N\omega_N \int_0^\gamma \sin^{N-1} t dt,$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. Equality sign holds in (4) if and only if Ω is a geodesic ball.

2. Properties of the Neumann Eigenvalues and Eigenfunctions of a Geodesic Ball

Let D_γ be a geodesic ball on $\mathbb{S}^N$ having radius γ. We think of this geodesic ball as the set of points of $\mathbb{S}^N$ with angle from the positive x_{N+1}-axis less than γ, that is a polar cap. By the standard separation of variables technique, we find that the eigenvalues of (2), with $\Omega = D_\gamma$, are the eigenvalues of the following one-dimensional problems

$$\begin{cases} -\dfrac{1}{\sin^{N-1}\theta}\dfrac{d}{d\theta}\left(\sin^{N-1}\theta\dfrac{dy}{d\theta}\right) + \dfrac{l(l+N-2)}{\sin^2\theta}y = \mu_{l,k}y \quad \text{in } (0,\gamma), \\ y(0) \text{ finite}, \quad y'(\gamma) = 0 \end{cases}$$

with $l \in \mathbb{N}_0, k \in \mathbb{N}$. Clearly, $\mu_1(D_\gamma) = \min\{\mu_{0,2}, \mu_{1,1}\}$. In [4] the authors show that $\mu_1(D_\gamma) = \mu_{1,1}$ at least if $\gamma \leq \frac{\pi}{2}$. Hence, an eigenfunction g (assumed positive) associated to $\mu_{1,1} = \mu_1(D_\gamma)$ satisfies

$$\begin{cases} -g'' - (N-1)\cot\theta g' + \dfrac{N-1}{\sin^2\theta}g = \mu_1(D_\gamma)g \quad \text{in } (0,\gamma), \\ g(0) = g'(\gamma) = 0. \end{cases} \tag{5}$$

Multiplying the equation in (5) by g and then integrating on D_γ yields

$$\mu_1(D_\gamma) = \frac{\displaystyle\int_{D_\gamma} \left[g'(\theta)^2 + (N-1)\frac{g(\theta)^2}{\sin^2\theta}\right] d\omega}{\displaystyle\int_{D_\gamma} g(\theta)^2 d\omega}. \tag{6}$$

The following properties are also proved in [4]:

(i) If $0 < \gamma \leq \frac{\pi}{2}$, then $g' > 0$ in $[0,\gamma)$, thus g is strictly increasing in $[0,\gamma]$,

(ii) $\mu_1(D_\gamma)$ is a strictly decreasing function of γ for $0 < \gamma \le \frac{\pi}{2}$,
(iii) $\mu_1(D_\gamma) > N = \mu_1(D_{\pi/2})$ for $0 < \gamma < \frac{\pi}{2}$.

We also recall that $\mu_1(D_\gamma)$ is N-fold degenerate, that is

$$\mu_1(D_\gamma) = \mu_2(D_\gamma) = \cdots = \mu_N(D_\gamma).$$

Now, define $G : [0, \frac{\pi}{2}] \to [0, +\infty)$ by

$$G(\theta) = \begin{cases} g(\theta) & \theta \le \gamma, \\ g(\gamma) & \theta > \gamma. \end{cases} \tag{7}$$

Then, we have the following.

Lemma 2.1. *The function* $\frac{G(\theta)}{\sin\theta}$ *is strictly decreasing in* $[0, \frac{\pi}{2}]$.

Proof. By Taylor–Frobenius expansion we have $G(\theta) = \theta - a\theta^3 + o(\theta^3)$, where

$$a = \frac{\mu_1(D_\gamma) - \frac{2}{3}(N-1)}{2N+4} > 0.$$

We explicitly observe that we are assuming $G'(0) = 1$. In order to get the claim it is enough to prove that

$$W(\theta) := G'(\theta) - G(\theta)\cot\theta < 0.$$

Using the behavior of $G(\theta)$ near $\theta = 0$, we have

$$W(\theta) = \left(\frac{1}{3} - 2a\right)\theta^2 + o(\theta^2) = \left(\frac{N - \mu_1(D_\gamma)}{N+2}\right)\theta^2 + o(\theta^2).$$

Property (iii) implies that $W(\theta) < 0$ is close to 0. We also know that $W(\gamma) < 0$. Assume by contradiction that $W(\theta)$ attained a positive maximum at a point $\tilde\theta \in (0, \gamma)$. Hence

$$W(\tilde\theta) > 0, \quad W'(\tilde\theta) = G''(\tilde\theta) - G'(\tilde\theta)\cot\tilde\theta + \frac{G(\tilde\theta)}{\sin^2\tilde\theta} = 0.$$

Using this last identity in (5), we obtain

$$N\left[G'(\tilde\theta)\cot\tilde\theta - \frac{G(\tilde\theta)}{\sin^2\tilde\theta}\right] = -\mu_1(D_\gamma)G(\tilde\theta),$$

that is

$$N[W(\tilde\theta)\cot\tilde\theta - G(\tilde\theta)] = -\mu_1(D_\gamma)G(\tilde\theta).$$

Since we are assuming that $W(\tilde\theta) > 0$, property (3) immediately gives a contradiction. □

Remark 2.2. (i) Notice that $\sin\theta \cdot W(\theta)$ is the Wronskian between the eigenfunction corresponding to the first nontrivial Neumann eigenvalue of $D_{\pi/2}$, whose derivative is equal to 1 at 0, and the extension of the eigenfunction corresponding to the first nontrivial Neumann eigenvalue of D_γ, $0 < \gamma < \pi/2$, introduced in (7).

(ii) Using the same method as above we can prove that $G(\theta)$ is concave and, in fact, that $G'(\theta)$ is decreasing. This, together with the result of Lemma 2.1, provides an alternative proof of the fact that the function $B(\theta)$ in [4, Theorem 4.1] is a decreasing function of θ.

3. Some Mathematical Tools Needed for the Proof of Theorem 1.1

For the proof of our main result, Theorem 1.1, it is convenient to parametrize the points of Ω in terms of the coordinates of their stereographic projection (see, for example, [7, 13]). For a point $P \in \Omega$, we denote by P' its stereographic projection from the south pole S onto the "equator" (as illustrated in Fig. 1).

For P' we use Cartesian coordinates $(x_1, x_2, \ldots, x_N, 0)$. We also use $s = \sqrt{\sum_{i=1}^{N} x_i^2}$, the Euclidean distance from P' to the origin O. As usual we denote by θ the azimuthal angle, i.e. the angle between ON and OP, where N stands for the north pole. Moreover, we denote by φ the angle between SN and SP. It is clear that $\theta = 2\varphi$ and $\tan\varphi = s$. Hence,

$$\theta = 2\arctan s \tag{8}$$

from which we immediately get

$$\frac{d\theta}{ds} = \frac{2}{1+s^2} = p(s), \tag{9}$$

the conformal factor associated to the differential structure on $\mathbb{S}^N$. In terms of the conformal factor p we can write

$$\nabla_{\mathbb{S}^N} = \frac{1}{p}\nabla_{\mathbb{R}^N},$$

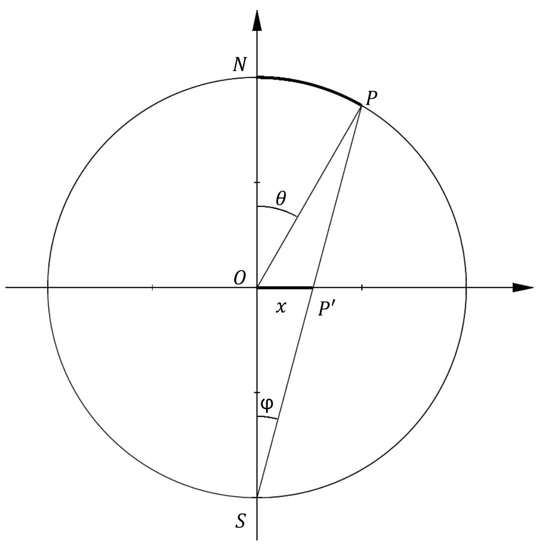

Fig. 1. Stereographic coordinates.

where $\nabla_{\mathbb{R}^N}$ is the standard gradient on the equator. We also have

$$-\Delta_{\mathbb{S}^N} = -p^{-N}\,\mathrm{div}(p^{N-2}\nabla_{\mathbb{R}^N} u).$$

Finally, from the figure (or directly from (8) and (9)) we also have that

$$\sin\theta = p\cdot s. \tag{10}$$

In the sequel we also need to compute $\theta_{,i} := \frac{\partial\theta}{\partial x_i}$. Using (9), the definition of s and the chain rule we have

$$\theta_{,i} = \frac{\partial\theta}{\partial s}\cdot s_{,i} = p\frac{x_i}{s}, \quad i = 1,\ldots,N,$$

and

$$\sum_{i=1}^{N}\theta_{,i}^2 = p^2. \tag{11}$$

With the notation introduced above, we define

$$\Phi_i(x) = G(\theta)\frac{x_i}{s}, \quad i = 1,\ldots,N, \tag{12}$$

where $G(\theta)$ is defined in (7). In order to use Φ_i as test function in (3), we need the following orthogonality conditions

$$\int_\Omega \Phi_i u_j d\omega = 0, \quad i = 1,\ldots,N,\ \ j = 0,\ldots,i-1, \tag{13}$$

where, as we said, u_j is an eigenfunction corresponding to $\mu_j(\Omega)$. To fulfill these conditions, we need a special "orientation" of the sphere $\mathbb{S}^N$. When $j = 0$, conditions (13) can be immediately deduced from [4, Theorem 2.1] via the following identity:

$$\int_\Omega \Phi_i d\omega = \int_\Omega G(\theta)\frac{x_i}{s}d\omega = \int_\Omega \frac{G(\theta)}{\sin\theta}y_i d\omega,$$

choosing $\tilde{G}(\theta) = \frac{G(\theta)}{\sin\theta}$. When $j > 0$, conditions (13) can be proved arguing in an analogous way as in the proof of [3, Theorem 2.1].

4. Proof of Theorem 1.1

Recalling the definition of Φ_i given in (12), we get

$$(\nabla\Phi_i)_j \equiv \Phi_{i,j} = G'(\theta)p\frac{x_i x_j}{s^2} + G(\theta)\frac{\delta_{ij}}{s} - G(\theta)\frac{x_i x_j}{s^3}, \quad j = 1,\ldots,N. \tag{14}$$

Using (11), the definition of s and (14) we have

$$\frac{1}{p^2}|\nabla\Phi_i|^2 = G'(\theta)^2\frac{x_i^2}{s^2} + G(\theta)^2\frac{1}{s^2p^2} - G(\theta)^2\frac{x_i^2}{p^2s^4}. \tag{15}$$

Hence, from (10) and (15),

$$\sum_{i=1}^{N} |\nabla_{\mathbb{S}^N} \Phi_i|^2 = \frac{1}{p^2} \sum_{i=1}^{N} |\nabla \Phi_i|^2 = G'(\theta)^2 + G(\theta)^2 \frac{N-1}{s^2 p^2} = G'(\theta)^2 + G(\theta)^2 \frac{N-1}{\sin^2 \theta}.$$

Taking into account the orthogonality conditions (13), we use Φ_i as test function in the variational characterization (3) of $\mu_i(\Omega)$, and we get

$$\begin{aligned}
\int_\Omega \Phi_i^2 d\omega &\le \frac{1}{\mu_i(\Omega)} \int_\Omega G'(\theta)^2 \frac{x_i^2}{s^2} d\omega + \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega \\
&= \frac{1}{\mu_i(\Omega)} \int_{\Omega \cap D_\gamma} G'(\theta)^2 \frac{x_i^2}{s^2} d\omega + \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega \\
&\le \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} G'(\theta)^2 \frac{x_i^2}{s^2} d\omega + \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega \\
&= \frac{1}{N\mu_i(\Omega)} \int_{D_\gamma} G'(\theta)^2 d\omega + \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega. \qquad (16)
\end{aligned}$$

Summing over $i = 1, \ldots, N$ we get

$$\int_\Omega G(\theta)^2 d\omega \le \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} G'(\theta)^2 d\omega + \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega.$$

Now notice that

$$\sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \left(1 - \frac{x_i^2}{s^2}\right) - \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} = \frac{1}{\mu_N(\Omega)} - \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \frac{x_i^2}{s^2} \le 0,$$

which follows from $\mu_i(\Omega) \le \mu_N(\Omega)$ for all $i = 1, \ldots, N-1$ and the definition of s. Hence,

$$\int_\Omega G(\theta)^2 d\omega \le \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} G'(\theta)^2 d\omega + \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} d\omega. \qquad (17)$$

By Lemma 2.1 we know that the function $\frac{G(\theta)}{\sin\theta}$ is decreasing in $(0, \gamma)$. Recalling that $|\Omega| = |D_\gamma|$, we get

$$\begin{aligned}
\int_\Omega \frac{G(\theta)^2}{\sin^2\theta} d\omega &= \int_{\Omega \cap D_\gamma} \frac{G(\theta)^2}{\sin^2\theta} d\omega + \int_{\Omega \setminus D_\gamma} \frac{G(\theta)^2}{\sin^2\theta} d\omega \\
&\le \int_{\Omega \cap D_\gamma} \frac{G(\theta)^2}{\sin^2\theta} d\omega + \frac{G(\gamma)^2}{\sin^2\gamma} |\Omega \setminus D_\gamma|
\end{aligned}$$

$$= \int_{\Omega\cap D_\gamma} \frac{G(\theta)^2}{\sin^2\theta} d\omega + \frac{G(\gamma)^2}{\sin^2\gamma} |D_\gamma \backslash \Omega|$$

$$\leq \int_{\Omega\cap D_\gamma} \frac{G(\theta)^2}{\sin^2\theta} d\omega + \int_{D_\gamma\backslash\Omega} \frac{G(\theta)^2}{\sin^2\theta} d\omega$$

$$= \int_{D_\gamma} \frac{g(\theta)^2}{\sin^2\theta} d\omega. \tag{18}$$

On the other side, since $G(\theta)$ is non-decreasing in $(0, \frac{\pi}{2})$, we have

$$\int_\Omega G(\theta)^2 d\omega = \int_{\Omega\cap D_\gamma} G(\theta)^2 d\omega + \int_{\Omega\backslash D_\gamma} G(\theta)^2 d\omega$$

$$\geq \int_{\Omega\cap D_\gamma} G(\theta)^2 d\omega + G(\gamma)^2 |\Omega\backslash D_\gamma|$$

$$= \int_{\Omega\cap D_\gamma} G(\theta)^2 d\omega + G(\gamma)^2 |D_\gamma\backslash\Omega|$$

$$\geq \int_{\Omega\cap D_\gamma} G(\theta)^2 d\omega + \int_{D_\gamma\backslash\Omega} g(\theta)^2 d\omega$$

$$= \int_{D_\gamma} g(\theta)^2 d\omega. \tag{19}$$

Using (17)–(19) and the monotonicity of the sequence $\{\mu_i(\Omega)\}_i$, we have

$$\int_{D_\gamma} g(\theta)^2 d\omega \leq \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} g'(\theta)^2 d\omega + \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} \frac{g(\theta)^2}{\sin^2\theta} d\omega$$

$$\leq \frac{1}{N-1} \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} \left[g'(\theta)^2 + (N-1)\frac{g(\theta)^2}{\sin^2\theta} \right] d\omega.$$

Finally, from (6) we conclude

$$\frac{1}{N-1} \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \geq \frac{1}{\mu_1(D_\gamma)}. \tag{20}$$

Since $G(\theta)/\sin\theta$ is strictly decreasing in $(0, \gamma)$, when $\gamma < \pi/2$, the equality sign holds in (20) if and only if Ω is a geodesic ball.

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