Monatshefte für Mathematik manuscript No.
(will be inserted by the editor)

On the integration of Riemann-measurable vector-valued functions

Diana Caponetti · Valeria Marraffa ·
Kirill Naralenkov

Received: date / Accepted: date

Abstract We confine our attention to convergence theorems and descriptive relationships within some subclasses of Riemann-measurable vector-valued functions that are based on the various generalizations of the Riemann definition of an integral.

Keywords Birkhoff, McShane, Henstock, and Pettis integrals · Lebesgue measurable gauge · Riemann-measurable function · almost uniform convergence · bounded variation · ACG_* and ACG_δ^* functions

Mathematics Subject Classification (2000) Primary 26A39 · 28B05; Secondary 46G10

1 Introduction

In this article we develop the vector-valued generalizations of the Henstock and McShane integrals within the *Riemann-measurable* function class, that is, in a certain sense under natural restrictions placed on the individual function rather than on the range space. The importance of Riemann measurability in

D. Caponetti
Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123 Palermo, Italy
E-mail: diana.caponetti@unipa.it

V. Marraffa
Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123 Palermo, Italy
E-mail: valeria.marraffa@unipa.it

K. Naralenkov
Moscow State Institute of International Relations, Department of Mathematics, Econometrics and Information Technologies, Vernadskogo Ave. 76, 119454 Moscow, Russian Federation
E-mail: naralenkov@gmail.com

the theory of vector-valued integration stems from the facts that (a) a function is $\mathscr{M}$-integrable ($\mathscr{H}$-integrable) (that is, the gauge in the definition of the McShane (Henstock) integral can be chosen to be Lebesgue measurable) if and only if the function is both McShane (Henstock) integrable and Riemann measurable; and (b) any bounded Riemann measurable function is necessarily $\mathscr{M}$-integrable [28]. It becomes clear from (a) and (b) that the $\mathscr{H}$- and $\mathscr{M}$-integrals possess a reasonable degree of generality. We should point out that the Riemann measurable function class is closely related to the theory of integration set forth by Kolmogorov [19] in the real-valued case and Birkhoff [1] in the vector-valued case. More precisely, in [36] it is shown that the $\mathscr{M}$-integral is equivalent to the Birkhoff integral. The *absolute* Birkhoff integral arises naturally in connection with (b) above. In Theorem 2 we have been able to extend (b) by showing that absolutely Birkhoff integrable functions are precisely those, which are Riemann measurable and have a Lebesgue integrable majorant. Furthermore, Theorem 1 and Corollary 2 give a partial descriptive characterization of the absolute Birkhoff integrable function class.

Several convergence theorems for the Birkhoff integral can be found in the literature [2,3,33–35]. However, the proofs are quite involved and their hypotheses are not easy to check in specific instances. We should emphasize at this point that, even for a uniformly bounded sequence of Riemann measurable functions, pointwise convergence does not imply the Riemann measurability of the limit function. Indeed, Rodríguez [32] gives an example of a uniformly bounded sequence of $\mathscr{M}$-integrable functions defined on $[0, 1]$ and assuming values in $c_0(\mathfrak{c})$ that converges pointwise to a non-$\mathscr{M}$-integrable function on $[0, 1]$. Theorem 3 makes it clear that *almost* uniform convergence is natural for the notion of Riemann measurability. Theorem 6 and Corollary 4 give two versions of the Dominated Convergence theorem for Riemann-measurable functions.

In the last section we follow the descriptive approach to the vector-valued Henstock and McShane integrals, that is, to describe the properties of the indefinite integrals and to see how the Henstock integral extends the McShane integral in terms of these properties. The material in this section is motivated by three well-known facts concerning real-valued functions defined on a closed interval of the real line. First, an indefinite McShane integral is merely an AC function whereas an indefinite Henstock integral is an ACG_* function (both indefinite integrals are differentiable to the integrand almost everywhere). Second, the ACG_δ function class appears to be the most natural generalized absolutely continuous function class related to the Henstock integral. It should be pointed out that the definition of the ACG_δ property, unlike the definition of the ACG_* property that is based on *constant* gauges, involves *arbitrary* gauges. As a result, the proof of the fact that the ACG_δ property implies the ACG_* property for an indefinite Henstock integral is indirect and relies heavily on the differentiability properties of the indefinite integral (see [16, Theorems 9.17 and 11.4]). Third, the domain of a Henstock integrable function can be written as a countable union of closed sets on each of which the integrand is McShane integrable. Once again, the proof is indirect and uses the theory of the Lebesgue integral, which is equivalent to the McShane integral in this case (see [16, Theorem 9.18]).

In the vector-valued case the situation changes dramatically. On the one hand a Henstock integrable c_0-valued function defined on a closed interval of the real line can be McShane integrable on no nondegenerate subinterval [27]; or in any infinite dimensional Banach space there exist *Bochner measurable* functions whose indefinite Pettis integrals are not weakly differentiable anywhere [4], while the Pettis integral and the McShane integral are equivalent within the Bochner measurable function class [10]. Thus, to put it more explicitly, there is no hope that the above descriptive results may be extended to vector-valued functions in their full generality using methods similar to the real-valued case.

The descriptive relationship between the Henstock and McShane integrals for vector-valued functions has received some attention in [7,26,27]. In [7], Fremlin proved that a function is McShane integrable if and only if it is both Henstock and Pettis integrable or equivalently, it is Henstock integrable and its indefinite Henstock integral is AC (see [26]). In [26], by means of Fremlin's criterion it was shown that in some classes of Banach spaces the domain of a Henstock integrable function can be written as a countable union of closed sets on each of which the integrand is McShane integrable and the corresponding indefinite McShane integrals converge to the indefinite Henstock integral in the Alexiewicz norm.

Our methods refine those in [26] so that we can get most of the results in a form that is as close as possible to the above real-valued case. Theorem 7 clarifies the relationship between two different absolute continuity concepts, the AC_* and AC^*_δ properties, defined in terms of constant and measurable gauges, respectively. Theorems 8 and 9 provide descriptive criterions for $\mathscr{M}$-integrability. In Theorems 10 and 11 we give a set of necessary and sufficient conditions for $\mathscr{M}$-integrability of a Henstock integrable function in terms of the absolute continuity properties of the indefinite integral. Theorems 12 and 13 contain sufficient conditions for the $\mathscr{H}$-integral to extend the $\mathscr{M}$-integral in the natural way as mentioned above in this introduction. General sufficient conditions to insure the convergence of a sequence of the indefinite McShane integrals on some closed subsets of the domain to the indefinite Henstock integral on the whole domain in the Alexiewicz norm are given in Theorems 14 and 15. We conclude this note with Theorem 16, which characterizes the $\mathscr{H}$- and $\mathscr{M}$-integrals in terms of the *'global smallness'* of the Riemann sums.

The reader who is not familiar with the extensions of the Riemann, McShane, and Henstock integrals to the case of vector-valued functions may wish to consult [5,6,11,14,15,7–10,20,22,24].

2 Notation and definitions

The following notation will be observed in the remainder of this note. We will restrict our attention throughout to a closed nondegenerate interval $[a, b]$ of the real line. In what follows, I will denote a closed nondegenerate subinterval of $[a, b]$. X denotes a real Banach space and X^* its dual. The symbol B_X stands for the *unit ball* of X. Given $F : [a, b] \to X$, $\Delta F(I)$ denotes the *increment* of F on I. Let E be a set and let t be a point, then $\mathrm{dist}(t, E)$ is the *distance* from t to E, ∂E, χ_E, and $\mu(E)$ will denote the *boundary* of E, the *characteristic function* of E, and *Lebesgue measure* of E, respectively. For ease of notation, we will drop the adjective Lebesgue and refer to *measurable* and *negligible* sets. Finally, a *gauge* on E is any positive function defined on E.

An extensive list of definitions appears next. We begin with the notion of Riemann measurability and then outline briefly a few definitions relevant to the Henstock and McShane integrals as well as to the Birkhoff integral and its absolute version referred to as the *Riemann-Lebesgue integral* in [17] and [18]. It is important to note that in the real-valued case the absolute Birkhoff coincides with the ordinary Lebesgue integral [17, Theorems 1.3 and 1.4]. The reader should consult [31] for a thorough discussion of other equivalent definitions of the Birkhoff integral and the absolute Birkhoff integral and [23] for an extension of the Birkhoff integral to locally convex spaces.

Definition 1 Let $f : [a, b] \to X$ and let $E \subset [a, b]$ be measurable. The function f is said to be *Riemann measurable* on E if for each $\varepsilon > 0$ a closed set $F \subset E$ with $\mu(E \setminus F) < \varepsilon$ and a positive number δ exist such that

$$\left\| \sum_{k=1}^{K} \{f(t_k) - f(t'_k)\} \cdot \mu(I_k) \right\| < \varepsilon$$

whenever $\{I_k\}_{k=1}^K$ is a finite collection of pairwise nonoverlapping intervals with $\max_{1 \le k \le K} \mu(I_k) < \delta$ and $t_k, t'_k \in I_k \cap F$.

Definition 2 (a) A *partial McShane partition* of $[a, b]$ is a finite collection $\mathscr{P} = \{(I_k, t_k)\}_{k=1}^K$ such that $\{I_k\}_{k=1}^K$ is a family of mutually nonoverlapping intervals and $t_k \in [a, b]$ for each k. $\mathscr{P}$ is *subordinate* to a gauge δ on $[a, b]$ if $I_k \subset (t_k - \delta(t_k), t_k + \delta(t_k))$ for each k. $\mathscr{P}$ is said to be a *McShane partition* of $[a, b]$ provided $\{I_k\}_{k=1}^K$ covers $[a, b]$.

We say that a function $f : [a, b] \to X$ is *McShane integrable* on $[a, b]$, with *McShane integral* $w \in X$, if for each $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$\left\| \sum_{k=1}^{K} f(t_k)\mu(I_k) - w \right\| < \varepsilon \tag{1}$$

whenever $\{(I_k, t_k)\}_{k=1}^K$ is a McShane partition of $[a, b]$ subordinate to δ.

(b) A *partial Henstock partition* (*Henstock partition*) of $[a, b]$ is a partial McShane partition (McShane partition) $\{(I_k, t_k)\}_{k=1}^K$ of $[a, b]$ with $t_k \in I_k$ for each k. A function $f : [a, b] \to X$ is *Henstock integrable* on $[a, b]$, with *Henstock integral* $w \in X$, if for each $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that (1) holds for each Henstock partition $\{(I_k, t_k)\}_{k=1}^K$ of $[a, b]$ subordinate to δ.

(c) A function $f : [a, b] \to X$ is said to be *$\mathscr{M}$-integrable* (*$\mathscr{H}$-integrable*) on $[a, b]$ if it is McShane (Henstock) integrable on $[a, b]$ and for each $\varepsilon > 0$ there exists a *measurable* gauge δ on $[a,b]$ that corresponds to ε in the definition of the McShane (Henstock) integral of f on $[a,b]$.

Definition 3 Let $E \subset [a,b]$ be measurable.

(a) A *Birkhoff partition* of E is any at most countable family $\Pi = \{E_k\}$ of mutually disjoint measurable sets that covers E.

(b) Let Γ and Π be two Birkhoff partitions of E. We say that Γ *refines* Π if each set in Γ is a subset of some set in Π.

(c) A function $f : E \to X$ is said to be *(absolutely) Birkhoff integrable* on E, with *(absolute) Birkhoff integral* $w \in X$, if for each $\varepsilon > 0$ there is a Birkhoff partition Π of E such that for any Birkhoff partition $\Gamma = \{E_k\}$ that refines Π the series $\sum_k f(t_k)\mu(E_k)$ is unconditionally (absolutely) summable and

$$\left\| \sum_k f(t_k)\mu(E_k) - w \right\| < \varepsilon \tag{2}$$

whenever $t_k \in E_k$ for each k.

We now define various notions of bounded variation (VB and VB_*), generalized bounded variation (VBG and VBG_*), absolute continuity (AC, AC_*, and AC^*_δ), and generalized absolute continuity (ACG, ACG_*, and ACG^*_δ) on a set.

Definition 4 Let $F : [a,b] \to X$ and let E be a nonempty subset of $[a,b]$.

(a) The function F is VB (resp. VB_*) on E if

$$\mathrm{V}(F,E) = \sup\left\{ \left\| \sum_{k=1}^{K} \Delta F(I_k) \right\| \right\} < \infty \text{ (resp. } \mathrm{V}_*(F,E))$$

where the supremum is taken over all finite collections $\{I_k\}_{k=1}^K$ of nonoverlapping intervals with $\partial I_k \subset E$ (resp. $\partial I_k \cap E \neq \varnothing$).

(b) The function F is said to be AC (resp. AC_*) on E if for each $\varepsilon > 0$ there exists a positive number η such that

$$\left\| \sum_{k=1}^{K} \Delta F(I_k) \right\| < \varepsilon \tag{3}$$

for each finite collection of pairwise nonoverlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \subset E$ (resp. $\partial I_k \cap E \neq \varnothing$) and

$$\sum_{k=1}^{K} \mu(I_k) < \eta. \tag{4}$$

(c) Suppose that the set E is measurable. The function F is said to be AC^*_δ on E if for each $\varepsilon > 0$ a positive number η and a measurable gauge δ on E exist such that (3) holds for each partial Henstock partition $\{(I_k, t_k)\}_{k=1}^K$ of $[a,b]$ with $t_k \in \partial I_k \cap E$ and (4) subordinate to δ.

(d) The function F is VBG (resp. VBG_*, ACG, ACG_*, ACG^*_δ) on E if E can be written as a countable union of sets on each of which F is VB (resp. VB_*, AC, AC_*, AC^*_δ).

Remark 1 If in Definition 4 the norm is placed *inside* the sum, then we obtain the corresponding *strong* function classes (the adjective strong serves as an indication that the norm is inside the sum). It can be easily shown that in the real-valued case the strong functions classes are equivalent to the ordinary ones.

Remark 2 A straightforward argument shows that a function is VB_* (resp. AC_*, AC^*_δ) on $E_1 \cup E_2$ whenever it is VB_* (resp. AC_*, AC^*_δ) on both E_1 and E_2.

Next, define the scalar derivatives (see [29] for some historical background behind this notion), the scalar equivalence, and the Pettis integral (see [38] for the general theory of this integral).

Definition 5 Let $F : [a, b] \to X$ and let $E \subset [a, b]$. A function $f : E \to X$ is a *scalar derivative* of F on E if for each x^* in X^* the function x^*F is differentiable almost everywhere on E and $(x^*F)' = x^*f$ almost everywhere on E (the exceptional set may vary with x^*).

Definition 6 Let $E \subset [a, b]$ and let $f, g : E \to X$. The function f is said to be *scalarly equivalent* to the function g on E if for each x^* in X^* $x^*f = x^*g$ almost everywhere on E (the exceptional set may vary with x^*).

Definition 7 Let $f : [a, b] \to X$.

(a) The function f is *Pettis integrable* on $[a, b]$ if for each measurable set E in $[a, b]$ there is a vector $\nu_f(E) \in X$ such that the Lebesgue integral $\int_E x^*f$ exists and is equal to $x^*(\nu_f(E))$ for all x^* in X^*.

(b) The function f is *Pettis integrable* on $[a, b]$ if there is an AC function $F : [a, b] \to X$ such that f is a scalar derivative of F on $[a, b]$.

A comment on the above definition is appropriate here. While part (a) is the original Pettis' definition [30], part (b) is a *descriptive* definition of the Pettis integral and the interested reader should refer to [11] or [25] for the details related to this definition.

Customarily, we say that a function f is McShane ($\mathscr{M}$-, Henstock, $\mathscr{H}$-, Pettis) integrable on a set $E \subset [a, b]$ if the function $f\chi_E$ is McShane ($\mathscr{M}$-, Henstock, $\mathscr{H}$-, Pettis) integrable on $[a, b]$ and $\int_E f = \int_a^b f\chi_E$. Standard arguments show that a McShane ($\mathscr{M}$-, Henstock, $\mathscr{H}$-) integrable function on $[a, b]$ is McShane ($\mathscr{M}$-, Henstock, $\mathscr{H}$-) integrable on any subinterval I of $[a, b]$. Moreover, a McShane ($\mathscr{M}$-, Birkhoff, absolutely Birkhoff) integrable function on $[a, b]$ is McShane ($\mathscr{M}$-, Birkhoff, absolutely Birkhoff) integrable on any measurable subset of $[a, b]$ (see [20, Theorem 9] and [17, Lemma 1.6]). If f is integrable on $[a, b]$, then it will be convenient to use the phrase *'indefinite integral'* to mean the function $F(t) = \int_a^t f$. In this case, it is easy to verify that F is continuous on $[a, b]$, the function f is a scalar derivative of F on $[a, b]$, and $\int_I f = \Delta F(I)$ for any subinterval I of $[a, b]$. At last, $\|f\|_A = \sup_{a<t\le b} \left\|\int_a^t f\right\|$ is the *Alexiewicz norm* of the function f.

Recall that the *upper integral* of a real-valued function f on a measurable set E is defined to be

$$\overline{\int}_E f = \inf\left\{\int_E \varphi : \varphi \text{ is summable on } E \text{ and } f \le \varphi \text{ on } E\right\}.$$

The function f is said to satisfy *condition* (N) on E if $\mu^*(f(A)) = 0$ for each negligible set $A \subset E$ (μ^* represents the usual *Lebesgue outer measure*).

If $(T, \mathscr{T})$ is a topological space, then $\operatorname{dens}(T, \mathscr{T})$ denotes the smallest cardinal for which there is a dense set of that cardinality. This cardinal is called the *density character* of $(T, \mathscr{T})$. Each cardinal number is identified with the first ordinal number of that cardinality.

The cardinal number $\varkappa(\mu)$ is defined to be the minimal cardinal number $\varkappa$ such that there exists the union of $\varkappa$ negligible sets of positive outer Lebesgue measure.

3 Characterizations of absolutely Birkhoff integrable functions

This section explores some necessary and sufficient conditions to distinguish absolutely Birkhoff integrable functions among Riemann measurable functions. It will be seen from Theorem 2 that the absolute Birkhoff integral is very similar to the Bochner and Talagrand integrals (see [10] and the references therein). Though Lemma 1 was stated implicitly in [17, p. 56] as well as Corollary 1 and Theorem 1 were stated in [18, p. 52] without any reference to a proof, we provide here complete proofs since we believe that these basic facts are not obvious.

Lemma 1 *Let $E \subset [a, b]$ be measurable and let $f : E \to X$. If f is absolutely Birkhoff integrable on E and a Birkhoff partition $\{E_k\}$ of E corresponds to some positive number ε in the definition of the absolute Birkhoff integral of f on E, then the function φ defined by*

$$\varphi = \sum_{k:\mu(E_k)>0} \sup_{E_k} \|f\| \cdot \chi_{E_k} + \|f\| \cdot \sum_{k:\mu(E_k)=0} \chi_{E_k}$$

is summable on E and $\|f\| \le \varphi$ on E.

Proof It is clear that $M_k = \sup_{E_k} \|f\| < \infty$ whenever $\mu(E_k) > 0$. For each k with $\mu(E_k) > 0$ pick a point t_k in E_k such that $\|f(t_k)\| \ge M_k - 1$. Since

$$\sum_{k:\mu(E_k)>0} M_k \cdot \mu(E_k) \le \sum_{k:\mu(E_k)>0} \|f(t_k)\| \cdot \mu(E_k) + \sum_{k:\mu(E_k)>0} \mu(E_k) < \infty,$$

the function φ is summable on E. Further, by the construction of the function φ, we have $\|f\| \le \varphi$ on E. The proof is complete.

Corollary 1 below, which is an immediate consequence of our lemma, shows that absolutely Birkhoff integrable functions are remarkably similar to Bochner and Talagrand integrable ones — the upper integral of the norm of an absolutely Birkhoff integrable function is finite.

Corollary 1 *Let $E \subset [a,b]$ be measurable and let $f : E \to X$. If f is absolutely Birkhoff integrable on E, then* $\overline{\int}_E \|f\| < \infty$.

Now we are ready to prove that the indefinite Birkhoff integral of an absolutely Birkhoff integrable function is sVB.

Theorem 1 *Let $f : [a,b] \to X$. If f is absolutely Birkhoff integrable on $[a,b]$, then the indefinite Birkhoff integral of f is sVB on $[a,b]$ and*

$$\mathrm{sV}\left(\int_a^{\cdot} f, [a,b]\right) \leq \overline{\int}_{[a,b]} \|f\|.$$

Proof Fix $\eta > 0$. By Corollary 1, there exists a summable function φ on $[a,b]$ such that $\|f\| \leq \varphi$ on $[a,b]$ and

$$\int_a^b \varphi < \overline{\int}_{[a,b]} \|f\| + \eta.$$

Let a Birkhoff partition Π of $[a,b]$ correspond to $\varepsilon = \eta$ in the definition of the absolute Birkhoff integral of φ on $[a,b]$.

Let $\{D_k\}_{k=1}^K$ be a finite collection of mutually disjoint measurable sets that covers $[a,b]$. For each k, let a Birkhoff partition $\Pi_k = \{E_{nk}\}$ of D_k correspond to $\varepsilon = \eta/K$ in the definition of the absolute Birkhoff integral of f on D_k. Clearly, we may assume that $\bigcup_{k=1}^K \Pi_k$ refines Π. For each pair of n and k, pick a point t_{nk} in E_{nk}. Consequently, we get

$$\sum_{k=1}^K \left\| \int_{D_k} f \right\| \leq \sum_{k=1}^K \left\| \int_{D_k} f - \sum_n f(t_{nk})\mu(E_{nk}) \right\| + \sum_{k=1}^K \sum_n \|f(t_{nk})\| \cdot \mu(E_{nk}) <$$
$$\eta + \sum_{k=1}^K \sum_n \varphi(t_{nk})\mu(E_{nk}) \leq 2\eta + \int_a^b \varphi < 3\eta + \overline{\int}_{[a,b]} \|f\|.$$

As $\eta > 0$ was arbitrary, we have the desired estimate for the strong variation of the indefinite Birkhoff integral of the function f. The proof is complete.

In Corollary 2 below we will be able to prove a partial converse to Theorem 1 under an additional assumption on the density character of the dual unit ball equipped with the w^*-topology. To do this, we need the following auxiliary fact.

Lemma 2 *Let $F : [a,b] \to X$. Suppose that* $\mathrm{dens}(B_{X^*}, w^*) < \varkappa(\mu)$. *If F is sVB on $[a,b]$ and f is a scalar derivative of F on $[a,b]$, then* $\overline{\int}_{[a,b]} \|f\| < \infty$.

Proof By [29, Theorem 3.1], there exists a summable function φ on $[a, b]$ such that $|x^* f| \leq \varphi$ almost everywhere on $[a, b]$ for each x^* in B_{X^*} (the exceptional set may vary with x^*).

Let λ denote $\mathrm{dens}(B_{X^*}, w^*)$ and let $\{x^*_\alpha\}_{\alpha<\lambda} \subset B_{X^*}$ be w^*-dense in B_{X^*}. For each $\alpha < \lambda$ define a set E_α by

$$E_\alpha = \{t \in [a, b] : |x^*_\alpha f(t)| \leq \varphi(t)\}.$$

Further, set

$$N = \bigcup_{\alpha<\lambda} [a, b] \setminus E_\alpha$$

and note that N is negligible. Now it follows that

$$\|f(t)\| = \sup_{\alpha<\lambda} |x^*_\alpha f(t)| \leq \varphi(t)$$

for each $t \notin N$. The proof is complete.

The key result of this section, Theorem 2, provides a complete description of the absolute Birkhoff integrability in terms of Riemann measurability.

Theorem 2 *Let $E \subset [a, b]$ be measurable and let $f : E \to X$. Then the function f is absolutely Birkhoff integrable on E if and only if f is Riemann measurable on E and $\overline{\int}_E \|f\| < \infty$.*

Proof The necessity part of the theorem results from the equivalence of the Birkhoff and $\mathscr{M}$-integrals and Corollary 1.

For the sufficiency, suppose that the function f is Riemann measurable on E and φ is a summable function on E such that $\|f\| \leq \varphi$ on E.

The Riemann measurability of f on E implies that there exists a sequence $\{F_n\}_{n=1}^\infty$ of pairwise disjoint closed subsets of E such that the set $N = E \setminus \bigcup_{n=1}^\infty F_n$ is negligible and f is bounded and Riemann measurable (and, consequently, Birkhoff integrable) on F_n for each n (see [28, Theorems 1 and 5]).

Fix $\varepsilon > 0$. Let a Birkhoff partition Π of $E \setminus N$ correspond to ε in the definition of the absolute Birkhoff integral of the function φ on $E \setminus N$. Then for each Birkhoff partition $\{E_k\}$ of $E \setminus N$ that refines Π we obtain the result that

$$\sum_k \|f(t_k)\| \cdot \mu(E_k) \leq \sum_k \varphi(t_k)\mu(E_k) < \infty$$

whenever $t_k \in E_k$.

For each n let $\Pi_n = \{E_{kn}\}$ be a Birkhoff partition of F_n such that

$$\left\| \sum_k f(t_{kn})\mu(E_{kn}) - \sum_k f(t'_{kn})\mu(E_{kn}) \right\| < \frac{\varepsilon}{2^n}$$

whenever $t_{kn}, t'_{kn} \in E_{kn}$. By part (RLC1) of Theorem 10 of [31], we may assume that $\bigcup_{n=1}^{\infty} \Pi_n$ refines Π. Consequently, we obtain

$$\left\| \sum_{n=1}^{\infty} \sum_k f(t_{kn})\mu(E_{kn}) - \sum_{n=1}^{\infty} \sum_k f(t'_{kn})\mu(E_{kn}) \right\| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon,$$

where all the sums are absolute whenever $t_{kn}, t'_{kn} \in E_{kn}$. Now part (FC) of Theorem 10 of [31] applies to the function f to show the absolute Birkhoff integrability of f on $E \setminus N$. This completes the proof.

Corollary 2 *Let $f : [a,b] \to X$. Suppose that* $\mathrm{dens}(B_{X^*}, w^*) < \varkappa(\mu)$. *If f is $\mathscr{M}$-integrable on $[a,b]$ and the indefinite McShane integral of f is sVB on $[a,b]$, then f is absolutely Birkhoff integrable on $[a,b]$.*

Proof Let F denote the indefinite McShane integral of f on $[a,b]$. Since f is a scalar derivative of the sVB function F on $[a,b]$, Lemma 2 yields that $\overline{\int}_{[a,b]} \|f\| < \infty$. As f is Riemann measurable on $[a,b]$, by the above theorem, f is absolutely Birkhoff integrable on $[a,b]$. The proof is complete.

4 Sequences of Riemann-measurable functions

In this section we consider several important properties of the sequences of Riemann measurable functions. For the reader's convenience, we recall that a function sequence $\{f_n\}$ defined on $[a,b]$ converges to a function f *almost uniformly* on a set $E \subset [a,b]$ if for each $\varepsilon > 0$ there is a measurable set $S \subset E$ with $\mu(S) < \varepsilon$ such that $\{f_n\}$ converges uniformly to f on $E \setminus S$. The proof of the following theorem is left to the reader.

Theorem 3 *Let $E \subset [a,b]$ be measurable and let $f_n : [a,b] \to X$, $n \in \mathbb{N}$, be Riemann measurable on E. If $\{f_n\}$ converges to a function f almost uniformly on E, then f is Riemann measurable on E.*

In the remainder of this section we address sequences of $\mathscr{H}$-integrable functions. We begin with the notions of uniformly $\mathscr{H}$-integrable function sequence and δ-Cauchy function sequence.

Definition 8 Let $f_n : [a,b] \to X$, $n \in \mathbb{N}$, be $\mathscr{H}$-integrable on $[a,b]$.

(a) The sequence $\{f_n\}$ is said to be *uniformly $\mathscr{H}$-integrable* on $[a,b]$ if for each $\varepsilon > 0$ there is a *measurable* gauge δ on $[a,b]$ such that

$$\left\| \sum_{k=1}^{K} f_n(t_k)\mu(I_k) - \int_a^b f_n \right\| < \varepsilon$$

for all n whenever $\{(I_k, t_k)\}_{k=1}^{K}$ is a Henstock partition of $[a,b]$ subordinate to δ.

(b) The sequence $\{f_n\}$ is said to be *δ-Cauchy* on $[a,b]$ if for each $\varepsilon > 0$ there is a *measurable* gauge δ on $[a,b]$ and a positive integer N such that

$$\left\| \sum_{k=1}^{K} f_m(t_k)\mu(I_k) - \sum_{k=1}^{K} f_n(t_k)\mu(I_k) \right\| < \varepsilon$$

for all $m,n \geq N$ whenever $\{(I_k,t_k)\}_{k=1}^K$ is a Henstock partition of $[a,b]$ subordinate to δ.

In point of fact, the above (a) and (b) turn out to be very close to one another, while (b) is sometimes easier to verify than (a). The proof of the next theorem is exactly the same as that of the corresponding fact in the real-valued case. The reader may want to see the relevant details in [16, Exercise 13.10].

Theorem 4 *Let $f_n : [a,b] \to X$, $n \in \mathbb{N}$, be $\mathscr{H}$-integrable on $[a,b]$. Then $\{f_n\}$ is δ-Cauchy sequence on $[a,b]$ if and only if the sequence $\{\int_a^b f_n\}$ converges and the sequence $\{f_n\}$ is uniformly $\mathscr{H}$-integrable on $[a,b]$.*

For convenience, we recall the statement of the Uniform Henstock lemma (see [26, Lemma 2]).

Lemma A *Let $f : [a,b] \to X$ be Henstock integrable on $[a,b]$ and let $\varepsilon > 0$. Suppose that a gauge δ on $[a,b]$ corresponds to ε in the definition of the Henstock integral of f on $[a,b]$. If $\{(I_k,t_k)\}_{k=1}^K$ is a partial Henstock partition of $[a,b]$ subordinate to δ, then*

$$\left\| \sum_{k=1}^{K} f(t_k)\chi_{I_k} - \sum_{k=1}^{K} f\chi_{I_k} \right\|_A \leq 2\varepsilon.$$

The following simple convergence theorem is extremely important to note.

Theorem 5 *Let $f_n : [a,b] \to X$, $n \in \mathbb{N}$. If $\{f_n\}$ is uniformly $\mathscr{H}$-integrable on $[a,b]$ and converges to a function f pointwise on $[a,b]$, then f is $\mathscr{H}$-integrable on $[a,b]$ and*

$$\int_a^b f_n \to \int_a^b f \text{ as } n \to \infty. \tag{5}$$

Moreover,

$$\|f_n - f\|_A \to 0 \text{ as } n \to \infty. \tag{6}$$

Proof We omit the proof of $\mathscr{H}$-integrability of the function f and (5) since it proceeds by the standard technique as that in [16, Theorem 13.16].

To prove (6), fix $\varepsilon > 0$ and let a measurable gauge δ on $[a,b]$ correspond to $\varepsilon/8$ in the definitions of the Henstock integral of f and f_n on $[a,b]$ for all n. Choose a Henstock partition $\{(I_k,t_k)\}_{k=1}^K$ of $[a,b]$ subordinate to δ (this is possible since $\{f_n\}$ is uniformly $\mathscr{H}$-integrable on $[a,b]$). As $\{f_n\}$ converges pointwise on $[a,b]$, there exists an integer N such that $\|f_n(t_k) - f_N(t_k)\| \leq$

$\varepsilon/4(b-a)$ for all $n \geq N$ and for all $k \in \{1, \dots, K\}$. Now, using Lemma A to yield the majorant $\varepsilon/4$ for the first and fourth terms on the right, compute

$$\|f_n - f\|_A \leq \left\| f_n - \sum_{k=1}^{K} f_n(t_k)\chi_{I_k} \right\|_A + \sum_{k=1}^{K} \|\{f_n(t_k) - f_N(t_k)\} \cdot \chi_{I_k}\|_A +$$
$$\sum_{k=1}^{K} \|\{f_N(t_k) - f(t_k)\} \cdot \chi_{I_k}\|_A + \left\| \sum_{k=1}^{K} f(t_k)\chi_{I_k} - f \right\|_A \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

for all $n \geq N$. This completes the proof.

A simple but very useful sufficient condition for the uniform $\mathscr{H}$-integrability of a function sequence reads as follows (cf. [13, Theorem 5]).

Theorem 6 *Let $f_n : [a,b] \to X$, $n \in \mathbb{N}$, be $\mathscr{H}$-integrable on $[a,b]$. Suppose that $\{f_n\}$ converges to a function f almost uniformly on $[a,b]$. If there exists a summable function φ on $[a,b]$ such that $\|f_m - f_n\| \leq \varphi$ on $[a,b]$ for all m and n, then $\{f_n\}$ is uniformly $\mathscr{H}$-integrable on $[a,b]$. Furthermore, f is $\mathscr{H}$-integrable on $[a,b]$ and (6) holds.*

Proof Fix $\varepsilon > 0$. By the absolute continuity of the indefinite Lebesgue integral of φ, choose a positive number η such that $\int_E \varphi < \varepsilon/3$ whenever $E \subset [a,b]$ is measurable and $\mu(E) < \eta$. Next, choose a measurable set $S \subset [a,b]$ with $\mu(S) < \varepsilon/3$ such that $\{f_n\}$ converges uniformly to f on $F = [a,b] \setminus S$. Clearly, we may assume that F is closed. Choose a positive integer N such that $\|f_m(t) - f_n(t)\| < \varepsilon/3(b-a)$ for all $m, n \geq N$ and for all t in F. Let a measurable gauge δ_0 on $[a,b]$ correspond to $\varepsilon/3$ in the definition of the Henstock integral of φ on $[a,b]$. Define a measurable gauge δ on $[a,b]$ by

$$\delta(t) = \begin{cases} \delta_0(t), \text{ if } t \in F, \\ \min(\delta_0(t), \operatorname{dist}(t, \partial S)), \text{ if } t \in S. \end{cases}$$

Suppose that $\{(I_k, t_k)\}_{k=1}^{K}$ is a Henstock partition of $[a,b]$ subordinate to δ, $m, n \geq N$ and, using the Saks-Henstock lemma (see e.g. [25, Lemma 1]), compute

$$\left\| \sum_{k=1}^{K} f_m(t_k)\mu(I_k) - \sum_{k=1}^{K} f_n(t_k)\mu(I_k) \right\| \leq \left\| \sum_{k:t_k \in F} f_m(t_k)\mu(I_k) - \right.$$
$$\left. \sum_{k:t_k \in F} f_n(t_k)\mu(I_k) \right\| + \left\| \sum_{k:t_k \in S} f_m(t_k)\mu(I_k) - \sum_{k:t_k \in S} f_n(t_k)\mu(I_k) \right\| <$$
$$\frac{\varepsilon}{3} + \sum_{k:t_k \in S} \varphi(t_k)\mu(I_k) \leq \frac{\varepsilon}{3} + \left| \sum_{k:t_k \in S} \varphi(t_k)\mu(I_k) - \right.$$
$$\left. \sum_{k:t_k \in S} (H)\int_{I_k} \varphi \right| + (L)\int_{\bigcup_{k:t_k \in S} I_k} \varphi < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This shows that the sequence $\{f_n\}$ is δ-Cauchy on $[a, b]$. By Theorem 4, the sequence $\{f_n\}$ is uniformly $\mathscr{H}$-integrable on $[a, b]$. Now Theorem 5 applies to the sequence $\{f_n\}$ to yield $\mathscr{H}$-integrability of f and the relation (6). The proof is complete.

Remark 3 See [37, p. 425] for a version of the above theorem dealing with Henstock integrable vector-valued functions.

Corollary 3 *Let $f_n : [a, b] \to X$, $n \in \mathbb{N}$, be $\mathscr{H}$-integrable on $[a, b]$. If $\{f_n\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ is uniformly $\mathscr{H}$-integrable on $[a, b]$.*

Remark 4 Compare [3, Proposition 5] where an analogous result is proven for Birkhoff integrable functions.

Remark 5 The function sequence $\{f_n = n\chi_{\{0\}}\}$ converges *almost* uniformly to 0 on $[0, 1]$. However, the sequence $\{f_n\}$ is *not* uniformly $\mathscr{H}$-integrable on $[0, 1]$ (see [16, p. 209]).

It is rather important to note that the hypotheses of the next corollary imply absolute Birkhoff integrability of all the functions.

Corollary 4 *Let $f_n : [a, b] \to X$, $n \in \mathbb{N}$, be Riemann measurable on $[a, b]$. Suppose that $\{f_n\}$ converges to a function f almost uniformly on $[a, b]$. If there is a summable function φ on $[a, b]$ such that $\|f_n\| \leq \varphi$ on $[a, b]$ for each n, then f_n for each n and f are absolutely Birkhoff integrable on $[a, b]$ and (6) holds.*

Proof By Theorem 3, the function f is Riemann measurable on $[a, b]$. Now it follows from Theorem 2 that f_n for each n and f are absolutely Birkhoff integrable on $[a, b]$. Now Theorem 6 applies to the sequence $\{f_n\}$ to yield the relation (6). The proof is complete.

Remark 6 Compare [35, Theorem 5.1] and [3, Theorem 7] where similar results are approached by using the Birkhoff definition of an integral.

5 The descriptive relationship between the $\mathscr{H}$- and $\mathscr{M}$-integrals

We begin with an examination of the AC_* and AC^*_δ function properties closely related to the $\mathscr{H}$- and $\mathscr{M}$-integrals. An easy proof makes it clear that these two properties are equivalent for functions defined on a closed interval. The general case however does not have obvious solutions. The first new result in this section, Theorem 7, relates these function properties to one another (cf. [39, Lemma 6]) in the case where the domain is an arbitrary measurable set. Before addressing the general situation, we first consider a simpler situation in the following lemma.

Lemma 3 *Let $F : [a,b] \to X$ and let $E \subset [a,b]$ be nonempty and measurable. If F is AC^*_δ on E, then for each $\varepsilon > 0$ an increasing sequence of measurable sets $\{E_n\}_{n=1}^\infty$ that cover E and a decreasing sequence of positive numbers $\{\eta_n\}_{n=1}^\infty$ exist such that, for each n, (3) holds for each finite collection of pairwise nonoverlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \cap E_n \neq \varnothing$ and*

$$\sum_{k=1}^{K} \mu(I_k) < \eta_n. \tag{7}$$

Proof Fix $\varepsilon > 0$. Let $\eta > 0$ and a measurable gauge δ on E correspond to ε in part (c) of Definition 4. For each positive integer n, set

$$E_n = \left\{ t \in E : \delta(t) > \frac{1}{n} \right\} \text{ and } \eta_n = \min\left(\eta, \frac{1}{n}\right).$$

Fix n for the moment. Let $\{I_k\}_{k=1}^K$ be a finite collection of pairwise nonoverlapping intervals with $\partial I_k \cap E_n \neq \varnothing$ and (7). For each k, pick a point $t_k \in \partial I_k \cap E_n$ and note that

$$\delta(t_k) > \frac{1}{n} \geq \eta_n.$$

On the other hand, since

$$\sum_{k=1}^{K} \mu(I_k) < \eta_n \leq \eta,$$

we have $\mu(I_k) < \eta_n$ for each k. It follows that $\{(I_k, t_k)\}_{k=1}^K$ is a partial Henstock partition of $[a,b]$ subordinate to δ. Consequently, (3) holds and the sequences $\{E_n\}_{n=1}^\infty$ and $\{\eta_n\}_{n=1}^\infty$ have all the desired properties.

Theorem 7 *Let $F : [a,b] \to X$ and let $E \subset [a,b]$ be measurable. If F is AC^*_δ on E, then for each $\sigma > 0$ there exists a closed set $P \subset E$ such that $\mu(E \setminus P) < \sigma$ and F is AC_* on P.*

Proof We may assume that $\mu(E) > 0$, since the other case is trivial. Fix a positive number $\sigma < \mu(E)$. By the preceding lemma, for each positive integer n a measurable set $H_n \subset E$ with $\mu(E \setminus H_n) < \sigma/2^{n+1}$ and $\eta_n > 0$ exist such that

$$\left\| \sum_{k=1}^{K} \Delta F(I_k) \right\| < \frac{1}{n}$$

whenever $\{I_k\}_{k=1}^K$ is a finite collection of pairwise nonoverlapping intervals with $\partial I_k \cap H_n \neq \varnothing$ and (7). Set

$$H = \bigcap_{n=1}^{\infty} H_n.$$

Note that

$$\mu(E \setminus H) = \mu\Big(\bigcup_{n=1}^{\infty}(E \setminus H_n)\Big) < \sum_{n=1}^{\infty} \frac{\sigma}{2^{n+1}} = \frac{\sigma}{2} < \frac{\mu(E)}{2}.$$

In particular, it is clear that H is nonempty.

We will prove that F is AC_* on H. Given a positive number ε, pick a positive integer n_0 so that $1/n_0 < \varepsilon$ and let $\eta = \eta_{n_0}$. Then we have

$$\Big\|\sum_{k=1}^{K} \varDelta F(I_k)\Big\| < \frac{1}{n_0} < \varepsilon$$

whenever $\{I_k\}_{k=1}^{K}$ is a finite collection of pairwise nonoverlapping intervals with $\partial I_k \cap H \neq \varnothing$ and (4). This shows that F is AC_* on H. Finally, choose a closed set $P \subset H$ so that $\mu(H \setminus P) < \sigma/2$. It follows that

$$\mu(E \setminus P) = \mu(E \setminus H) + \mu(H \setminus P) < \frac{\sigma}{2} + \frac{\sigma}{2} = \sigma < \mu(E).$$

Of course $\mu(P) > 0$ and F is AC_* on P. The proof is complete.

Corollary 5 *Let $F : [a, b] \to X$ and let $E \subset [a, b]$ be a measurable. If F is ACG_δ^* on E, then there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of closed subsets of E such that the set $E \setminus \bigcup_{n=1}^{\infty} P_n$ is negligible and F is AC_* on P_n for each n.*

Proof Suppose that E is the union of a sequence $\{E_m\}_{m=1}^{\infty}$ of measurable sets such that F is AC_δ^* on E_m for each m. It follows from Theorem 7 that for each m there exists a sequence $\{P_{lm}\}_{l=1}^{\infty}$ of closed subsets of E_m such that the set $E_m \setminus \bigcup_{l=1}^{\infty} P_{lm}$ is negligible and F is AC_* on P_{lm} for each l. For each positive integer n, set

$$P_n = \bigcup_{l,m: l+m \leq n+1} P_{lm}.$$

It is now easily seen that the sequence $\{P_n\}_{n=1}^{\infty}$ has all the desired properties.

Now we give a descriptive criterion to distinguish $\mathscr{M}$-integrable functions among $\mathscr{H}$-integrable functions defined on $[a, b]$ (cf. Fremlin's criterion, [7, Theorem 8]).

Theorem 8 *Let $f : [a, b] \to X$. Then f is $\mathscr{M}$-integrable on $[a, b]$ if and only if f is $\mathscr{H}$-integrable on $[a, b]$ and its indefinite Henstock integral is AC on $[a, b]$.*

Proof If f is $\mathscr{M}$-integrable on $[a, b]$, then $\mathscr{H}$-integrability of f is obvious. By Lemma 6 of [20], the indefinite McShane integral of f is AC on $[a, b]$.

If f is $\mathscr{H}$-integrable on $[a, b]$ and its indefinite Henstock integral is AC on $[a, b]$, then combining Theorem 4 of [28] and Theorem 3 of [26] shows that f must be both Riemann measurable and McShane integrable on $[a, b]$. Lastly, Theorem 7 of [28] applies to f to show that f is in fact $\mathscr{M}$-integrable on $[a, b]$. The proof is complete.

The next step is to show that the $\mathscr{M}$- and Pettis integrals are equivalent within the Riemann measurable function class. To do this, we will extend Corollary 4 of [28] that relates the $\mathscr{M}$-integral to the McShane integral by replacing McShane integrability with Pettis integrability.

Theorem 9 *Let $f : [a,b] \to X$ and let $E \subset [a,b]$ be measurable. Then f is $\mathscr{M}$-integrable on E if and only if f is both Riemann measurable and Pettis integrable on E.*

Proof The necessity part of the theorem is obvious.

Suppose that f is both Riemann measurable and Pettis integrable on E. Since f is Riemann measurable on E, by part (d) of Theorem 1 of [28], there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of pairwise disjoint closed subsets of E such that the set $N = E \setminus \bigcup_{n=1}^{\infty} F_n$ is negligible and f is bounded on F_n for each n. Now it follows from Theorem 5 of [28] that f is $\mathscr{M}$-integrable on F_n for each n. Since the Birkhoff integral [1] and the $\mathscr{M}$-integral are equivalent, f is Birkhoff integrable on each of these sets. Finally, note that f is Birkhoff integrable to zero on N. Thus, E is covered by a sequence of measurable sets on each of which f is Birkhoff integrable. As f is Pettis integrable on E, by Fremlin's lemma on the Birkhoff integral (see [9, §9]) f must be Birkhoff integrable on E, which is equivalent to $\mathscr{M}$-integrability of f on E. The proof is complete.

Corollary 6 *Let $f : [a,b] \to X$. If f is $\mathscr{H}$-integrable on $[a,b]$ and is sVB on $[a,b]$, then f is $\mathscr{M}$-integrable on $[a,b]$.*

Proof Let F be the indefinite Henstock integral of f on $[a,b]$. For each x^* in X^*, since the function x^*F is both ACG_* and VB on $[a,b]$, it must be AC on $[a,b]$. It follows that f is Dunford integrable on $[a,b]$ and $(D)\int_I f = (H)\int_I f$ for each I. As the function F is sVB on $[a,b]$, the series $\sum_{k=1}^{\infty} \Delta F(I_k)$ converges absolutely whenever $\{I_k\}_{k=1}^{\infty}$ is a sequence of mutually nonoverlapping intervals. By Proposition 2B of [10], f is Pettis integrable on $[a,b]$. Now Theorem 9 applies to show that f is $\mathscr{M}$-integrable on $[a,b]$. The proof is complete.

Corollary 7 *Let $f : [a,b] \to X$. Suppose that $\mathrm{dens}(B_{X^*}, w^*) < \varkappa(\mu)$. If f is $\mathscr{H}$-integrable on $[a,b]$ and the indefinite Henstock integral of f is sVB on $[a,b]$, then f is absolutely Birkhoff integrable on $[a,b]$.*

The next theorem provides a descriptive sufficient condition for $\mathscr{M}$-integrability on a closed set (cf. [26, Theorem 5]).

Theorem 10 *Let $f : [a,b] \to X$ be $\mathscr{H}$-integrable on $[a,b]$ and let $F : [a,b] \to X$ be the indefinite Henstock integral of f. Suppose that E is a nonempty closed subset of $[a,b]$. If F is AC_* on E, then f is $\mathscr{M}$-integrable on E.*

Proof By Theorem 5 of [26], f is McShane integrable on E. Since f is Riemann measurable on E, Theorem 7 of [28] applies to f. This completes the proof.

In addition, we have a necessary condition for $\mathscr{M}$-integrability on a measurable set, the AC^*_δ property of the indefinite integral. The proof follows essentially as that of Theorem 6 in [26] and is omitted.

Theorem 11 *Let $f : [a,b] \to X$ be $\mathscr{H}$-integrable on $[a,b]$ and let $F : [a,b] \to X$ be the indefinite Henstock integral of f. Suppose that E is a nonempty measurable subset of $[a,b]$. If f is $\mathscr{M}$-integrable on E, then F is AC^*_δ on E.*

Corollary 8 *If $f : [a,b] \to X$ is $\mathscr{H}$-integrable on $[a,b]$, then the indefinite Henstock integral of f is ACG^*_δ on $[a,b]$.*

Proof Since f is Riemann measurable on $[a,b]$, by part (d) of Theorem 1 of [28], there exists a negligible subset N of $[a,b]$ such that $[a,b]\setminus N$ can be written as the union of a sequence $\{F_n\}_{n=1}^{\infty}$ of pairwise disjoint closed sets on each of which f is bounded. Note that f is certainly $\mathscr{M}$-integrable to zero on N. By Theorem 11, the indefinite Henstock integral of f is AC^*_δ on N. Combining Theorem 5 of [28] and Theorem 11 shows that the indefinite Henstock integral of f is also AC^*_δ on each of the sets F_n, which is what we desired.

Corollary 9 *If $f : [a,b] \to X$ is $\mathscr{H}$-integrable on $[a,b]$, then there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of closed subsets of $[a,b]$ such that the set $[a,b] \setminus \bigcup_{n=1}^{\infty} P_n$ is negligible and the indefinite Henstock integral of f is AC_* on P_n for each n.*

Remark 7 According to Example 1 of [27], there exists a c_0-valued Henstock integrable function defined on $[a,b]$ that is not McShane integrable on any nondegenerate subinterval of $[a,b]$ and, as a result of this fact, its indefinite Henstock integral is not even ACG on $[a,b]$ (cf. Corollary 5).

However, in Banach spaces that do not contain an isomorphic copy of c_0 we establish the following somewhat stronger result (cf. Theorem 10 of [26] and Corollary 8).

Theorem 12 *Suppose that X does not contain an isomorphic copy of c_0. If $f : [a,b] \to X$ is $\mathscr{H}$-integrable on $[a,b]$, then $[a,b]$ can be written as a countable union of closed sets on each of which the indefinite Henstock integral is both VB_* and AC as well as the function f is $\mathscr{M}$-integrable.*

Proof By Corollary 4.4 of [25], there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of closed sets such that $[a,b] = \bigcup_{n=1}^{\infty} F_n$ and the indefinite Henstock integral of f is both VB_* and AC on F_n for each n. As the indefinite Henstock integral of f is scalarly differentiable to $f|_{F_n}$ on F_n, Corollary 5.1 of [25] yields the Pettis integrability of f on F_n. Now it follows from Theorem 9 that f is $\mathscr{M}$-integrable on F_n for each n. The proof is complete.

In general, with no restrictions either on the integrand or the range space an indefinite Henstock integral is at most VBG_* [25, Theorem 3.3]. In the next theorem, we are able to remove from Theorem 12 the assumption that the range space does not contain an isomorphic copy of c_0 by placing a stronger restriction on the indefinite integral. We first need an elementary fact that we have been unable to find in print.

Lemma 4 *Let $F : [a,b] \to X$. Suppose that E is a closed subset of $[a,b]$ with $a, b \in E$, F is sVB on E, and $F|_E$ is continuous on E. Then the linear extension of F from E to $[a,b]$ is continuous on $[a,b]$.*

Proof Let G be the linear extension of F from E to $[a,b]$ and let $\{I_n\}_{n=1}^{\infty}$ be the sequence of intervals contiguous to E. We make note of the facts that (i) as G is linear on each I_n,

$$\|G(b_n) - G(a_n)\| \leq \|\Delta F(I_n)\|$$

whenever $[a_n, b_n] \subset I_n$ for each n; (ii) as F is sVB on E,

$$\sum_{n=1}^{\infty} \|\Delta F(I_n)\| < \infty. \tag{8}$$

Without any loss of generality, we may assume that t is a right-hand limit point of E and will prove that the function G is right-hand continuous at t. Fix $\varepsilon > 0$. Let a positive number δ correspond to $\varepsilon/2$ in the definition of the continuity of $F|_E$ at t. By (8), there exists a positive integer N such that

$$\sum_{n>N} \|\Delta F(I_n)\| < \frac{\varepsilon}{2}.$$

Since t is a right-hand limit point of E, we can choose a positive number $\delta_1 \leq \delta$ so that

$$\bigcup_{n=1}^{N} I_n \cap (t, t+\delta_1) = \varnothing.$$

We seek to estimate the norm $\|F(t') - F(t)\|$ for each $t' \in (t, t+\delta_1)$. Suppose that $t' \notin E$ since the other case is obvious. Then $t' \in I_{n'}$ for some $n' > N$. Denote $a_{n'} = \inf I_{n'}$ and compute

$$\|F(t') - F(t)\| \leq \|F(t') - F(a_{n'})\| + \|F(a_{n'}) - F(t)\| \leq \\ \|\Delta F(I_{n'})\| + \|F(a_{n'}) - F(t)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is complete.

Theorem 13 *Let $f : [a,b] \to X$ be $\mathscr{H}$-integrable on $[a,b]$. If the indefinite Henstock integral of f is $sVBG$ on $[a,b]$, then $[a,b]$ can be written as a countable union of closed sets on each of which the indefinite Henstock integral of f is both sVB and AC as well as the function f is $\mathscr{M}$-integrable.*

Proof Let F denote the indefinite Henstock integral of f. Since F is continuous and $sVBG$ on $[a,b]$, it follows from Theorem 2 of [12] that

$$[a,b] = \bigcup_{i=1}^{\infty} P_i$$

where P_i is closed and F is sVB on P_i for each i. Fix a positive integer i. Let $a_i = \inf P_i$, $b_i = \sup P_i$ and let F_i be the linear extension of F from P_i to $[a_i, b_i]$. By Theorem 3 of [12], F_i is sVB $[a_i, b_i]$. As F is scalarly differentiable to $f|_{P_i}$ on P_i, part (f) of Theorem 3.1 of [25] applies to the function F_i to show that F_i is scalarly differentiable on $[a_i, b_i]$ to a function f_i such that $f_i|_{P_i} = f|_{P_i}$. Now, by Theorem 3.2 of [29], we obtain that f_i is Pettis integrable on $[a_i, b_i]$ and so is the function f on P_i. Thus, f is both Riemann measurable and Pettis integrable on P_i. This means, by Theorem 9, that f is $\mathscr{M}$-integrable on P_i.

To see that F is AC on P_i, fix x^* in X^*. We claim that the real-valued function x^*F_i is AC on $[a_i, b_i]$. Indeed, the function x^*F_i is VB on $[a_i, b_i]$ and satisfies condition (N) on $[a_i, b_i]$. The second fact is valid since $x^*F_i|_{P_i} = x^*F|_{P_i}$ satisfies condition (N) on P_i and x^*F_i satisfies condition (N) on each interval where it is linear. Furthermore, by Lemma 4, the function F_i is continuous on $[a_i, b_i]$. Now the Banach-Zarecki Theorem [16, Theorem 6.16] applies to x^*F_i to show that the function x^*F_i is AC on $[a_i, b_i]$. Consequently, we have $x^*(\varDelta F_i(I)) = \int_I x^* f_i$ for each interval I in $[a_i, b_i]$ so that the function F_i differs from the indefinite Pettis integral of f_i on $[a_i, b_i]$ by constant. This means that F_i is AC on $[a_i, b_i]$ and so is F on P_i. The proof is complete.

In what follows, we will need two more auxiliary results that rely on Lemma A. The statement of the first, which is Lemma 3 of [26], is included for completeness.

Lemma B *Let $f : [a,b] \to X$ be Henstock integrable on $[a,b]$, let f be Henstock integrable on a measurable subset E of $[a,b]$, and let $\varepsilon > 0$. Then there is a gauge δ on $[a,b]$ such that*

$$\left\| \sum_{k=1}^{K} f\chi_{I_k \cap E} - \sum_{k=1}^{K} f\chi_{I_k} \right\|_A \leq \varepsilon$$

whenever $\{(I_k, t_k)\}_{k=1}^K$ is a partial Henstock partition of $[a,b]$ subordinate to δ with $t_k \in E$ for each k.

Lemma 5 *Let $f : [a,b] \to X$ be Henstock integrable on $[a,b]$, let N be a negligible subset of $[a,b]$, and let $\varepsilon > 0$. Then there is a gauge δ on $[a,b]$ such that*

$$\left\| \sum_{k=1}^{K} f\chi_{I_k} \right\|_A \leq \varepsilon$$

whenever $\{(I_k, t_k)\}_{k=1}^K$ is a partial Henstock partition of $[a,b]$ subordinate to δ with $t_k \in N$ for each k.

Proof Clearly, f is Henstock integrable on N and $\int_a^b f\chi_{I\cap N} = 0$ for all I. Fix $\varepsilon > 0$. By Lemma B, there exists a gauge δ on $[a, b]$ such that

$$\left\| \sum_{k=1}^{K} f\chi_{I_k\cap N} - \sum_{k=1}^{K} f\chi_{I_k} \right\|_A \leq \varepsilon$$

whenever $\{(I_k, t_k)\}_{k=1}^K$ is a partial Henstock partition of $[a, b]$ subordinate to δ with $t_k \in N$ for each k. Consequently, we get

$$\left\| \sum_{k=1}^{K} f\chi_{I_k} \right\|_A \leq \sum_{k=1}^{K} \|f\chi_{I_k\cap N}\|_A + \left\| \sum_{k=1}^{K} f\chi_{I_k\cap N} - \sum_{k=1}^{K} f\chi_{I_k} \right\|_A \leq \varepsilon.$$

The proof is complete.

We are now in a position to give a more involved approximation property of the Henstock integral including the convergence in the Alexiewicz norm. This approximation property, which is an improvement of Theorem 8 of [26] that removes the need for the *whole* domain to be covered by closed sets on each of which the integrand is McShane integrable, has considerably broadened the present paper. The proof uses a technique similar to that of [26], Theorem 8. The relevant details appear below.

Theorem 14 *Let $f : [a, b] \to X$ be Henstock integrable on $[a, b]$. If there exists a negligible subset N of $[a, b]$ such that $[a, b] \setminus N$ can be written as a countable union of closed sets on each of which f is McShane integrable, then $[a, b] \setminus N$ can be written as the union of an increasing sequence $\{F_n\}_{n=1}^\infty$ of closed sets on each of which f is McShane integrable and*

$$\|f\chi_{F_n} - f\|_A < \frac{1}{n} \tag{9}$$

Proof With no loss of generality, we may assume that

$$[a, b] \setminus N = \bigcup_{i=1}^{\infty} P_i,$$

where $P_i \subset P_{i+1}$, P_i is closed, and f is *McShane integrable* on P_i for each i.

Fix $\varepsilon > 0$ and a positive integer n. We will prove that a closed set F exists such that $P_n \subset F \subset P_{i(n,\varepsilon)}$ for some $i(n, \varepsilon) > n$ and $\|f\chi_F - f\|_A < \varepsilon$. For each i, let a gauge δ_i on $[a, b]$ correspond to P_i and $\varepsilon/2^{i+1}$ in Lemma B and let a gauge δ_0 correspond to N and $\varepsilon/2$ in Lemma 5. Define a gauge δ on $[a, b]$ by

$$\delta(t) = \begin{cases} \delta_n(t), \text{ if } t \in P_n, \\ \min(\delta_i(t), \mathrm{dist}(t, P_{i-1})), \text{ if } t \in P_i \setminus P_{i-1} \text{ for some } i > n, \\ \delta_0(t), \text{ if } t \in N. \end{cases}$$

Choose a Henstock partition $\{(I_k,t_k)\}_{k=1}^K$ of $[a,b]$ subordinate to δ. Let $D_n = P_n$, $D_i = P_i \setminus P_{i-1}$ for each $i > n$, and

$$F = \bigcup_{i=n}^{\infty} \bigcup_{k:t_k \in D_i} I_k \cap P_i.$$

It is easy to check that F is closed, $P_n \subset F$, and $F \subset P_{i(n,\varepsilon)}$ for some $i(n,\varepsilon) > n$. Using Lemma B, Lemma 5, and the fact that for each k with $t_k \in N$ the set $F \cap I_k$ is either empty or a singleton, we get

$$\|f\chi_F - f\|_A \le \sum_{i=n}^{\infty}\Big\|\sum_{k:t_k\in D_i}(f\chi_{F\cap I_k} - f\chi_{I_k})\Big\|_A + \Big\|\sum_{k:t_k\in N}(f\chi_{F\cap I_k} - f\chi_{I_k})\Big\|_A =$$
$$\sum_{i=n}^{\infty}\Big\|\sum_{k:t_k\in D_i}(f\chi_{F\cap I_k} - f\chi_{I_k})\Big\|_A + \Big\|\sum_{k:t_k\in N} f\chi_{I_k}\Big\|_A \le$$
$$\sum_{i=n}^{\infty}\Big\|\sum_{k:t_k\in D_i}(f\chi_{P_i\cap I_k} - f\chi_{I_k})\Big\|_A + \frac{\varepsilon}{2} < \sum_{i=n}^{\infty}\frac{\varepsilon}{2^{i+1}} + \frac{\varepsilon}{2} \le \varepsilon.$$

Define inductively a sequence $\{F_n\}_{n=1}^{\infty}$ of sets and a sequence $\{i_n\}_{n=1}^{\infty}$ of positive integers as follows. Let F_1 be a closed set such that $P_1 \subset F_1 \subset P_{i_1}$ for some $i_1 > 1$ and $\|f\chi_{F_1} - f\|_A < 1$. For each $n > 1$, let F_n be a closed set such that $P_{i_{n-1}} \subset F_n \subset P_{i_n}$ for some $i_n > i_{n-1}$ and $\|f\chi_{F_n} - f\|_A < n^{-1}$. Evidently, the sequence $\{F_n\}_{n=1}^{\infty}$ has all the desired properties. The proof is complete.

Corollary 10 *Let $f : [a,b] \to X$ be $\mathscr{H}$-integrable on $[a,b]$. Suppose that at least one of the following statements holds:*

(i) X does not contain an isomorphic copy of c_0;

(ii) the indefinite Henstock integral of f is sVBG on $[a,b]$.

Then $[a,b]$ can be written as the union of an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of closed sets on each of which f is $\mathscr{M}$-integrable and (9) holds.

Theorem 15 *If $f : [a,b] \to X$ is $\mathscr{H}$-integrable on $[a,b]$, then a negligible subset N of $[a,b]$ exists such that $[a,b] \setminus N$ can be written as the union of an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of closed sets on each of which f is bounded and (9) holds.*

Proof Since f is Riemann measurable on $[a,b]$, by part (d) of Theorem 1 of [28], a negligible set N exists such that the set $[a,b] \setminus N$ can be written as a countable union of closed sets on each of which the function f is *bounded.* As a bounded Riemann measurable function is $\mathscr{M}$-integrable (see [28, Theorem 5]), the remainder of the proof runs the same lines as our proof of Theorem 14.

The last result of this note is related to the concepts of *globally small Riemann sums* and *functionally small Riemann sums* as defined in [21].

Theorem 16 *Let $f : [a,b] \to X$.*

(a) If f is $\mathscr{H}$-integrable ($\mathscr{M}$-integrable) on $[a,b]$, then for each $\varepsilon > 0$ a closed set $F \subset [a,b]$ and a measurable gauge δ on $[a,b]$ exist such that $\mu([a,b] \setminus F) < \varepsilon$, f is bounded on F, and

$$\left\| \sum_{k:t_k \notin F} f(t_k)\mu(I_k) \right\| < \varepsilon \tag{10}$$

whenever $\{(I_k, t_k)\}_{k=1}^K$ is a Henstock (McShane) partition of $[a,b]$ subordinate to δ.

(b) If for each $\varepsilon > 0$ a measurable set $E \subset [a,b]$ and a measurable gauge δ on $[a,b]$ exist such that f is $\mathscr{H}$-integrable ($\mathscr{M}$-integrable) on E and (10) holds for each Henstock (McShane) partition $\{(I_k, t_k)\}_{k=1}^K$ of $[a,b]$ subordinate to δ, then f is $\mathscr{H}$-integrable ($\mathscr{M}$-integrable) on $[a,b]$.

Proof We will prove the $\mathscr{H}$-integrability case since in the other case the proof is similar.

Suppose first that f is $\mathscr{H}$-integrable on $[a,b]$ and let $\varepsilon > 0$. By the preceding theorem, there exists a closed set $F \subset [a,b]$ such that $\mu([a,b] \setminus F) < \varepsilon$, f is bounded on F, and $\| \int_a^b f - \int_F f \| < \varepsilon/3$. Let a measurable gauge δ on $[a,b]$ correspond to $\varepsilon/3$ in the definitions of the Henstock integral of f on $[a,b]$ and on F. Let $\{(I_k, t_k)\}_{k=1}^K$ be a Henstock partition of $[a,b]$ subordinate to δ and compute

$$\left\| \sum_{k:t_k \notin F} f(t_k)\mu(I_k) \right\| \le \left\| \sum_{k=1}^K f(t_k)\mu(I_k) - \int_a^b f \right\| +$$
$$\left\| \int_a^b f - \int_F f \right\| + \left\| \int_a^b f\chi_F - \sum_{k=1}^K f\chi_F(t_k)\mu(I_k) \right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

For (b), let $\varepsilon > 0$. Choose a measurable set $E \subset [a,b]$ and a measurable gauge δ_1 on $[a,b]$ such that f is $\mathscr{H}$-integrable on E and

$$\left\| \sum_{k:t_k \notin E} f(t_k)\mu(I_k) \right\| < \frac{\varepsilon}{3}$$

whenever $\{(I_k, t_k)\}_{k=1}^K$ is a Henstock partition of $[a,b]$ subordinate to δ_1. Since $f\chi_E$ is $\mathscr{H}$-integrable on $[a,b]$, a measurable gauge $\delta \le \delta_1$ on $[a,b]$ exists such that

$$\left\| \sum_{k:t'_k \in E} f(t'_k)\mu(I'_k) - \sum_{k:t''_k \in E} f(t''_k)\mu(I''_k) \right\| < \frac{\varepsilon}{3}$$

whenever $\{(I'_k, t'_k)\}_{k=1}^{K'}$ and $\{(I''_k, t''_k)\}_{k=1}^{K''}$ are Henstock partitions of $[a, b]$ subordinate to δ. For such partitions, we have

$$\left\|\sum_{k=1}^{K'} f(t'_k)\mu(I'_k) - \sum_{k=1}^{K''} f(t''_k)\mu(I''_k)\right\| \leq \left\|\sum_{k:t'_k\notin E} f(t'_k)\mu(I'_k)\right\| + \left\|\sum_{k:t'_k\in E} f(t'_k)\mu(I'_k) - \sum_{k:t''_k\in E} f(t''_k)\mu(I''_k)\right\| + \left\|\sum_{k:t''_k\notin E} f(t''_k)\mu(I''_k)\right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

By the Cauchy criterion for $\mathscr{H}$-integrability, the function f is $\mathscr{H}$-integrable on $[a, b]$. The proof is complete.

We close the paper with a few comments on the context of our results. It remains unclear to us whether our assumption on the w^*-density character of the dual unit ball can be removed from the hypotheses of Corollary 2 and Lemma 2. We also do not know if there is a Riemann measurable function that is scalarly equivalent to zero, but not absolutely Birkhoff integrable. We should note that if such a function exists, it is certainly both unbounded and Birkhoff integrable. Unfortunately, we have been unable to completely remove the use of gauges from our characterization of an indefinite Henstock integral in terms of the various generalized absolute continuity properties. In particular, it would be interesting to find some classes of infinite dimensional Banach spaces (or some conditions on the particular function) in which the indefinite Henstock integral is necessarily ACG_*.

We thank the referee for many helpful insights toward improving our original manuscript. The third author expresses his appreciation to the University of Palermo Department of Mathematics and Computer Science for its hospitality in summer 2015, during which part of this research was carried out.

References

1. G. Birkhoff, Integration of functions with values in a Banach space. Trans. Amer. Math. Soc. **38** (1935), no. 2, 357-378. MR1501815
2. M. Balcerzak and K. Musiał, A convergence theorem for the Birkhoff integral. Funct. Approx. Comment. Math. **50** (2014), no. 1, 161-168. MR3189505
3. M. Balcerzak and M. Potyrała, Convergence theorems for the Birkhoff integral. Czechoslovak Math. J. **58** (2008), no. 4, 1207-1219. MR2471177
4. S. J. Dilworth and M. Girardi, Nowhere weak differentiability of the Pettis integral. Quaestiones Math. **18** (1995), no. 4, 365-380. MR1354118
5. L. Di Piazza and V. Marraffa, An equivalent definition of the vector-valued McShane integral by means of partitions of unity. Studia Math. **151** (2002), no. 2, 175-185. MR1917952
6. L. Di Piazza and V. Marraffa, The McShane, PU and Henstock integrals of Banach valued functions. Czechoslovak Math. J. **52** (2002), no. 3, 609-633. MR1923266
7. D. H. Fremlin, The Henstock and McShane integrals of vector-valued functions. Illinois J. Math. **38** (1994), no. 3, 471-479. MR1269699

8. D. H. Fremlin, The generalized McShane integral. Illinois J. Math. **39** (1995), no. 1, 39-67. MR1299648
9. D. H. Fremlin, The McShane and Birkhoff integrals of vector-valued functions, University of Essex Mathematics Department Research Report 92-10, version of 18.5.07 available at URL http://www.essex.ac.uk/maths/people/fremlin/preprints.htm.
10. D. H Fremlin and J. Mendoza, On the integration of vector-valued functions. Illinois J. Math. **38** (1994), no. 1, 127-147. MR1245838
11. R. A. Gordon, Integration and differentiation in a Banach space. Ph.D. Thesis, University of Illinois at Urbana-Champaign, 1987. MR2635774
12. R. A. Gordon, The Denjoy extension of the Bochner, Pettis, and Dunford integrals. Studia Math. **92** (1989), no. 1, 73-91. MR0984851
13. R. A. Gordon, Another look at a convergence theorem for the Henstock integral. Real Anal. Exchange **15** (1989/90), no. 2, 724-728. MR1059433
14. R. A. Gordon, The McShane integral of Banach-valued functions. Illinois J. Math. **34** (1990), no. 3, 557-567. MR1053562
15. R. Gordon, Riemann integration in Banach spaces. Rocky Mountain J. Math. **21** (1991), no. 3, 923-949. MR1138145
16. R. A. Gordon, The integrals of Lebesgue, Denjoy, Perron, and Henstock. Graduate Studies in Mathematics, Vol. 4, American Mathematical Society, Providence RI, 1994. MR1288751
17. V. M. Kadets and L. M. Tseytlin, On "integration" of non-integrable vector-valued functions. Mat. Fiz. Anal. Geom. **7** (2000), no. 1, 49-65. MR1760946
18. V. Kadets, B. Shumyatskiy, R. Shvidkoy, L. Tseytlin and K. Zheltukhin, Some remarks on vector-valued integration. Mat. Fiz. Anal. Geom. **9** (2002), no. 1, 48-65. MR1911073
19. A. Kolmogoroff, Untersuchungen über den Integralbegriff. (German) Math. Ann. **103** (1930), no. 1, 654-696. MR1512641
20. J. Kurzweil and Š. Schwabik, On McShane integrability of Banach space-valued functions. Real Anal. Exchange **29** (2003/04), no. 2, 763-780. MR2083811
21. Lu Shi Pan and Lee Peng Yee, Globally small Riemann sums and the Henstock integral. Real Anal. Exchange **16** (1990/91), no. 2, 537-545. MR1112049
22. V. Marraffa, A descriptive characterization of the variational Henstock integral. Proceedings of the International Mathematics Conference (Manila, 1998). Matimyás Mat. **22** (1999), no. 2, 73-84. MR1770168
23. V. Marraffa, A Birkhoff type integral and the Bourgain property in a locally convex space. Real Anal. Exchange **32** (2007), no. 2, 409-427. MR2369853
24. K. M. Naralenkov, Asymptotic structure of Banach spaces and Riemann integration. Real Anal. Exchange **33** (2007/08), no. 1, 111-124. MR2402867
25. K. Naralenkov, On Denjoy type extensions of the Pettis integral. Czechoslovak Math. J. **60** (2010), no. 3, 737-750. MR2672413
26. K. Naralenkov, Several comments on the Henstock-Kurzweil and McShane integrals of vector-valued functions. Czechoslovak Math. J. **61** (2011), no. 4, 1091-1106. MR2886259
27. K. M. Naralenkov, A Henstock-Kurzweil integrable vector-valued function which is not McShane integrable on any portion. Quaest. Math. **35** (2012), no. 1, 11-21. MR2931302
28. K. M. Naralenkov, A Lusin type measurability property for vector-valued functions. J. Math. Anal. Appl. **417** (2014), no. 1, 293-307. MR3191427
29. K. M. Naralenkov, Some comments on scalar differentiation of vector-valued functions. Bull. Aust. Math. Soc. **91** (2015), no. 2, 311-321. MR3314150
30. B. J. Pettis, On integration in vector spaces. Trans. Amer. Math. Soc. **44** (1938), no. 2, 277-304. MR1501970
31. M. Potyrała, Some remarks about Birkhoff and Riemann-Lebesgue integrability of vector valued functions. Tatra Mt. Math. Publ. **35** (2007), 97-106. MR2372438
32. J. Rodríguez, On the existence of Pettis integrable functions which are not Birkhoff integrable. Proc. Amer. Math. Soc. **133** (2005), no. 4, 1157-1163. MR2117218
33. J. Rodríguez, Pointwise limits of Birkhoff integrable functions. Proc. Amer. Math. Soc. **137** (2009), no. 1, 235-245. MR2439446
34. J. Rodríguez, Convergence theorems for the Birkhoff integral. Houston J. Math. **35** (2009), no. 2, 541-551. MR2519546

35. D. O. Snow, On integration of vector-valued functions. Canad. J. Math. **10** (1958), 399-412. MR0095242
36. A. P. Solodov, On the limits of the generalization of the Kolmogorov integral. (in Russian) Mat. Zametki **77** (2005), no. 2, 258-272; translation in Math. Notes 77 (2005), no. 1-2, 232-245. MR2157094
37. C. Swartz, Norm convergence and uniform integrability for the Henstock-Kurzweil integral. Real Anal. Exchange **24** (1998/99), no. 1, 423-426. MR1691761
38. M. Talagrand, Pettis integral and measure theory. Mem. Amer. Math. Soc. **51** (1984), No. 307. MR0756174
39. Wang Pujie, Equi-integrability and controlled convergence for the Henstock integral. Real Anal. Exchange **19** (1993/94), no. 1, 236-241. MR1268849