A note on the admissibility of modular function spaces

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Abstract

In this paper we prove the admissibility of modular function spaces E_ρ considered and defined by Kozłowski in [17]. As an application we get that any compact and continuous mapping $T : E_\rho \to E_\rho$ has a fixed point. Moreover, we prove that the same holds true for any retract of E_ρ.

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1. Introduction

The notion of admissibility, introduced by Klee in [14], allows one to approximate the identity on compact sets by finite-dimensional mappings. Locally convex spaces are admissible (see [24]), and a large literature is devoted to prove that particular classes of non-locally convex function spaces are admissible, among others we mention [7, 21, 25, 26, 27]. Recently, in [2] it has been proved the admissibility of spaces of functions determined by finitely additive set functions. It is important to notice that not all non-locally convex spaces are admissible, in [3] Cauty provides an example of a metric linear space in which the admissibility fails. Here we prove the admissibility of modular function spaces in the framework defined by Kozłowski in [17] (see also [15, 16]). Modular function spaces are a natural generalization of both functions and sequence variants of Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderón-Lozanovskii spaces and many others. Our interest in the admissibility of modular function spaces lies in the possibility of applying the result to the fixed point theory. The fixed point theory in modular function spaces was initiated by Khamsi, Kozłowski and Reich [12], and it is a topic of interest in the theory of nonlinear operators, see e.g. [1, 4, 5, 8, 9, 10, 11, 18, 20, 22] and references therein. For more information about the current state of the theory the reader is referred to [13]. One of the advantages of the theory, as observed for example in [5], is that even in absence of a metric, many problems in metric fixed point theory can be formulated in modular spaces. We recall the definition of admissibility.

Definition 1. [14] Let E be a Haudorff topological vector space. A subset Z of E is said to be *admissible* if for every compact subset K of Z and for every neighborhood V of zero in E there exists a continuous mapping $H : K \to Z$ such that $\dim(\text{span}\,[H(K)]) < \infty$ and $f - Hf \in V$ for every $f \in K$. If $Z = E$ we say that the space E is *admissible.*

2. Preliminaries

We start by introducing modular function spaces, following [17]. Let X be a nonempty set and $\mathcal{P}$ a nontrivial δ-ring of subsets of X, i.e. a ring closed under countable intersections. Let Σ be the smallest σ-algebra of subsets of X such that $\mathcal{P}$ is contained in Σ. Let us assume that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$, and

$$X = \bigcup_{n=1}^{\infty} X_n, \tag{1}$$

where $X_n \subset X_{n+1}$ and $X_n \in \mathcal{P}$ for any $n \in \mathbb{N}$. Let $(W, \|\cdot\|)$ be a Banach space. By a $\mathcal{P}$-simple function on X with values in W we mean a function of the form

$$g = \sum_{i=1}^{n} w_i \chi_{E_i},$$

where $\omega_i \in W$, $E_i \in \mathcal{P}$, $E_i \cap E_j = \emptyset$ for $i \neq j$, and by $\mathcal{E}$ we denote the linear space of all $\mathcal{P}$-simple functions. A function $f : X \to W$ is called *measurable* if there exists a sequence of $\mathcal{P}$-simple functions $\{s_n\}$ such that $s_n(x) \to f(x)$ for any $x \in X$. By $M(X, W)$ we denote the set of all measurable functions.

Definition 2. A functional $\rho : \mathcal{E} \times \Sigma \to [0, +\infty]$ is called a *function modular* if it satisfies the following properties:

(P1) $\rho(0, E) = 0$ for every $E \in \Sigma$;

(P2) $\rho(f, E) \leq \rho(g, E)$ whenever $\|f(x)\| \leq \|g(x)\|$ for all $x \in E$ and any $f, g \in \mathcal{E}$ ($E \in \Sigma$);

(P3) $\rho(f, \cdot) : \Sigma \to [0, +\infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$;

(P4) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where for $\alpha > 0$

$$\rho(\alpha, A) = \sup\{\rho(r\chi_A, A) : r \in W, \|r\| \leq \alpha\};$$

(P5) there is $\alpha_0 \geq 0$ such that $\sup_{\beta>0} \rho(\beta, A) = 0$ whenever $\sup_{\alpha>\alpha_0} \rho(\alpha, A) = 0$;

(P6) $\rho(\alpha, \cdot)$ is order continuous on $\mathcal{P}$ for every $\alpha > 0$, that is $\rho(\alpha, A_n) \to 0$ for any sequence $\{A_n\} \subset \mathcal{P}$ decreasing to $\emptyset$.

Then for $f \in M(X, W)$ we set

$$\rho(f, E) = \sup\{\rho(g, E) : g \in \mathcal{E}, \|g(x)\| \leq \|f(x)\| \text{ for all } x \in E\}.$$

Definition 3. A set $E \in \Sigma$ is said to be ρ-*null* if $\rho(\alpha, E) = 0$ for every $\alpha > 0$, and a property is said to hold ρ-*almost everywhere* (briefly ρ-*a.e.*) if the set where it fails to hold is ρ-null.

Then the functional $\rho : M(X, W) \to [0, +\infty]$ defined by $\rho(f) = \rho(f, X)$ is a semimodular, that is

(i) $\rho(\lambda f) = 0$ for any $\lambda > 0$ iff $f = 0$ ρ-a.e.;

(ii) $\rho(\alpha f) = \rho(f)$ if $|\alpha| = 1$ and $f \in M(X, W)$;

(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$ ($\alpha, \beta \geq 0$) and $f, g \in M(X, W)$.

Given the semimodular ρ we consider the modular space

$$L_\rho = \{f \in M(X, W) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0\},$$

endowed with the F-norm

$$\|f\|_\rho = \inf\left\{\lambda > 0 : \ \rho\left(f/\lambda\right) \leq \lambda\right\}.$$

Recall that $\|f_n - f\|_\rho \to 0$ is equivalent to $\rho(\alpha(f_n - f)) \to 0$ for all $\alpha > 0$. We are interested in the closed subspace E_ρ of L_ρ defined by

$$E_\rho = \{f \in M(X, W) : \rho(\alpha f, \cdot) \text{ is order continuous for every } \alpha > 0\}.$$

For any set S in E_ρ we denote by clS the closure of S with respect to $\|\cdot\|_\rho$. We recall that $E_\rho = cl\mathcal{E}$ (see [17, Theorem 2.4.8]). Also, for $f \in E_\rho$, we set

$$F_n f = f\chi_{X_n}, \tag{2}$$

where $X = \bigcup_{n=1}^\infty X_n$ as in (1). We denote by $\mathbb{N}$ the set of natural numbers.

3. Preliminary results

Throughout this section we assume that $X \in \mathcal{P}$ and we consider finite partitions of X whose elements are disjoint and not ρ-null sets in $\mathcal{P}$. Given a partition $\Pi = \{A_1, ...A_n\}$ of X we denote by S_Π the set of $\mathcal{P}$-simple functions generated by Π. Given the partitions $\Pi_k = \{E_1, ...E_l\}$ and $\Pi_n = \{F_1, ..., F_m\}$ $(k, n \in \mathbb{N})$ of X, we write $\Pi_k \leq \Pi_n$ if each $E_i \in \Pi_k$ can be written as

$$E_i = \sum_{j=1}^{m_i} F_{ij} \tag{3}$$

with $F_{ij} \in \Pi_n$, for $j = 1, ..., m_i$.

Aim of this section is that of defining, corresponding to a given sequence of partitions of X, an equicontinuous sequence of operators which approximates uniformly the functions of a given compact subset of E_ρ (Theorem 1). To this end, given any two partitions $\Pi_k = \{E_1, ...E_l\}$ and $\Pi_n = \{F_1, ..., F_m\}$, whenever $\Pi_k \leq \Pi_n$, we define $P_{kn} : S_{\Pi_n} \to S_{\Pi_k}$ by setting, for $s = \sum_{j=1}^m w_j \chi_{F_j}$,

$$P_{kn}s = \sum_{i=1}^{l} \frac{\sum_{j=1}^{m_i} w_j}{m_i} \chi_{E_i}, \tag{4}$$

where, for each $i = 1, 2, \cdots, l$, the indices $1, 2, \cdots, m_i$ satisfy (3). Given a sequence of partitions $Q = \{\Pi_n\}$ of X such that $\Pi_1 \leq \Pi_2 \leq ...\Pi_n \leq ...$ and $\Pi_1 = \{X\}$, we put

$$S(Q) = \bigcup_{j=1}^{\infty} S_{\Pi_j}.$$

Then, corresponding to Q, for any $k \in \mathbb{N}$, we define $P_k : S(Q) \to S_{\Pi_k}$ by setting

$$P_k s = \begin{cases} s & \text{if } s \in \bigcup_{j=1}^{k} S_{\Pi_j} \\ P_{kn} s & \text{if } s \in \bigcup_{j=k+1}^{\infty} S_{\Pi_j}, \end{cases}$$

where for $s \in \bigcup_{j=k+1}^{\infty} S_{\Pi_j}$ we choose $n > k$ such that $s \in S_{\Pi_n}$. Observe that by (4) the above formula does not depend on the choice of the n for which $s \in S_{\Pi_n}$, therefore the operator P_k is well-defined.

In the following, if not specified otherwise, we consider $Q = \{\Pi_n\}$ a fixed sequence of partitions of X such that $\Pi_1 \leq \Pi_2 \leq ...\Pi_n \leq ...$ and $\Pi_1 = \{X\}$.

Lemma 1. *Let $s_1, s_2 \in S(Q)$ such that $\sup\{\|s_i(x)\| : x \in X, i = 1, 2\} \leq a < \infty$. If for a given set $D \in \mathcal{P}$ and a given $\delta > 0$*

$$\sup\{\|s_1(x) - s_2(x)\| : x \in D\} \leq \delta,$$

then, for any $k \in \mathbb{N}$, we have

$$\rho(P_k s_1 - P_k s_2, X) \leq \rho(\delta, D) + \rho(2a, X \setminus D).$$

Proof. Fix $k \in \mathbb{N}$. Choose $n \geq k$, such that $s_1, s_2 \in S_{\Pi_n}$. Assume first $n > k$. Let $x \in D$ and fix $E_i \in \Pi_k$ such that $x \in E_i$. Assume that $E_i = \bigcup_{j=1}^{m_i} F_{ij}$, $F_{ij} \in \Pi_n$ for $j = 1, ..., m_i$. Put $w_{ij}^1 = s_1(x)$ and $w_{ij}^2 = s_2(x)$ for $x \in F_{ij}$. Then

$$\|(P_k s_1)(x) - (P_k s_2)(x)\| = \Big\| \sum_{j=1}^{m_i} \frac{w_{ij}^1 - w_{ij}^2}{m_i} \Big\| \leq \sum_{j=1}^{m_i} \frac{\|w_{ij}^1 - w_{ij}^2\|}{m_i} \leq \delta.$$

By (P2), it follows $\rho(P_k s_1 - P_k s_2, D) \leq \rho(\delta, D)$. On the other hand, for any $x \in X \setminus D$, we have

$$\|(P_k s_1)(x) - (P_k s_2)(x)\| \leq \sup\{\|s_1(y)\| + \|s_2(y)\| : y \in X\} \leq 2a,$$

which implies $\rho(P_k s_1 - P_k s_2, X \setminus D) \leq \rho(2a, X \setminus D)$. Therefore, by (P3), we obtain

$$\begin{aligned} \rho(P_k s_1 - P_k s_2, X) &\leq \rho(P_k s_1 - P_k s_2, D) + \rho(P_k s_1 - P_k s_2, X \setminus D) \\ &\leq \rho(\delta, D) + \rho(2a, X \setminus D), \end{aligned}$$

as required. If $k = n$, then $P_k s_i = s_i$, for $i = 1, 2$, and the last inequality is immediate. □

Now for $a > 0$ we put

$$S_a(Q) = \{s \in S(Q) : \sup\{\|s(x)\| : x \in X\} \le 2a\}.$$

As $E_\rho = cl\mathcal{E}$, given a function $f \in E_\rho$ there exists a sequence $\{s_n\} \subset \mathcal{E}$ such that $\|s_n(x)\| \le \|f(x)\|$ for any $x \in X$ and $\|f - s_n\|_\rho \to 0$, and in the sequel we will use the fact that the sequence $\{s_n\}$ can be chosen in $S_a(Q)$ whenever $\sup\{\|f(x)\| : x \in X\} \le a < \infty$.

Lemma 2. *Let f be a function in $cl(S(Q))$. If $\sup\{\|f(x)\| : x \in X\} \le a < \infty$, then, for any $k \in \mathbb{N}$ and any sequence $\{s_n\} \subset S_a(Q)$ such that $\|f - s_n\|_\rho \to 0$, the limit $\lim_n P_k s_n$ exists, in the norm $\|\cdot\|_\rho$, and does not depend on $\{s_n\}$.*

Proof. Assume there exist $k \in \mathbb{N}$, $\{s_n\} \subset S_a(Q)$ with $\|f - s_n\|_\rho \to 0$ such that $\lim_n P_k s_n$ does not exist. Since E_ρ is complete, passing to a subsequence if necessary, we can assume that there is $d > 0$ such that $\|P_k s_m - P_k s_n\|_\rho > d > 0$ for any $n, m \in \mathbb{N}$, and moreover, in vitue of [17, Corollary 2.4.6 and Proposition 2.3.5], we can also assume that $s_n(x) \to f(x)$ ρ-a.e. Then by a version of the Egoroff Theorem [17, Theorem 2.3.4] there exists a nondecreasing sequence of sets $D_m \in \mathcal{P}$ with $\bigcup_{m=1}^\infty D_m = X$ such that $\{s_n\}$ converges uniformly to f on every D_m.

Next let $\varepsilon = d/3$ and, applying (P4) and (P6), find $m_o \in \mathbb{N}$ and $\delta > 0$ such that $\rho(\delta, D_{m_o}) < \varepsilon$ and $\rho(2a, X \setminus D_{m_o}) < \varepsilon$. Since $\{s_n\}$ converges uniformly to f on D_{m_o}, we can choose $m_1, m_2 \in \mathbb{N}$ such that $\sup\{\|s_{m_1}(x) - s_{m_2}(x)\| : x \in D_{m_o}\} \le 2\varepsilon\delta$. By Lemma 1, we have

$$\rho\Big(\frac{P_k s_{m_1} - P_k s_{m_2}}{2\varepsilon}, X\Big) \le \rho(\delta, D_{m_o}) + \rho(2a, X \setminus D_{m_o}) < 2\varepsilon.$$

Therefore, by the definition of $\|\cdot\|_\rho$, we find

$$\|P_k s_{m_1} - P_k s_{m_2}\|_\rho \le 2\varepsilon < d$$

which is a contradiction. To show that, for any $k \in \mathbb{N}$, $\lim_n P_k s_n$ is independent on the choice of $\{s_n\}$, let $\{s_n\}$ and $\{v_n\}$ be two sequences in $S_a(Q)$ such that $\|f - s_n\|_\rho \to 0$ and $\|f - v_n\|_\rho \to 0$. Put $z_{2n} = s_n$ and $z_{2n+1} = v_n$. Applying the above reasoning to the sequence $\{z_n\}$ we find $\lim_n P_k s_n = \lim_n P_k v_n$, which completes our proof. □

Now, corresponding to the given sequence of partitions Q, for any function $f \in cl(S(Q))$ such that $\sup\{\|f(x)\| : x \in X\} \leq a < \infty$, for any $k \in \mathbb{N}$, we define $P_k f$, in S_{Π_k}, by setting

$$P_k f = \lim_n P_k s_n, \tag{5}$$

where $\{s_n\} \subset S_a(Q)$ is any sequence satisfying $\|f - s_n\|_\rho \to 0$ and the limit is meant in the norm $\| \cdot \|_\rho$. The last definition, due to Lemma 2, is well-posed. Our next goal is that of proving the equicontinuity of the sequence of operators $\{P_k\}$ defined in (5).

Lemma 3. *Let $\{f_n\}$ be a sequence in $cl(S(Q))$ and $f \in cl(S(Q))$. If $\sup\{\|f_n(x)\| : x \in X,\ n \in \mathbb{N}\} \leq a < \infty$, $\sup\{\|f(x)\| : x \in X\} \leq a < \infty$ and $\|f - f_n\|_\rho \to 0$, then*

$$\lim_n (\sup\{\|P_k f_n - P_k f\|_\rho : k \in \mathbb{N}\}) = 0. \tag{6}$$

Proof. Suppose that (6) is not satisfied. Then passing to a subsequence, if necessary, we can assume that there exists $d > 0$ such that for any $n \in \mathbb{N}$

$$\sup\{\|P_k f_n - P_k f\|_\rho : k \in \mathbb{N}\} > d, \tag{7}$$

and, applying [17, Proposition 2.3.3 and Proposition 2.3.5], we can also assume that $f_n \to f$ ρ-a.e.

Let $\varepsilon = d/4$. Then by [17, Theorem 2.3.4] and property (P6) there exists $D \in \mathcal{P}$ such that $\rho(2a, X \setminus D) < \varepsilon$ and $\{f_n\}$ converges uniformly to f on D. By (P4), we can choose $\delta > 0$ such that $\rho(\delta, D) < \varepsilon$.
Now fix $n_o \in \mathbb{N}$ with

$$\sup\{\|f_{n_o}(x) - f(x)\| : x \in D\} \leq \varepsilon\delta.$$

Choose two sequences $\{s_l\}$ and $\{v_l\}$ in $S_a(Q)$ such that $\|f_{n_o} - v_l\|_\rho \to 0$ and $\|f - s_l\|_\rho \to 0$. Again, by [17, Theorem 2.3.4] and (P6), passing to a subsequence if necessary, we can find $D_1 \in \mathcal{P}$, $D_1 \subset D$ (so that $\rho(\delta, D_1) < \varepsilon$) and $l_o \in \mathbb{N}$ such that for $l \geq l_o$ we have

$$\sup\{\|s_l(x) - f(x)\| : x \in D_1\} \leq \varepsilon\delta/2,$$
$$\sup\{\|v_l(x) - f_{n_o}(x)\| : x \in D_1\} \leq \varepsilon\delta/2,$$

and $\rho(2a, D \setminus D_1) < \varepsilon$. Hence, by the triangle inequality,

$$\sup\{\|s_l(x) - v_l(x)\| : x \in D_1\} \leq 2\varepsilon\delta$$

for $l \geq l_o$. Consequently by Lemma 1, for $l \geq l_o$ and any $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\rho\Big(\tfrac{P_k s_l - P_k v_l}{2\varepsilon}, X\Big) &\leq \rho(\delta, D_1) + \rho(2a, X \setminus D_1)) \\ &\leq \rho(\delta, D_1) + \rho(2a, X \setminus D) + \rho(2a, D \setminus D_1) < 3\varepsilon,\end{aligned}$$

and, by the definition of $\|\cdot\|_\rho$, we get

$$\|P_k s_l - P_k v_l\|_\rho \leq 3\varepsilon.$$

Since $\|f_{n_o} - v_l\|_\rho \to 0$ and $\|f - s_l\|_\rho \to 0$, by Lemma 2 and (5), for any $k \in \mathbb{N}$, it follows

$$\|P_k f_{n_o} - P_k f\|_\rho \leq 3\varepsilon.$$

Consequently, by the choice of ε, we find $\sup\{\|P_k f_{n_o} - P_k f\|_\rho : k \in \mathbb{N}\} < d$, which contradicts (7). □

Theorem 1. *Let K be a compact subset of E_ρ such that*

$$\sup\{\|f(x)\| : x \in X, f \in K\} \leq a < \infty.$$

Then there exists a sequence $\Pi_1 \leq \Pi_2 \leq ...\Pi_n \leq ...$, which will be denoted by Q, of partitions of X such that $\Pi_1 = \{X\}$ and $K \subset cl(S(Q))$.
Moreover, for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$\sup\{\|f - P_k f\|_\rho : f \in K\}) \leq \varepsilon,$$

where $\{P_k\}$ is the sequence of operators corresponding to the sequence of partitions Q.

Proof. Since K is a compact subset of E_ρ, K is separable. Let $\{f_n\}$ be a fixed countable and dense subset of K. Since $K \subset E_\rho$, for any $n \in \mathbb{N}$ there exists a sequence $\Pi_1^n \leq \Pi_2^n \leq ...\Pi_k^n \leq ...$ of partitions of X, which will be denoted by Q_n, such that $\Pi_1^n = \{X\}$ and $f_n \subset cl(S(Q_n))$. Put $R = \bigcup_{n\in\mathbb{N}} Q_n$ and $S^R = \bigcup_{n\in\mathbb{N}} S(Q_n)$. Since $\{f_n\}$ is dense in K, we have $K \subset cl(S^R)$. Moreover, as R is a countable set of partitions, we can write $R = \{\Gamma_n\}_{n=1}^\infty$, where each Γ_n is a partition of X and we can assume $\Gamma_1 = \{X\}$.
Now we construct by induction a sequence of partitions $\{\Pi_n\}$. To this end define $\Pi_1 = \{X\}$ and assume that we have defined Π_j for $j = 1, ..., n-1$. Set

$$\Pi_n = \{E \cap F : E \in \Gamma_n, F \in \Pi_j \text{ for some } j < n\}.$$

It is clear that $\Pi_j \leq \Pi_k$ for $j \leq k$ and $S^R \subset S(Q)$, where $Q = \{\Pi_n\}$. Since $K \subset cl(S^R)$, it follows $K \subset cl(S(Q))$, which shows that Q is the required sequence of partitions.

Let $\{P_k\}$ be the sequence of operators determined by the sequence Q. First we show that for any $f \in K$

$$\lim_k \|f - P_k f\|_\rho \to 0. \tag{8}$$

Fix a sequence of simple functions $\{s_n\}$ in $S_a(Q)$ such that $\|f - s_n\|_\rho \to 0$. Passing to a subsequence, if necessary, we can assume that $s_n \to f$ ρ-a.e. Fix $\varepsilon > 0$. Applying [17, Theorem 2.3.4] we can assume that there exists $D \subset X$ such that for any $\delta > 0$ we can find $n_o \in \mathbb{N}$ satisfying

$$\sup\{\|f(x) - s_n(x)\| : x \in D, n \in \mathbb{N}, n \geq n_o\} \leq \varepsilon\delta.$$

and $\|f - s_n\|_\rho < \varepsilon$ for $n \geq n_o$. Hence

$$\sup\{\|s_n(x) - s_{n_o}(x)\| : x \in D, n \in \mathbb{N}, n \geq n_o\} \leq 2\varepsilon\delta.$$

By Lemma 1, for any $n \geq n_o$ and $k \in \mathbb{N}$,

$$\rho\Big(\frac{P_k s_{n_o} - P_k s_n}{2\varepsilon}, X\Big) \leq \rho(\delta, D) + \rho(2a, X \setminus D).$$

Now fix $D \subset X$, $D \in \mathcal{P}$, such that $\rho(2a, X \setminus D) \leq \varepsilon$ and $\delta > 0$ satisfying $\rho(\delta, D) < \varepsilon$. Then we have

$$\rho\Big(\frac{P_k s_{n_o} - P_k s_n}{2\varepsilon}, X\Big) \leq \rho(\delta, D) + \rho(2a, X \setminus D) < 2\varepsilon$$

for $n \geq n_o$ and consequently, $\|P_k s_n - P_k s_{n_o}\|_\rho \leq 2\varepsilon$ for any $k \in \mathbb{N}$ and $n \geq n_o$. By (5), as n goes to infinity, $\|P_k f - P_k s_{n_o}\|_\rho \leq 2\varepsilon$ for any $k \in \mathbb{N}$. Finally, since $P_{n_o} s_{n_o} = s_{n_o}$, we have

$$\|f - P_{n_o} f\|_\rho \leq \|f - s_{n_o}\|_\rho + \|P_{n_o} s_{n_o} - P_{n_o} f\|_\rho \leq 3\varepsilon,$$

which proves (8).
Now assume on the contrary that there exists $\varepsilon > 0$ such that, passing to a subsequence if necessary, for any $k \in \mathbb{N}$,

$$\sup\{\|f - P_k f\|_\rho : f \in K\}) > \varepsilon.$$

Then for any $k \in \mathbb{N}$ there exists $f_{n_k} \in K$ such that

$$\|f_{n_k} - P_k f_{n_k}\|_\rho > \varepsilon.$$

By the compactness of K, we can assume that $\|f_{n_k} - f\|_\rho \to 0$ for some $f \in K$. Note that for any $k \in \mathbb{N}$

$$\begin{aligned}\|f_{n_k} - P_k f_{n_k}\|_\rho &\leq \|f_{n_k} - f\|_\rho + \|f - P_k f\|_\rho + \|P_k f - P_k f_{n_k}\|_\rho \\ &\leq \|f_{n_k} - f\|_\rho + \|f - P_k f\|_\rho + \sup\{\|P_l f - P_l f_{n_k}\|_\rho : l \in \mathbb{N}\}.\end{aligned}$$

Since $\|f_{n_k} - f\|_\rho \to 0$, by Lemma 3, we have

$$\|f_{n_k} - f\|_\rho + \sup\{\|P_l f - P_l f_{n_k}\|_\rho : l \in \mathbb{N}\} \leq \varepsilon/2$$

for n_k sufficiently large. By the previous part of the proof there is $k_o \in \mathbb{N}$ such that $\|f - P_k f\|_\rho \leq \varepsilon/2$ for $k \geq k_o$. Hence for $k \geq k_o$

$$\|f_{n_k} - P_k f_{n_k}\|_\rho \leq \varepsilon,$$

which is a contradiction. The proof of our theorem is completed. □

4. Admissibility

In this section we prove the admissibility of the space E_ρ. The method of the proof is similar to that of [2]. We begin with the following proposition by showing that the space of $\mathcal{P}$-simple functions generated by a given partition of X is admissible.

Proposition 1. *Let $\Pi = \{A_1, \cdots, A_n\}$ be a partition of X. Then the subspace*

$$S_\Pi = \left\{ s \in E_\rho : s = \sum_{i=1}^{n} w_i \chi_{A_i},\ w_i \in W \right\}$$

of E_ρ is admissible.

Proof. Let K be a compact subset of S_Π. For each $g \in K$ we can write

$$g = \sum_{i=1}^{n} w_i(g) \chi_{A_i}$$

for suitable elements $w_i(g)$ of the Banach space W. Note that by (P2), for any $i = 1, ..., n$ the linear mapping p_i defined in S_Π by $p_i(g) = w_i(g)\chi_{A_i}$ satisfies

$\|p_i(g)-p_i(h)\|_\rho \leq \|g-h\|_\rho$ for any $g, h \in S_\Pi$ and consequently it is continuous. Then we prove that for any $i = 1, ..., n$ the mapping $g \to w_i(g)$ is continuous with respect to $\|\cdot\|_\rho$. Assume on the contrary that there exists $d > 0$, $\{g_n\} \subset S_\Pi$ and $g \in S_\Pi$ such that $\|g_n - g\|_\rho \to 0$ and $\|w_i(g_n) - w_i(g)\| \geq d$. Again by (P2), we have

$$\rho(w_i(g)\chi_{A_i} - w_i(g_n)\chi_{A_i}, A_i) \geq \rho(d, A_i).$$

Since A_i is not a ρ-null set, by (P5) we get $\rho(d, A_i) > 0$, which contradicts the continuity of p_i.

Consequently, for any fixed $i = 1, \cdots, n$, the set $C_i = \{w_i(g) : \ g \in K\}$ is a compact subset of W, and $C = \cup_{i=1}^n C_i$ as well. Let $\delta > 0$ be fixed. Then by the admissibility of the Banach space W, there exist a finite dimensional space $Z_\delta = \text{span}[z_1, \cdots, z_m]$ in W and a continuous mapping $H_\delta : \ C \to Z_\delta$ such that

$$\|w - H_\delta(w)\| \leq \delta \ \text{ for all } \ w \in C.$$

Then for each $i \in \{1, \cdots, n\}$, $g \in K$ and for suitable real numbers $w_j^i(g)$, $j = 1, \cdots, m$, we can write

$$H_\delta(w_i(g)) = \sum_{j=1}^m w_j^i(g) z_j.$$

We denote again by $H_\delta : K \to S_\Pi$ the continuous mapping defined by

$$H_\delta g = \sum_{i=1}^n H_\delta(w_i(g))\chi_{A_i} = \sum_{i=1}^n \Big(\sum_{j=1}^m w_j^i(g) z_j\Big)\chi_{A_i},$$

then

$$H_\delta(K) \subseteq \ \text{span}[\chi_{A_i} z_j, i = 1, \cdots, n; \ j = 1, \cdots, m]$$

and $\dim(\text{span}[H_\delta(K)]) < \infty$. On the other hand for every $g \in K$ we have

$$\begin{aligned} \|g - H_\delta g\|_\rho &= \Big\| \textstyle\sum_{i=1}^n w_i(g)\chi_{A_i} - \sum_{i=1}^n \Big(\sum_{j=1}^m w_j^i(g) z_j\Big)\chi_{A_i}\Big\|_\rho \\ &\leq \textstyle\sum_{i=1}^n \Big\| \Big(w_i(g) - \sum_{j=1}^m w_j^i(g) z_j\Big)\chi_{A_i}\Big\|_\rho. \end{aligned}$$

By (P2) we have

$$\sum_{i=1}^n \Big\| \Big(w_i(g) - \sum_{j=1}^m w_j^i(g) z_j\Big)\chi_{A_i}\Big\|_\rho \leq \sum_{i=1}^n \|\delta\chi_{A_i}\|_\rho.$$

Since $\|\cdot\|_\rho$ is an F-norm, given $\varepsilon > 0$ we can choose $\delta > 0$ such that $\sum_{i=1}^{n} \|\delta\chi_{A_i}\|_\rho < \varepsilon$. Therefore we have proved $\|g - H_\delta g\|_\rho < \varepsilon$, and hence the admissibility of S_Π in E_ρ. □

In order to prove our main result (Theorem 2) we need the following two lemmas.

Lemma 4. *For $f, g \in E_\rho$ and $n \in \mathbb{N}$, we have*

$$\|F_n f - F_n g\|_\rho \leq \|f - g\|_\rho,$$

where F_n is defined by (2). Moreover, for any $\varepsilon > 0$ and any compact subset K of E_ρ there exists $n_o \in \mathbb{N}$ such that for each $n \geq n_o$ we have

$$\sup\{\|f - F_n f\|_\rho : f \in K\} < \varepsilon.$$

Proof. Let $f, g \in E_\rho$, then by property (P2) we have

$$\rho(\alpha(F_n f - F_n g), X) \leq \rho(\alpha(f - g), X),$$

for every $\alpha > 0$. Hence, by definition of $\|\cdot\|_\rho$,

$$\|F_n f - F_n g\|_\rho \leq \|f - g\|_\rho.$$

Set $Z_n = X \setminus X_n$, then $\{Z_n\} \subset \mathcal{P}$ is a decreasing sequence of sets. Since by [17, Theorem 2.5.1], for every $\alpha > 0$ the set function $\sup\{\rho(\alpha f, \cdot) : f \in K\}$ is order continuous, given $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ such that for each $n \geq n_o$

$$\sup\{\rho(f/\varepsilon, Z_n) : f \in K\} \leq \varepsilon.$$

Consequently, by the definition of $\|\cdot\|_\rho$, as required, we find for every $f \in K$

$$\|f - F_n f\|_\rho = \|f\chi_{Z_n}\|_\rho \leq \varepsilon.$$

□

Now, for $a > 0$, we denote by R_a the radial projection, of the Banach space W onto its closed ball $B_a(W)$ of radius a, defined for $w \in W$ by

$$R_a(w) = \begin{cases} w & \text{if} \quad \|w\| \leq a \\ a\,\frac{w}{\|w\|} & \text{if} \quad \|w\| > a. \end{cases}$$

Then we define the mapping $T_a : E_\rho \to E_\rho$ by setting for $x \in X$

$$(T_a f)(x) = R_a(f(x)).$$

Lemma 5. *For any $a > 0$ and $f, g \in E_\rho$, we have*

$$\|T_a f - T_a g\|_\rho \leq 2\|f - g\|_\rho.$$

Moreover, for any $\varepsilon > 0$ and for any compact subset K of E_ρ there exists $a > 0$ such that

$$\sup\{\|f - T_a f\|_\rho : f \in K\} \leq \varepsilon.$$

Proof. Fix $a > 0$. Note that by the definition of T_a for any $d > 0$,

$$\rho(d(T_a f - T_a g), X) \leq \rho(2d(f - g), X).$$

as R_a is a Lipschitz mapping with constant 2. Hence we obtain

$$\|T_a f - T_a g\|_\rho \leq \|2(f - g)\|_\rho \leq 2\|(f - g)\|_\rho.$$

Now let K be a compact subset of E_ρ and $f \in K$. Note that for any $x \in X$, $\lim_n (T_n f)(x) = f(x)$ and $\|(T_n f)(x)\| \leq \|f(x)\|$ for any $n \in \mathbb{N}$. By [17, Theorem 2.4.7], we get

$$\|f - T_n f\|_\rho \to 0. \tag{9}$$

To prove our second assert assume by contradiction that there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there exists $f_{k_n} \in K$ such that

$$\|f_{k_n} - T_n f_{k_n}\|_\rho > \varepsilon.$$

Without loss of generality, passing to a convergent subsequence if necessary, we can assume that there exists $f \in E_\rho$ such that $\|f_{k_n} - f\|_\rho \to 0$. Note that for any $a > 0$

$$\begin{aligned} \|f_{k_n} - T_a f_{k_n}\|_\rho &\leq \|f - T_a f\|_\rho + \|T_a f - T_a f_{k_n}\|_\rho + \|f - f_{k_n}\|_\rho \\ &\leq 3\|f_{k_n} - f\|_\rho + \|f - T_a f\|_\rho. \end{aligned}$$

Hence by (9), $\|f_{k_n} - T_a f_{k_n}\|_\rho \leq \varepsilon$ for $a \in \mathbb{N}$ sufficiently large which proves our claim. □

Theorem 2. *The space E_ρ is admissible.*

Proof. Fix K a compact set in E_ρ, and $\varepsilon > 0$. Since K is compact, by Lemma 4, we can find $n \in \mathbb{N}$ such that

$$\sup\{\|f - F_n f\|_\rho : f \in K\} \leq \varepsilon/4.$$

Moreover, F_n is continuous. By Lemma 5 applied to the compact set $F_n(K)$, there exists $a > 0$ such that

$$\sup\{\|F_n f - T_a F_n f\|_\rho : f \in K\} \leq \varepsilon/4.$$

Since T_a is a continuous mapping, by Theorem 1 applied to $(T_a \circ F_n)(K)$ there exists a sequence Q of partitions of X, $\Pi_1 \leq \Pi_2 \leq ...\Pi_n \leq ...$ and $\Pi_1 = \{X\}$, for which $(T_a \circ F_n)(K) \subset cl(S(Q))$. Moreover we can choose $k \in \mathbb{N}$ such that

$$\sup\{\|T_a F_n f - P_k T_a F_n f\|_\rho : f \in K\} \leq \varepsilon/4,$$

were $\{P_k\}$ is the sequence of operators associated to Q. By Lemma 1, P_k is a continuous mapping for any $k \in \mathbb{N}$. Then $V = (P_k \circ T_a \circ F_n)(K)$ is a compact subset of S_{Π_k}, hence by Proposition 1 there exists $H_\varepsilon : V \to E_\rho$ such that $\text{span}[H_\varepsilon(V)]$ is finite-dimensional and

$$\sup\{\|H_\varepsilon P_k T_a F_n f - P_k T_a F_n f\|_\rho : f \in K\} \leq \varepsilon/4.$$

Now we consider the continuous mapping $H = H_\varepsilon \circ P_k \circ T_a \circ F_n$. We have $\dim[\text{span}[H(K)] < \infty$. Moreover, by the above facts, for any $f \in K$ we have

$$\begin{aligned} \|f - Hf\|_\rho &\leq \|f - F_n f\|_\rho + \|F_n f - T_a F_n f\|_\rho + \|T_a F_n f - P_k T_a F_n f\|_\rho \\ &+ \|P_k T_a F_n f - H_\varepsilon P_k T_a F_n f\|_\rho \leq \varepsilon, \end{aligned}$$

hence the admissibility of E_ρ is proved. □

Now we present one important consequence of admissibility and Theorem 2. The proof works for any admissible Hausdorff topological vector space and it is well-known (see [19], we also refer to [6]). We give the proof for sake of completeness.

Theorem 3. *Let $T : L_\rho \to E_\rho$ be a compact and continuous mapping. Then there exists $f \in E_\rho$ such that $Tf = f$.*

Proof. Without loss of generality we can assume that $T : E_\rho \to E_\rho$. Since T is a compact mapping, $K = cl[T(E_\rho)]$ is a compact set in E_ρ. Hence by Theorem 2 for any $\varepsilon > 0$ there exists a continuous mapping $H_\varepsilon : K \to E_\rho$ such that $\dim(\text{span}[H_\varepsilon(K)]) < \infty$ and $\sup\{\|f - H_\varepsilon f\|_\rho : f \in K\} \leq \epsilon$.
Let $T_\varepsilon = H_\varepsilon \circ T$. Then $T_\varepsilon(E_\rho) \subset H_\epsilon(K)$ and consequently $T_\varepsilon[conv(H_\varepsilon(K))] \subset conv(H_\epsilon(K))$. Also T_ε is a continuous mapping. Since the dimension of

span$[H_\varepsilon(K)]$ is finite, by the Carathéodory Theorem, $conv(H_\varepsilon(K))$ is a compact set. By the Brouwer Theorem there exists $f_\varepsilon \in E_\rho$ such that $T_\varepsilon f_\varepsilon = f_\varepsilon$. Hence for any $n \in \mathbb{N}$,

$$\|Tf_{1/n} - f_{1/n}\|_\rho = \|Tf_{1/n} - T_{1/n}f_{1/n}\|_\rho = \|Tf_{1/n} - H_{1/n}Tf_{1/n}\|_\rho \leq 1/n,$$

since $Tf_{1/n} \in K$.
By the compactness of K we can assume that $\|Tf_{1/n} - f\|_\rho \to 0$ for some $f \in K$. Hence by the above estimate and the triangle inequality, we obtain $\|f_{1/n} - f\|_\rho \to 0$. By the continuity of T, $\|Tf_{1/n} - Tf\|_\rho \to 0$, which gives that $Tf = f$. □

Now we show that Theorem 3 also holds true for any retract of E_ρ. Recall that a set $A \subset E_\rho$ is called a retract of E_ρ if there exists a continuous mapping $r : E_\rho \to A$ such that $r|_A = id_A$. The mapping r is called a retraction.

Theorem 4. *Let $A \subset E_\rho$ be a retract of E_ρ with a retraction r. Then any compact and continuous mapping $T : A \to A$ has a fixed point.*

Proof. We apply Theorem 3. Observe that $T \circ r : E_\rho \to E_\rho$ is a compact and continuous mapping. Applying Theorem 3 to $T \circ r$ we get that there exists $f \in E_\rho$ such that $(T \circ r)f = f$. Since T maps A into A, $f \in A$. Consequently, $r(f) = f$ and $Tf = f$, as required. □

The following remark provides some interesting cases of modular function spaces in which our results can be applied.

Remark 1. By [17, Chapter 4], Musielak-Orlicz spaces of vector-valued functions determined by a σ-finite measure are modular function spaces. The same applies to Lorentz-type L^p spaces determined by a σ-finite measure (see also [17, Chapter 5], where countably modulared function spaces are considered). It is clear that in general these spaces are not locally convex, which makes our result original.

Finally, we have $E_\rho = L_\rho$ if and only if the function modular ρ satisfies the Δ_2-condition:
$\sup_n \rho(2f_n, A_k) \to 0$ as $k \to \infty$, whenever $\{f_n\}$ is a sequence in $M(X, W)$, $A_k \in \Sigma$, $A_k \to \emptyset$ and $\sup_n \rho(f_n, A_k) \to 0$ as $k \to \infty$.
Hence if the Δ_2-condition is satisfied, we can apply our results to L_ρ. Notice that in the case of classical Musielak-Orlicz spaces the above Δ_2-condition

coincides with the classical Orlicz Δ_2-condition (see [17, Proposition 4.1.10]). For more information about Orlicz spaces and Musielak-Orlicz spaces see [23].

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