SET VALUED INTEGRABILITY IN NON SEPARABLE FRÉCHET SPACES AND APPLICATIONS

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Abstract. We focus on measurability and integrability for set valued functions in non-necessarily separable Fréchet spaces. We prove some properties concerning the equivalence between different classes of measurable multifunctions. We also provide useful characterizations of Pettis set-valued integrability in the announced framework. Finally, we indicate applications to Volterra integral inclusions.

1. Introduction

Due to tremendous interest in applications (e.g. control theory, optimization or mathematical economics), a wide theory was developed for the set-valued integrability in separable Banach spaces ([9], [32] and reference inside, [17], [18], [20], [39], [40], [50], [53]). Besides, as seen in [6], a similar theory for non-separable case is necessary; several authors studied the set-valued integrability without separability assumptions in Banach spaces (see [3], [4, 7], [19] and [42]).
On the other side, it is well known that evolution partial differential equations in finite- or infinite-dimensional case can be treated as ordinary differential equations in infinite dimensional non-normable locally convex spaces. Also, the study of various differential problems on infinite intervals or that of infinite systems of differential equations naturally lead to the framework of Fréchet spaces (as in [21], [43], [27] or [11], to cite only a few). Therefore, the study of ordinary differential equations (and inclusions) in general locally convex spaces has serious motivations.
But the way in this direction is paved with difficulties; indeed, even in the single-valued case, as described in [37, Theorem 1], the classical Peano Theorem fails in all infinite-dimensional Fréchet spaces. Also, a straightforward generalization of Picard-Lindelöf Theorem to Fréchet spaces fails; existence results were obtained only under very strong assumptions on the function governing the equation (we

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refer to [37], [36], [46] or [24]).
As for the set-valued case, the situation is even more complicated since the literature concerning set-valued measure and integration in this setting is quite poor ([9, 14], [48], [49], [47]). Differential inclusions in Fréchet spaces have been studied, as far as we know, only in the separable case (see [9] and the references therein). One can although mention several works on differential inclusions involving at some point Fréchet spaces, e.g. [43] or [27], and also the papers [22], [23] where set differential equations using the notion of Hukuhara derivative have been considered. In all previously mentioned papers, such a space is viewed as a projective limit of a sequence of Banach spaces.
In the present work we focus on measurability and integrability for set valued functions in non-necessarily separable Fréchet spaces. There are many notions of set-valued measurability and integrability available in locally convex spaces (see [1] , [9], [30, 32], [48]); while, without separability, the connections between them are not always clearly stated. Therefore, in Section 3 and in Section 4 we obtain some results concerning relations between various concepts of measurability and integrability for multifunctions defined on a probability space with non empty closed, bounded, convex values of a Fréchet space (in view of applications, we will consider only multifunctions defined in $[0,1]$, but all results hold true also in case of multifunctions defined in a complete probability space).

We obtain equivalence between different classes of measurable multifunctions (see Theorem 3.5). In these framework we prove a version of the Pettis Measurability Theorem for set-valued functions, that, as far as we know, is new also in case the target space is a Banach space (see Theorem 3.6). We also provide useful characterizations of Pettis set-valued integrability (see Theorem 4.4 and Proposition 4.6).
Afterwards, in Section 5 we consider applications to integral inclusions and provide an existence result via Kakutani fixed point Theorem for the Volterra integral inclusion

$$x(t) \in x_0 + \int_0^t k(t,s)F(s,x(s))ds, t \in [0,1]. \tag{1}$$

2. Preliminaries

Let $[0,1]$ be the unit interval of the real line equipped with the usual topology and the Lebesgue measure μ and let denote by $\mathcal{L}$ the collection of all measurable sets in $[0,1]$. Throughout this paper X is a Fréchet space, i.e. a metrizable, complete, locally convex space. It is well known that there exists a sequence $(p_i)_{i\in\mathbb{N}}$ of seminorms which is sufficient (that is, $p_i(x) = 0$ for all $i \in \mathbb{N}$ implies that $x = 0$) and increasing such that the metric

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_i(x-y)}{1+p_i(x-y)}$$

is invariant to translations and generates an equivalent topology on X.

By X^* we denote the topological dual of X. The symbol $c(X)$ stands for the family of all nonempty closed convex subsets of X, while its subfamilies consisting of bounded (resp. compact) subsets will be denoted by $cb(X)$ (resp. $ck(X)$).

We consider on $c(X)$ the Minkowski addition ($A \oplus B := \overline{\{a+b : a \in A,\, b \in B\}}$) and the standard multiplication by scalars. Our unexplained terminology related to Fréchet spaces can be found in the monographs [35] or [51].

Let H be the corresponding Hausdorff metric on $cb(X)$, i.e.

$$H(A,B) = \max(e_d(A,B), e_d(B,A))$$

where the excess $e_d(A,B)$ of the set A over the set B is defined as

$$e_d(A,B) = \sup\{d(a,B) : a \in A\} = \sup\{\inf_{b \in B} d(a,b) : a \in A\}.$$

In a similar way (see [9] or [48]), for each seminorm p_i we can define the semimetric

$$H_i(A,B) = \max(e_i(A,B), e_i(B,A))$$

where the excess $e_i(A,B)$ of the set A over the set B is defined as

$$e_i(A,B) = \sup\{p_i(a,B), a \in A\} = \sup\{\inf_{b \in B} p_i(a-b) : a \in A\}.$$

The families $cb(X)$ and $ck(X)$ endowed with the Hausdorff metric are complete metric spaces and the family of semimetrics $(H_i)_{i \in \mathbb{N}}$ generates the Hausdorff metric H (see [10]). If $A \subseteq X$, by the symbol A^o we denote the *polar* of A, i.e. the set $\{x^* \in X^* : |\langle x^*, x\rangle| \leq 1 \text{ for all } x \in A\}$.

For every $C \in c(X)$ the *support function of* C is denoted by $\sigma(\cdot, C)$ and defined on X^* by $\sigma(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}$, for each $x^* \in X^*$. For each $i \in \mathbb{N}$, let U_i be the set $U_i := \{x \in X : p_i(x) \leq 1\}$. It is useful to recall the Hörmander equality (see [9, Theorem II-18]):

$$H_i(A,B) = \sup\{|\sigma(x^*, A) - \sigma(x^*, B)| : x^* \in U_i^o\} \tag{2}$$

for all $A, B \subseteq c(X)$.

Any map $\Gamma : [0,1] \to c(X)$ is called a *multifunction.* A function $f : [0,1] \to X$ is called a *selection of* Γ if

$$f(t) \in \Gamma(t) \text{ a.e in } [0,1].$$

A multifunction $\Gamma : [0,1] \to cb(X)$ is a *simple multifunction* if $\Gamma = \sum_{i=1}^p \chi_{A_i} C_i$ where $A_i \in \mathcal{L}$ and $C_i \in cb(X)$ are pairwise disjoint. According to [10] for every $E \in \mathcal{L}$ we define $\int_E \Gamma(t)dt := \sum_{i=1}^p \mu(A_i \cap E) C_i$.

3. Measurability for multifunctions

In this section, we focus on measurability of multifunctions. The measurability has been defined in many different ways in the literature concerned with the set-valued theory. We will consider only several of these notions and try to clarify the relationships among them.

Definition 3.1. Let $\Gamma : [0,1] \to cb(X)$ be a multifunction.

i) Γ is said to be *Bochner measurable* (see [30]) if it is Bochner-measurable when seen as a single-valued function having values in the space $(cb(X), H)$ i.e. if there exists a sequence of simple multifunctions $\Gamma_n : [0,1] \to cb(X)$ such that

$$\lim_{n\to\infty} H(\Gamma_n(t), \Gamma(t)) = 0 \text{ for almost all } t \in [0,1]$$

ii) Γ is said to be *totally measurable* (see [48]) if there exists a sequence of simple multifunctions $\Gamma_n : [0,1] \to cb(X)$ such that for every $i \in \mathbb{N}$:

$$\lim_{n\to\infty} H_i(\Gamma_n(t), \Gamma(t)) = 0 \text{ for almost all } t \in [0,1] \tag{3}$$

iii) Γ is said to be *measurable by seminorm* or *p-measurable* (as in [48] or [50]), if for every $i \in \mathbb{N}$ there exists a sequence of simple multifunctions $\Gamma_n^i : [0,1] \to cb(X)$ such that

$$\lim_{n\to\infty} H_i(\Gamma_n^i(t), \Gamma(t)) = 0 \text{ for almost all } t \in [0,1]$$

iv) Γ is said to be *Borel-measurable* (see [30]) if it is Borel-measurable when seen as a single-valued function having values in the space $(cb(X), H)$.

v) Γ is said to be *lower-measurable* (in [30]) (or *measurable* (in [32]) or *Effros measurable* (in [1]) if for every open subset V of X,

$$\Gamma^-(V) = \{t \in [0,1] : \Gamma(t) \cap V \neq \emptyset\}$$

is measurable.

vi) Γ is said to be *upper-measurable* (see [30]), or *strongly-measurable* (see [32])) if for every closed subset D of X,

$$\Gamma^-(D) = \{t \in [0,1] : \Gamma(t) \cap D \neq \emptyset\}$$

is measurable.

vii) Γ is said to be *scalarly-measurable* (see [9]) if for every $x^* \in X^*$, the support functional $t \to \sigma(x^*, \Gamma(t))$ is measurable.

Remark 3.2. Observe that if Γ is measurable by seminorm, then for each i and $n \in \mathbb{N}$, the function $H_i(\Gamma_n^i(\cdot), \Gamma(\cdot)) : [0,1] \to \mathbb{R}$ is measurable. Indeed fixed $\overline{n} \in \mathbb{N}$, we get that

$$\left| H_i(\Gamma^i_{\overline{n}}(t), \Gamma(t)) - H_i(\Gamma^i_{\overline{n}}(t), \Gamma^i_j(t)) \right| \leq H_i(\Gamma^i_j(t), \Gamma(t))$$

and the measurability of $H_i(\Gamma^i_{\frac{i}{n}}(\cdot), \Gamma(\cdot))$ follows from the fact that $H_i(\Gamma^i_{\frac{i}{n}}(\cdot), \Gamma^i_j(\cdot))$ are measurable being $\Gamma^i_j(\cdot)$ simple multifunctions for each i and j and $\lim_{j\to\infty} H_i(\Gamma^i_j(t), \Gamma(t)) = 0$.

Remark 3.3. We recall that when the multifunction is a function and X is a general locally convex space, then the Bochner measurability implies the measurability by seminorm, while the converse implication is true if the topology is generated by a countable family of seminorms. In particular the two concepts are the same in Fréchet spaces (see [25, p. 247]). Also, the measurability by seminorm implies the scalar measurability (see [25, p. 237]), while according to Pettis measurability Theorem in locally convex spaces ([2, Theorem 2.2] and [25, p. 248]), the scalar measurability implies the measurability by seminorms if the range of the function is separable for seminorm. Concerning the Borel measurability, simple functions are Borel measurable. However Bochner measurable functions need not be Borel measurable in general, unless X is metrizable (see [12]), while if a function f is measurable by seminorm then the inverse images of semiballs are measurable (see [25, p. 242]).

In the more general setting of multifunctions, by definition, every totally measurable multifunction is measurable by seminorm.

Moreover, it is known that

i) For multifunctions defined on $[0, 1]$ and, more generally, on a measure space with the disjoint hereditary additive property, the Bochner measurability coincides with the Borel measurability (see [30] Lemma 2.5 and p. 126).

ii) Every Borel measurable multifunction is lower measurable (see [30, p. 131]).

iii) Every upper-measurable multifunction is lower measurable (see [9, Proposition III.11]); the reciprocal holds for compact-valued multifunctions, as shown in [9, Proposition III.12].

iv) The Borel measurability in general does not imply the upper-measurability ([30, p. 131]).

v) As a consequence of [32, Proposition 2.32], every lower measurable multifunction is also scalarly measurable.

vi) The measurability by seminorm is stronger than the scalar measurability (see Remark 3.2 and the properties of the semimetrics H_i given in [9, p. 49]).

Remark also that

vii) In separable metric spaces there are additional links between the notions presented before. In fact, Theorem III.30 in [9] shows that the lower-measurability and the upper measurability are equivalent to the fact that for every $x \in X$, the real function $t \to d(x, \Gamma(t))$ is measurable and also to the fact that the graph is measurable; Theorem III.2 in [9] states that for compact-valued multifunctions, the Bochner measurability, the lower measurability and the upper measurability coincide; besides, by Theorem III.15 in [9], for multifunctions with convex compact values, the lower measurability coincides with the scalar measurability.

viii) For Banach space-valued multifunctions without separability hypothesis, the measurability of multifunctions was investigated in [5], [6], [8]; also, the lower measurability and the scalar measurability are studied in [1]; in particular, an example is provided to show that the lower measurability is strictly stronger that the scalar measurability.

ix) In [28] some other notions of measurability of Bochner-type (via simple multifunctions) were discussed, with respect to different topologies on the hyperspace of closed sets; in particular, the Vietoris topology was considered.

□

In order to go further into the study of various types of measurability for multifunctions in non-necessarily separable Fréchet spaces, we need the following auxiliary result.

Lemma 3.4. (see [35, p. 206]) *Let $(\alpha_i)_{i\in\mathbb{N}}$ be an increasing sequence of positive numbers and let*

$$\alpha = \sum_{i=1}^{\infty} \frac{1}{2^i}\frac{\alpha_i}{1+\alpha_i}.$$

Then:
i) If for some $k \in \mathbb{N}$, $\alpha_k < \frac{1}{2^k}$, then $\alpha < \frac{1}{2^{k-1}}$.
ii) If for some $m, k \in \mathbb{N}$, $\alpha < \frac{1}{2^{m+k+1}}$, then $\alpha_m < \frac{1}{2^k}$.

We prove now that for multifunctions in Fréchet spaces, the Bochner measurability, the totally measurability and the measurability by seminorm are equivalent, as in case of Fréchet valued functions.

Theorem 3.5. *Let $\Gamma : [0,1] \to cb(X)$ be a multifunction. Then the following are equivalent:*

i) Γ *is Bochner measurable;*
ii) Γ *is totally measurable;*
iii) Γ *is measurable by seminorm.*

Proof. Let us begin by proving the equivalence of *i)* and *ii)*. Suppose Γ is totally measurable. Then there exists a sequence of simple multifunctions $\Gamma_n : [0,1] \to cb(X)$ such that for every $i \in \mathbb{N}$ there exists $N_i \subset [0,1]$ with $\mu(N_i)$=0 and

$$\lim_{n\to\infty} H_i(\Gamma_n(t), \Gamma(t)) = 0, \forall t \in [0,1] \setminus N_i.$$

Fix $t \in [0,1] \setminus (\bigcup_{i=1}^{\infty} N_i)$ and let $k \in \mathbb{N}$. Since $\lim_{n\to\infty} H_k(\Gamma_n(t), \Gamma(t)) = 0$, there exists $\overline{n}(t,k) \in \mathbb{N}$ such that

$$H_k(\Gamma_n(t), \Gamma(t)) < \frac{1}{2^k},\ \forall n > \overline{n}(t,k).$$

Otherwise said, for every $n > \overline{n}(t,k)$,

$$e_k(\Gamma_n(t), \Gamma(t)) < \frac{1}{2^k} \ \text{ and } \ e_k(\Gamma(t), \Gamma_n(t)) < \frac{1}{2^k}.$$

It follows that for each $x \in \Gamma_n(t)$, there exists $\overline{y} \in \Gamma(t)$ satisfying

$$p_k(x - \overline{y}) < \frac{1}{2^k}$$

whence, by applying Lemma 3.4, we get

$$d(x, \overline{y}) < \frac{1}{2^{k-1}}$$

and so,

$$e_d(\Gamma_n(t), \Gamma(t)) \leq \frac{1}{2^{k-1}}.$$

Likewise,

$$e_d(\Gamma(t), \Gamma_n(t)) \leq \frac{1}{2^{k-1}},$$

therefore

$$H(\Gamma_n(t), \Gamma(t)) \leq \frac{1}{2^{k-1}},\ \forall n > \overline{n}(t,k).$$

The Bochner measurability is thus proved.
Suppose now that Γ is Bochner measurable. One can find a sequence of simple multifunctions $\Gamma_n : [0,1] \to cb(X)$ such that there exists $N \subset [0,1]$ with $\mu(N)$=0 and

$$\lim_{n\to\infty} H(\Gamma_n(t), \Gamma(t)) = 0, \forall t \in [0,1] \setminus N.$$

Fix now $i \in \mathbb{N}$ and $t \in [0,1] \setminus N$. For every $k \in \mathbb{N}$, there exists $\overline{n}(i,k,t) \in \mathbb{N}$ such that

$$H(\Gamma_n(t), \Gamma(t)) < \frac{1}{2^{i+k+1}},\ \forall n > \overline{n}(i,k,t).$$

It means that

$$e_d(\Gamma_n(t), \Gamma(t)) < \frac{1}{2^{i+k+1}} \ \text{ and } \ e_d(\Gamma(t), \Gamma_n(t)) < \frac{1}{2^{i+k+1}}.$$

So, for each $x \in \Gamma_n(t)$ one can find $\overline{y} \in \Gamma(t)$ with

$$d(x, \overline{y}) < \frac{1}{2^{i+k+1}}$$

whence, by Lemma 3.4,

$$p_i(x - \overline{y}) < \frac{1}{2^k}$$

and so,

$$e_i(\Gamma_n(t), \Gamma(t)) \leq \frac{1}{2^k}.$$

Similarly,

$$e_i(\Gamma(t), \Gamma_n(t)) \leq \frac{1}{2^k},$$

whence

$$H_i(\Gamma_n(t), \Gamma(t)) \leq \frac{1}{2^k}, \ \forall n > \overline{n}(i, k, t).$$

Therefore

$$H_i(\Gamma_n(t), \Gamma(t)) \to 0 \ a.e.$$

and the totally measurability is proved. In order to show the equivalence of *i*) and *ii*) with *iii*), it is enough to prove that *iii*) implies *ii*) . Suppose thus that Γ is measurable by seminorm. Then for every $i \in \mathbb{N}$, there exists a sequence of simple multifunctions $\Gamma_n^i : [0,1] \to cb(X)$ such that

$$\lim_{n\to\infty} H_i(\Gamma_n^i(t), \Gamma(t)) = 0 \ \forall t \in [0,1] \quad a.e.$$

We need to prove the existence of a sequence $(G_n)_n$ of simple multifunctions on $[0,1]$ such that for each $i \in \mathbb{N}$, $\lim_{n\to\infty} H_i(G_n(t), \Gamma(t)) \to 0$ a.e.
By Remark 3.2, for each i and $n \in \mathbb{N}$, $H_i(\Gamma_n^i(t), \Gamma(t))_n$ is measurable and also

$$\frac{H_i(\Gamma_n^i(t), \Gamma(t))}{1 + H_i(\Gamma_n^i(t), \Gamma(t))} < 1.$$

Fix $i \in \mathbb{N}$. The sequence $(H_i(\Gamma_n^i(t), \Gamma(t)))_n$ converges to zero a.e., thus it converges in measure to zero. Therefore for each i there exist a simple multifunction $\Gamma_{\overline{n}(i)}^i$ and a measurable set $E_i \subset [0,1]$ with $\mu(E_i) < \frac{1}{2^i}$ such that $H_i(\Gamma_{\overline{n}(i)}^i, \Gamma(t)) < \frac{1}{2^i}$ for all $t \notin E_i$.

Then

$$\int_0^1 \frac{H_i(\Gamma_{\overline{n}(i)}^i(t), \Gamma(t))}{1 + H_i(\Gamma_{\overline{n}(i)}^i(t), \Gamma(t))} dt = \int_{[0,1]\setminus E_i} \frac{H_i(\Gamma_{\overline{n}(i)}^i(t), \Gamma(t))}{1 + H_i(\Gamma_{\overline{n}(i)}^i(t), \Gamma(t))} dt + \int_{E_i} \frac{H_i(\Gamma_{\overline{n}(i)}^i(t), \Gamma(t))}{1 + H_i(\Gamma_{\overline{n}(i)}^i(t), \Gamma(t))} dt < \frac{1}{2^{i-1}},$$

which implies that

$$\lim_{i\to\infty} \int_0^1 \frac{H_i(\Gamma_{\overline{n}(i)}^i(t), \Gamma(t))}{1 + H_i(\Gamma_{\overline{n}(i)}^i(t), \Gamma(t))} dt = 0.$$

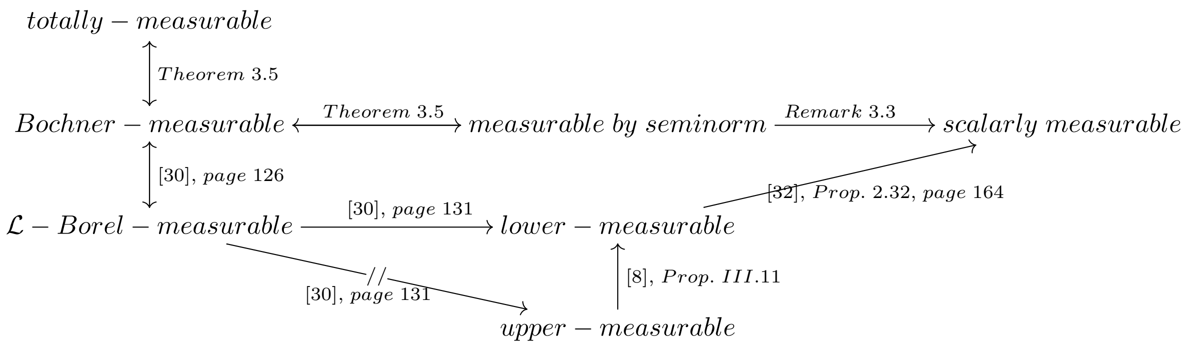
Then it is possible to find a subsequence $(\Gamma_{\overline{n}(i_k)}^{i_k})_k$ of $(\Gamma_{\overline{n}(i)}^i)_i$ such that $\lim_{k\to\infty}(H_{i_k}(\Gamma_{\overline{n}(i_k)}^{i_k}(t), \Gamma(t)) = 0$ a.e.
Defining $G_k := \Gamma_{\overline{n}(i_k)}^{i_k}$ we get a sequence of simple multifunctions such that $\lim_{k\to\infty}(H_k(G_k(t), \Gamma(t)) = 0$, for all $t \in [0,1] \setminus N$, with $\mu(N) = 0$.
Now fix $i \in \mathbb{N}$, $t \in [0,1] \setminus N$ and $\varepsilon > 0$. We can find $\overline{k} = \overline{k}(i, \varepsilon, t)$ such that if $k > \overline{k}$, we get

$$H_i(G_k(t), \Gamma(t)) \leq H_k(G_k(t), \Gamma(t)) < \varepsilon.$$

Therefore the totally measurability follows and the proof is over.

□

The SCHEMA below summarizes our discussion concerning the measurability of $cb(X)$-valued multifunctions defined on $[0,1]$ when X is a general separable Fréchet space:



By Pettis Measurability Theorem (see [2, Theorem 2.2] for its locally convex space version) it is known that any single-valued function which is strongly measurable has almost separable range. As far as we know, the following is the first result of this kind in the framework of the set-valued functions taking values in Fréchet spaces. Related to this we note only that, in [30, Lemma 2.5], it is shown that the range of a Bochner-measurable multifunction seen like a single-valued function having values in the space $(cb(X), H)$ is almost separable in that space.

Theorem 3.6. *Let $\Gamma : [0,1] \to ck(X)$ be Bochner measurable. Then there is a set $N \subseteq [0,1]$ with $\mu(N) = 0$ such that $\bigcup_{t\in[0,1]\setminus N} \Gamma(t) \subseteq X$ is separable.*

Proof. Since Γ is Bochner measurable there is a set $N \subseteq [0,1]$ with $\mu(N) = 0$ and a sequence of simple multifunctions $\Gamma_n : [0,1] \to ck(X)$ such that for every $t \in [0,1] \setminus N$

$$\lim_{n\to\infty} H(\Gamma_n(t), \Gamma(t)) = 0.$$

Remark at the first step that for each simple multifunction $G : [0,1] \to ck(X)$, $\bigcup_{t\in[0,1]} G(t)$ is separable. Indeed, if $G(t) = \sum_{j=1}^{l} K_j \chi_{\Omega_j}$ where, for each j, K_j is compact of X and $\Omega_j \in \mathcal{L}$, then $\bigcup_{t\in[0,1]} G(t) = \bigcup_{j=1}^{l} K_j$. Hence it is separable (by the separability of compacts in a metrizable space).

At the next step, we point out that, for each n, $\bigcup_{t\in[0,1]}\Gamma_n(t)$ is separable and then there is a sequence $(x^n_m)_{m\in\mathbb{N}}\subseteq X$ dense in $\bigcup_{t\in[0,1]}\Gamma_n(t)$.
We claim that

$$\bigcup_{t\in[0,1]\setminus N}\Gamma(t)\subseteq\overline{D},\ \text{where } D=\{x^n_m:\quad n,m\in\mathbb{N}\}.$$

Indeed, let $x\in\bigcup_{t\in[0,1]\setminus N}\Gamma(t)$. Then there is $t\in[0,1]\setminus N$ so that $x\in\Gamma(t)$, and for each $i\in\mathbb{N}$, there is $n_i\in\mathbb{N}$ such that

$$H(\Gamma_{n_i}(t),\Gamma(t))<\frac{1}{i},$$

which implies that there is $x_{n_i}\in\Gamma_{n_i}(t)$ such that $d(x,x_{n_i})<\frac{1}{i}$. Corresponding to this x_{n_i} there is $\overline{x}\in D$, such that $d(\overline{x},x_{n_i})<\frac{1}{i}$. Thus $d(\overline{x},x)<\frac{2}{i}$ and so, the separability is proved. □

Remark 3.7. An important aspect when dealing with multifunctions is the existence of selections with appropriate properties (depending on the properties of the multifunction). In this direction, when speaking about measurability, the mostly used result is the classical Kuratowski-Ryll-Nardzewski Theorem that ensures the existence of measurable selections for measurable multifunctions in complete metrizable spaces that are separable. Without separability assumptions, in Banach spaces the existence of measurable selectors was obtained in [5] and [6]. In non-normable non-separable spaces, we cite the works [29], [38], [28] or [30].
The result that will be applied in the sequel is Theorem 2.9 in [30] which states that: *in a metric space any Bochner-measurable multifunction taking convex closed bounded values has Bochner-measurable selections.* In [30] the metric is supposed to be bounded but the result is available without this assumption (see [28, Remark 3.7]).
From now on S_Γ will denote the family of all Bochner-measurable selections of Γ.

4. Integrability for multifunctions

In what follows, we are going to consider the integrability matter. In view of the applications from now on we will consider only multifunctions taking values in $ck(X)$. We observe that propositions 4.2 and 4.3 hold true also in case of multifunctions taking values in $cb(X)$. Let us recall the following definitions:

Definition 4.1. Let $\Gamma:[0,1]\to ck(X)$ be a multifunction.

j) Γ is said to be *integrable* (see [48]) if it is totally measurable and there exists a sequence of simple multifunctions $\Gamma_n:[0,1]\to ck(X)$ that satisfies $\lim_{n\to\infty} H_i(\Gamma_n(t), \Gamma(t)) = 0$ for almost all $t \in [0,1]$ and such that for every $i \in \mathbb{N}$:

$$\lim_{n,m\to\infty} \int_0^1 H_i(\Gamma_n(t), \Gamma_m(t))dt = 0. \tag{4}$$

Then, for each measurable $E \subset [0,1]$, $(\int_E \Gamma_n(t)dt)_n$ is a Cauchy sequence ([48, p. 373]) in the Hausdorff metric. Therefore there is $x_E \in ck(X)$ such that for each $i \in \mathbb{N}$,

$$\lim_{n\to\infty} H_i\left(\int_E \Gamma_n(t)dt, x_E\right) = 0, \tag{5}$$

jj) Γ is said to be *integrable by seminorm* or *p-integrable* (see [48]), if it is measurable by seminorm and for every $i \in \mathbb{N}$ there exists a sequence of simple multifunctions $\Gamma_n^i : [0,1] \to ck(X)$ such that
jj$_a$) $H_i(\Gamma_n^i(t), \Gamma(t)) \in L^1([0,1])$,
jj$_b$) $H_i(\Gamma_n^i(t), \Gamma(t))$ converges to zero in μ-measure
jj$_c$) for each measurable $E \subset [0,1]$,

$$\lim_{n\to\infty} \int_E H_i(\Gamma_n^i(t), \Gamma(t))dt = 0,$$

and there is a unique $x_E \in ck(X)$ such that

$$\lim_{n\to\infty} H_i\left(\int_E \Gamma_n^i(t)dt, x_E\right) = 0.$$

jjj) Γ is said to be *scalarly integrable* if for every $x^* \in X^*$, the function $\sigma(x^*, \Gamma(\cdot))$ is Lebesgue-integrable.

jv) Γ is said to be *Pettis-integrable* in $ck(X)$ (see [9]) if Γ is scalarly integrable and for every $E \in \mathcal{L}$, there exists $x_E \in ck(X)$ such that

$$\sigma(x^*, x_E) = \int_E \sigma(x^*, \Gamma(t))dt, \quad \text{for all } \ x^* \in X^*.$$

In the case of Pettis integrability the set x_E is denoted by $(P)\int_E \Gamma(t)dt$ and is called the *Pettis integral over the measurable set* E, while in the case of integrability or p-integrability will be denoted by $\int_E \Gamma(t)dt$ and called the *integral of* Γ *over the measurable set* E.

If X is a Banach space the definitions of integrability and of p-integrability coincide. We will now prove that this is true, more generally, even in Fréchet spaces (extending, in this way, the result known in the single valued case, [2, Theorem 2.12]).

Proposition 4.2. *A multifunction* $\Gamma : [0,1] \to ck(X)$ *is integrable if and only if it is p-integrable. In such a case, the two integrals coincide.*

Proof. The "only if" part follows by the definition. We have to prove only the "if" part. By Theorem 3.5, there is a sequence of simple multifunctions $(\Gamma_n)_n$ such that for each $i \in \mathbb{N}$, $H_i(\Gamma_n(t), \Gamma(t))$ converges to zero. Since Γ is p-integrable, for each $i \in \mathbb{N}$, there is a simple multifunction S_i such that $H_i(S_i(t), \Gamma(t)) \in L^1([0,1])$, and for each $E \in \mathcal{L}$,

$$\int_E H_i(S_i(t), \Gamma(t))\ dt < \frac{1}{2^i}, \tag{6}$$

and moreover

$$H_i\left(\int_E S_i(t)dt, x_E\right) < \frac{1}{2^i},$$

where $x_E \in ck(X)$ is the p-integral of Γ over E. We claim that this sequence $(S_n)_n$ satisfies condition (4) of Definition 4.1 j). Fix k and $\varepsilon > 0$. There is an l such that $l > k$ and $\frac{1}{2^l} < \varepsilon$. So for each $n, m \geq l$ we have

$$\begin{aligned} H_k(S_n(t), S_m(t)) &\leq H_k(S_n(t), \Gamma(t)) + H_k(\Gamma(t), S_m(t)) \\ &\leq H_n(S_n(t), \Gamma(t)) + H_m(\Gamma(t), S_m(t)) \end{aligned} \tag{7}$$

which implies that $H_k(S_n(t), S_m(t)) \in L^1([0,1])$ and by (6) and (7)

$$\int_E H_k(S_n(t), S_m(t)) < \frac{1}{2^n} + \frac{1}{2^m} < 2\varepsilon. \tag{8}$$

So condition (4) is satisfied. We are proving now that the integrals coincide. So again fix k and $\varepsilon > 0$. By [48, Proposition 5] for each $n, m \in \mathbb{N}$ and for each $E \in \mathcal{L}$, we have,

$$H_k\left(\int_E S_n(t)dt, \int_E S_m(t)dt\right) \leq \int_E H_k(S_n(t), S_m(t))dt.$$

Let $l \in \mathbb{N}$ be such that $l > k$ and $\frac{1}{2^l} < \varepsilon$. For each $n, m \geq l$ by (6) and (7) we infer

$$\begin{aligned} H_k\left(\int_E S_n(t)dt, x_E\right) &\leq H_k\left(\int_E S_n(t)dt, \int_E S_m(t)dt\right) + H_k\left(x_E, \int_E S_m(t)dt\right) \\ &\leq \int_E H_k\left(S_n(t), S_m(t)\right)dt + H_m\left(x_E, \int_E S_m(t)dt\right) < 3\varepsilon. \end{aligned} \tag{9}$$

Therefore the integrals coincide. □

As for the relationship with the Pettis integrability, it is not difficult to see that if Γ is integrable, then it is also Pettis integrable.

Proposition 4.3. *Let $\Gamma : [0,1] \to ck(X)$ be integrable. Then it is also Pettis integrable in $ck(X)$ and the two integrals coincide.*

Proof. By [48, Proposition 1], for every seminorm p_i, the scalar function $H_i(\Gamma(\cdot),\{0\})$ is integrable. By (2) the scalar integrability of Γ follows. In order to obtain the Pettis integrability, it is enough to prove that the integral of Γ in the sense of Definition 4.1 j) satisfies the equality

$$\sigma\left(x^*,\int_E \Gamma(t)dt\right)=\int_E \sigma(x^*,\Gamma(t))dt,$$

for every $E\in\mathcal{L}$ and for every $x^*\in X^*$. Since Γ is Bochner measurable, there exists a sequence of simple multifunctions $\Gamma_n=\sum_{i=1}^{p_n}\chi_{A_i^n}C_i^n$ where $A_i^n\in\mathcal{L}$ and $C_i^n\in ck(X)$ like in Definition 4.1 j). Fix $x^*\in X^*$ and let i such that $x^*\in U_i^0$, where U_i^0 denotes the polar of the set U_i. Then taking in account that for each i, n, and for each $E\in\mathcal{L}$, $H_i(\int_E\Gamma(t),\int_E\Gamma_n(t))\le\int_E H_i(\Gamma(t),\Gamma_n(t))$ (see [48, Proposition 5]), by (2), (4) and (5) we get

$$\begin{aligned}
\sigma\left(x^*,\int_E \Gamma(t)dt\right) &= \lim_{n\to\infty}\sigma\left(x^*,\int_E \Gamma_n(t)dt\right)\\
&= \lim_{n\to\infty}\sigma\left(x^*,\sum_{i=1}^{p_n}\mu(A_i^n\cap E)C_i^n\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{p_n}\mu(A_i^n\cap E)\sigma\left(x^*,C_i^n\right)\\
&= \lim_{n\to\infty}\int_E\sigma\left(x^*,\Gamma_n(t)\right)dt\\
&= \int_E\sigma\left(x^*,\Gamma(t)\right)dt.
\end{aligned}$$

□

In applications to differential inclusions we will need other results, which are well known for Banach valued multifunctions (see [20]) and also for single-valued Pettis integrable functions in Fréchet spaces ([33]).

We recall that a map $M:\mathcal{L}\to ck(X)$ is called a *weak multimeasure* or simply a *multimeasure* if $\sigma(x^*,M(\cdot))$ is a measure, for every $x^*\in X^*$.

In order to prove next proposition we will use the following characterization of the $ck(X)$ sets (see [41, Section 7] for the general case and [20, Proposition 1.5] for the Banach valued case): *a nonempty set $C\in c(X)$ belongs to $ck(X)$ if and only if, for every 0-neighborhood U in X, the restriction of $\sigma(\cdot,C)$ to the polar U^o of U is weak*-continuous.*

Theorem 4.4. *Let $\Gamma:[0,1]\to ck(X)$ be Bochner-measurable. Then Γ is Pettis integrable in $ck(X)$ if and only if for each equicontinuous set $K\subset X^*$, the set $\{\sigma(x^*,\Gamma(\cdot)):x^*\in K\}$ is uniformly integrable.*

Proof. Assume first that the multifunction Γ is Pettis integrable in $ck(X)$. In order to prove that for each equicontinuous set $K\subset X^*$, the set $\{\sigma(x^*,\Gamma(t)):$

$x^* \in K\}$ is uniformly integrable, it is enough to show that for each i the set $\{\sigma(x^*, \Gamma(t)) : x^* \in U_i^o\}$ is uniformly integrable (see [35, p. 258]). So fix i. For each $A \in \mathcal{L}$, let

$$M(A) = (P)\int_A \Gamma(t)dt.$$

Then M is a weak multimeasure with convex compact values and by [26, Proposition 3] M is a normal multimeasure. Therefore M is σ-additive with respect to the uniformity of the Hausdorff distance, that is if $A = \cup_{i=1}^{\infty} A_i$, with $(A_i)_{i\in\mathbb{N}}$ pairwise disjoint, then

$$\lim_{n\to\infty} H\left(M(A), \sum_{j=1}^{n} M(A_j)\right) = 0,$$

where $M(A) + M(B) = M(A \cup B)$, if $A \cap B = \emptyset$. So by Lemma 3.3 also for each i

$$\lim_{n\to\infty} H_i\left(M(A), \sum_{j=1}^{n} M(A_j)\right) = 0.$$

By the Hörmander equality (2) we have

$$H_i\left(M(A), \sum_{j=1}^{n} M(A_j)\right) = \sup\left\{\left|\sigma(x^*, M(A)) - \sigma\left(x^*, \sum_{j=1}^{n} M(A_j)\right)\right| : x^* \in U_i^o\right\}.$$

Therefore

$$\lim_{n\to\infty}\left[\sup\left\{\left|\sigma(x^*, M(A)) - \sigma\left(x^*, \sum_{j=1}^{n} M(A_j)\right)\right| : x^* \in U_i^o\right\}\right] = 0.$$

Since

$$\begin{aligned}
&\lim_{n\to\infty}\left[\sup\left\{\left|\sigma(x^*, M(A)) - \sigma\left(x^*, \sum_{j=1}^{n} M(A_j)\right)\right| : x^* \in U_i^o\right\}\right] \\
&= \lim_{n\to\infty}\left[\sup\left\{\left|\sigma\left(x^*, M\left(\bigcup_{j=n+1}^{\infty} A_j\right)\right)\right| : x^* \in U_i^o\right\}\right] \qquad (10)\\
&= \lim_{n\to\infty}\left[\sup\left\{\left|\sigma\left(x^*, \int_{\bigcup_{j=n+1}^{\infty} A_j} \Gamma(t)dt\right)\right| : x^* \in U_i^o\right\}\right]
\end{aligned}$$

it follows that

$$\lim_{n\to\infty}\left[\sup\left\{\left|\sigma\left(x^*, \int_{\bigcup_{j=n+1}^{\infty} A_j} \Gamma(t)dt\right)\right| : x^* \in U_i^o\right\}\right] = 0.$$

This implies that the collection of scalar measures $\{\sigma(x^*, M(\cdot)) : x^* \in U_i^o\}$ is uniformly σ-additive. Moreover, for every $x^* \in X^*$, $\sigma(x^*, M(\cdot))$ is a real signed measure. Now let $M(\mathcal{L}) := \bigcup_{A\in\mathcal{L}} M(A)$ and let B^+ and $B^- \in \mathcal{L}$ the corresponding Hahn decomposition of $[0,1]$. Then we have

$$\begin{aligned}\sigma(x^*, M(\mathcal{L})) &= \sigma(x^*, \cup_{A\in\mathcal{L}} M(A)) = \sup_{A\in\mathcal{L}} \sigma(x^*, M(A))\\ &= \sigma(x^*, M(B^+)) =< x^*, \overline{x} >,\end{aligned}$$

where $\overline{x} \in M(B^+)$ and the last equality follows from the fact that $M(B^+) \in ck(X)$. Since $x^* \in X^*$ is arbitrary, by the James' Theorem we get that $\overline{M(\mathcal{L})}$ is weakly compact. Then $M(\mathcal{L})$ is bounded. Therefore there is a constant β_i such that, for each $x \in M(\mathcal{L})$, $p_i(x) \le \beta_i$. This yields that for each $x^* \in U_i^o$

$$|\sigma(x^*, M(\mathcal{L}))| = \sup\{|x^*(x)| : x \in M(\mathcal{L})\} \le \beta_i$$

and the family $\{\sigma(x^*, M(\cdot)) : x^* \in U_i^o\}$ is bounded. Moreover, for every $x^* \in U_i^o$, the measure $\sigma(x^*, M(\cdot))$ is absolutely continuous with respect to μ. It follows from [15, Corollary I.2.5] that the measures of the family are also uniformly absolutely continuous with respect to μ. Since $\sigma(x^*, M(A)) = \int_A \sigma(x^*, \Gamma(t))dt$ for every $A \in \mathcal{L}$, the uniform integrability of the set $\{\sigma(x^*, \Gamma(t)) : x^* \in U_i^o\}$ follows.

To prove the converse, we observe that in order to obtain the Pettis integrability of Γ in $ck(X)$, we have to show that there exists $C \in ck(X)$ such that $\int_0^1 \sigma(x^*, \Gamma(t))\, dt = \sigma(x^*, C)$ for all functionals x^*. We shall prove first that the sublinear function $a : X^* \to (-\infty, +\infty)$ given by $a(x^*) := \int_0^1 \sigma(x^*, \Gamma(t))\, dt$ is w^*-lower semi-continuous, i.e. that for each real α the set $Q(\alpha) := \{x^* \in X^* : a(x^*) \le \alpha\}$ is w^*-closed. By [51, Theorem 6.4] it suffices to show that, for each i, $Q(\alpha) \cap U_i^o$ is w^*-closed. Since Γ is Bochner measurable, by Theorem 3.6 there is a set $N \subseteq [0,1]$ with $\mu(N) = 0$ such that $\bigcup_{t\in[0,1]\setminus N} \Gamma(t) \subseteq X$ is separable. Thus restricting to the closed linear span of the set $\bigcup_{t\in[0,1]\setminus N} \Gamma(t)$, by [35, p. 259] it is enough to prove that $Q(\alpha) \cap U_i^o$ is sequentially w^*-closed. So let $x_n^* \in Q(\alpha) \cap U_i^o$ be such that $x_n^* \to x_0^*$ in $\sigma(X^*, X)$. Since $\Gamma(t) \in ck(X)$, applying the w^*-continuity of all $\sigma(\cdot, \Gamma(t))$, we get the pointwise convergence of $\sigma(x_n^*, \Gamma(t))$ to $\sigma(x_0^*, \Gamma(t))$. As by the Alaoglu-Bourbaki Theorem U_i^o is compact in the w^*-topology, then it is also equicontinuous. So by hypothesis the set $\{\sigma(x^*, \Gamma(t)) : x^* \in U_i^o\}$ is uniformly integrable. Therefore by the Vitali convergence Theorem we have

$$\begin{aligned}a(x_0^*) &= \int_0^1 \sigma(x_0^*, \Gamma(t))\, dt = \int_0^1 \lim_n \sigma(x_n^*, \Gamma(t))\, dt\\ &= \lim_n \int_0^1 \sigma(x_n^*, \Gamma(t))\, dt = \lim_n a(x_n^*) \le \alpha\end{aligned} \tag{11}$$

So, a is w^*-lower semi-continuous, and according to [31, Theorem 5], there exists a closed convex set $C \subset X$ such that $a(x^*) = \sigma(x^*, C)$. Moreover if $x_n^* \in U_i^o$ is such that $x_n^* \to x_0^*$ in $\sigma(X^*, X)$, then proceeding as in the previous part, we get that $a(x_0^*) = \lim_n a(x_n^*)$. Since each polar of a 0-neighborhood is metrizable in the $\sigma(X^*, X)$ topology, this means that the restriction of $\sigma(\cdot, C)$ to U_i^o is weak*-continuous. Then the compactness of C follows and Γ is Pettis integrable in $ck(X)$. □

Remark 4.5. We note that, in the proof of the previous characterization of Pettis integrability, the Bochner measurability is not needed in order to prove the uniform integrability of the set $\{\sigma(x^*, \Gamma(\cdot)) : x^* \in K\}$ when the multifunction $\Gamma(\cdot)$ is Pettis integrable.

Without the Bochner measurability hypotheses, we can prove the following characterization:

Proposition 4.6. *Let $\Gamma : [0, 1] \to ck(X)$ be scalarly measurable. Then the following are equivalent:*

i) *Γ is Pettis integrable in $ck(X)$;*
ii) *for each $A \in \mathcal{L}$ and for each 0-neighborhood U_i the map*

$$\phi_A^\Gamma : X^* \to (-\infty, +\infty), \qquad x^* \to \int_A \sigma(x^*, \Gamma(\cdot))dt$$

restricted to the polar U_i^o of U_i is weak-continuous.*

Proof. Assume first that Γ is Pettis integrable in $ck(X)$. Then, for each $A \in \mathcal{L}$, there exists $C_A \in ck(X)$ such that for each $x^* \in X^*$

$$\sigma(x^*, C_A) = \int_A \sigma(x^*, \Gamma(\cdot))dt.$$

Since $C_A \in ck(X)$, for every 0-neighborhood U_i in X the restriction of $\sigma(\cdot, C_A)$ to the polar U_i^o of U_i is weak*-continuous. By previous equality the thesis follows. Conversely assume that $ii)$ holds. We have to show that for each $A \in \mathcal{L}$, there exists $C_A \in ck(X)$ such that $\sigma(x^*, C_A) = \int_A \sigma(x^*, \Gamma(\cdot))dt$ for each $x^* \in X^*$. Fix $A \in \mathcal{L}$ and U_i in X. At first we want to prove that the sublinear map $\phi_A^\Gamma : X^* \to (-\infty, +\infty)$ is w^*-lower semi-continuous. To do this we have to show that, for each real α, the set $Q(\alpha) := \{x^* \in X^* : \phi_A^\Gamma(x^*) \leq \alpha\}$ is w^*-closed. By [51, Theorem 6.4] it suffices to show that, for each i, $Q(\alpha) \cap U_i^o$ is w^*-closed and this follows by the hypothesis that the map ϕ_A^Γ restricted to U_i^o is weak*-continuous. Therefore, ϕ_A^Γ is w^*-lower semi-continuous, and according to [31, Theorem 5], there exists a closed convex set $C_A \subset X$ such that $\phi_A^\Gamma(x^*) = \sigma(x^*, C_A)$. Since the restriction of $\sigma(\cdot, C_A)$ to U_i^o is weak*-continuous, the compactness of C_A follows and Γ is Pettis integrable in $ck(X)$. □

Corollary 4.7. *Let $\Gamma : [0,1] \to ck(X)$ be Pettis integrable in $ck(X)$ and $G : [0,1] \to ck(X)$ be scalarly measurable and satisfying the condition*

$$G(t) \subset \Gamma(t),\ \forall t \in [0,1].$$

Then the multifunction G is Pettis integrable in $ck(X)$.

Proof. Since we have

$$-\sigma(-x^*, \Gamma(t)) \leq \sigma(x^*, G(t)) \leq \sigma(x^*, \Gamma(t)),$$

the integrability of $\sigma(x^*, G)$ follows. Now fix $A \in \mathcal{L}$. The map φ_A^G is subadditive and satisfies $\varphi_A^G(x^*) \leq \varphi_A^\Gamma(x^*)$, for all $x^* \in X^*$. Hence

$$-\varphi_A^\Gamma(y^* - x^*) \leq -\varphi_A^G(y^* - x^*) \leq \varphi_A^G(x^*) - \varphi_A^G(y^*) \leq \varphi_A^G(x^* - y^*) \leq \varphi_A^\Gamma(x^* - y^*).$$

Since Γ is Pettis integrable, by Proposition 4.6 the map φ_A^Γ is weak*-continuous so, it is weak*-continuous at the origin and then the previous inequality implies that also the map φ_A^G is weak*-continuous. Applying again Proposition 4.6 we get the Pettis integrability in $ck(X)$ of G. □

By previous result we get at once

Corollary 4.8. *Let $\Gamma : [0,1] \to ck(X)$ be Pettis integrable in $ck(X)$. Then every scalarly measurable selection of Γ is Pettis integrable.*

A property similar to Corollary 4.7 can be proved for integrability in a stronger sense.

Proposition 4.9. *Let $\Gamma : [0,1] \to ck(X)$ be integrable and $G : [0,1] \to ck(X)$ be Bochner measurable and satisfying the condition*

$$G(t) \subset \Gamma(t),\ \forall t \in [0,1].$$

Then the multifunction G is integrable.

Proof. By Theorem 3.5 it follows that Γ is measurable by seminorm. Since for each $i \in \mathbb{N}$

$$0 \leq H_i(G(\cdot), \{0\}) \leq H_i(\Gamma(\cdot), \{0\}),$$

and since from [48, Proposition 1] the function $H_i(\Gamma(\cdot), \{0\})$ is integrable, by [48, Theorem 3], the integrability of the multifunction G follows. □

When studying multivalued differential problems, a special attention must be paid to integrable selections of the multifunction on the right hand side. The results proved below are therefore important in this framework.

Proposition 4.10. *Let $\Gamma : [0,1] \to ck(X)$ be Bochner-measurable and Pettis integrable in $ck(X)$. Then for each $E \in \mathcal{L}$ the set*

$$I_E(\Gamma) := \left\{ (P)\int_E f(t)dt : f \text{ Pettis }-\text{integrable selection of } \Gamma \right\}$$

is closed.

Proof. By Corollary 4.8 for any measurable E, the set $I_E(\Gamma)$ is nonempty. Since Γ is Bochner-measurable, by Theorem 3.6 there is a set $N \subseteq [0,1]$ with $\mu(N) = 0$ such that $\bigcup_{t\in[0,1]\setminus N} \Gamma(t) \subseteq X$ is separable, and being Γ Pettis integrable, we can assume that $\bigcup_{t\in[0,1]} \Gamma(t) \subseteq X$ is separable. Thus restricting to the closed linear span of the set $\bigcup_{t\in[0,1]} \Gamma(t)$, by [35, p. 259] the proof of the closeness of the set $I_E(\Gamma)$ follows as in [20, Proposition 5.2] with suitable changes. □

Theorem 4.11. *Let $\Gamma : [0,1] \to ck(X)$ be Bochner-measurable and Pettis integrable in $ck(X)$. Then for each $E \in \mathcal{L}$*

$$\int_E \Gamma(t)dt = \left\{ (P)\int_E f(t)dt : f \text{ Pettis }-\text{integrable selection of } \Gamma. \right\}.$$

Proof. By Lemma 4.10 for any measurable E, the set

$$I_E(\Gamma) := \left\{ (P)\int_E f(t)dt : f \text{ Pettis -integrable selection of } \Gamma. \right\}$$

is closed. Now we want to prove that for each $x^* \in X^*$ and for each $E \in \mathcal{L}$,

$$\sigma(x^*, I_E(\Gamma)) = \int_E \sigma(x^*, \Gamma(t))dt.$$

Let $x^* \in X^*$ and $f \in S_\Gamma$. Then $\langle x^*, f(t)\rangle \leq \sigma(x^*, \Gamma(t))$ for every $t \in [0,1]$, therefore

$$\int_E \langle x^*, f(t)\rangle dt \leq \int_E \sigma(x^*, \Gamma(t))dt.$$

This gives that

$$\sigma(x^*, I_E(\Gamma)) \leq \int_E \sigma(x^*, \Gamma(t))dt. \tag{12}$$

To prove the reverse inequality, let us fix x^* and consider the multifunction $G : [0,1] \to ck(X)$ defined by

$$G(t) := \{x \in \Gamma(t) : \langle x^*, x\rangle = \sigma(x^*, \Gamma(t))\}.$$

Since Γ has compact convex values, for each $t \in [0,1]$, $G(t)$ is nonempty. By [52, Lemma 3], we infer that G is scalarly measurable. For each $t \in [0,1]$, $G(t) \subset \mathit{\Gamma}(t)$, then by Corollary 4.7 G is Pettis integrable. Let g be a Pettis integrable selection of G and then also of Γ. Clearly, for all $t \in [0,1]$ it satisfies the equality

$$\langle x^*, g(t)\rangle = \sigma(x^*, \Gamma(t)).$$

Therefore

$$\int_E \langle x^*, g(t)\rangle dt = \int_E \sigma(x^*, \Gamma(t))dt,$$

whence

$$\sigma(x^*, I_E(\Gamma)) \geq \int_E \langle x^*, g(t)\rangle dt = \int_E \sigma(x^*, \Gamma(t))dt. \tag{13}$$

By (12) and (13) the assertion follows. □

5. Volterra integral inclusions in Fréchet spaces

In this section we apply previously obtained results to provide existence results for the integral problem (1).

The first one deals with the Pettis integrability notion, therefore it seems useful to recall some basic properties of the primitive of a Pettis integrable function: $t \in [0,1] \mapsto (\mathrm{P}) \int_0^t f(s)ds$. It is, by definition, weakly continuous. But thanks to the characterization given in [33] (which was generalized by us to the set-valued case in Theorem 4.4), it is in fact continuous. Moreover, it is pseudo-differentiable in the sense described below:

Definition 5.1. A function $F : [0,1] \to X$ is said to be *pseudo-differentiable* with a pseudo-derivative f if, for every $x^* \in X^*$, there exists $N(x^*) \subset [0,1]$ of null measure such that $\langle x^*, F\rangle$ is differentiable on $[0,1] \setminus N(x^*)$ and its derivative is $\langle x^*, F(t)\rangle' = \langle x^*, f(t)\rangle$, for every $t \in [0,1] \setminus N(x^*)$.

Note that the weak differentiability of the Pettis primitive (namely, the existence of a null-measure set N independent of x^* except which the property in the preceding definition holds) is not valid even in separable Banach spaces (as proved e.g. in [16]).

We proceed to give the first existence result. Our proof uses a technique similar to that applied for a particular case in [9, Theorem VI-7], where X is separable. It is based on Kakutani-Ky Fan Theorem ([45, Theorem 6.5.19]). We do not impose the separability to the Fréchet space X, but applying Theorem 3.6, we get by this lack by considering Bochner-measurable multifunctions in a general Fréchet space.

Theorem 5.2. *Let $\Gamma : [0,1] \to ck(X)$ be Bochner-measurable and Pettis-integrable in $ck(X)$ and $F : [0,1] \times X \to ck(X)$ satisfy the following assumptions:*
1) for every $x \in C([0,1], X)$, the multifunction $t \mapsto F(t, x(t))$ is Bochner-measurable;

2) for every $t \in [0,1]$, $x \mapsto F(t,x)$ is upper semi-continuous;
3) for every $t \in [0,1]$ and $x \in X$, $F(t,x) \subset \Gamma(t)$.
Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be such that for each $t \in [0,1]$, the function $k(t,\cdot) \in L^\infty([0,1])$ and $t \mapsto k(t,\cdot)$ is L^∞-continuous. Then the integral problem (1) has solutions $x \in C([0,1],X)$.

Proof. Consider

$$\mathcal{X} = \left\{ z : [0,1] \to X : \ z(t) = x_0 + (\mathrm{P})\int_0^t k(t,s)f(s)ds, \ \forall t \in [0,1], \ f \in S_\Gamma \right\}$$

where we recall that S_Γ is the family of all Bochner-measurable selections and, (by Corollary 4.8, of all Pettis-integrable selections) of Γ. By Theorem 2.9 in [30], S_Γ is non-empty and, as $k(t,\cdot) \in L^\infty([0,1])$, it follows that for each $f \in S_\Gamma$, $k(t,\cdot)f(\cdot)$ is Pettis-integrable ([33, Theorem 2.3]) and so, $\mathcal{X}$ is non-empty as well. We obtain by the particular single-valued case of Theorem 4.4 (see [33]) that $\mathcal{X} \subset C([0,1],X)$ since for every seminorm p_i,

$$\begin{aligned}
&\sup_{x^* \in U_i^0} \langle x^*, z(t) - z(t_0) \rangle \\
&\le \sup_{x^* \in U_i^0} \int_0^t |\langle x^*, (k(t,s) - k(t_0,s))f(s) \rangle| \, ds + \sup_{x^* \in U_i^0} \int_{t_0}^t |\langle x^*, k(t_0,s)f(s) \rangle| \, ds \\
&\le \|k(t,\cdot) - k(t_0,\cdot)\|_\infty \sup_{x^* \in U_i^0} \int_0^t |\langle x^*, f(s) \rangle| \, ds + \sup_{t \in [0,1]} \|k(t,\cdot)\|_\infty \sup_{x^* \in U_i^0} \int_{t_0}^t |\langle x^*, f(s) \rangle| \, ds.
\end{aligned}$$

We intend to prove now that $\mathcal{X}$ is compact in the topology of uniform convergence. By Ascoli's Theorem ([34, p. 233]), it is enough to prove that:
i) $\mathcal{X}$ is equi-continuous;
ii) for every $t \in [0,1]$, the subset $\{z(t) : \ z \in \mathcal{X}\} \subset X$ is relatively compact;
iii) $\mathcal{X}$ is closed.
In order to prove the condition i), see that by Theorem 4.4 for every seminorm p_i and every $\varepsilon > 0$ there exists $\delta_{\varepsilon,p_i} > 0$ such that for all $t_1 < t_2 \in [0,1]$ with $t_2 - t_1 < \delta_{\varepsilon,p_i}$, $\|k(t_1,\cdot) - k(t_2,\cdot)\|_\infty \le \varepsilon$ and

$$\int_{t_1}^{t_2} |\sigma(x^*, \Gamma(s))| ds < \varepsilon, \quad \forall \ x^* \in U_i^0.$$

It follows that for all $f \in S_\Gamma$,

$$\int_{t_1}^{t_2} |\langle x^*, f(s) \rangle| \, ds \le \max \left(\int_{t_1}^{t_2} |\sigma(x^*, \Gamma(s))| ds, \int_{t_1}^{t_2} |\sigma(-x^*, \Gamma(s))| ds \right) < \varepsilon$$

whenever $|t_1 - t_2| < \delta_{\varepsilon,p_i}$ for all $x^* \in U_i^0$. So, for every $z \in \mathcal{X}$ and every such $t_1 < t_2 \in [0,1]$,

$$\begin{aligned}
&\sup_{x^* \in U_i^0} \langle x^*, z(t_2) - z(t_1) \rangle \\
&\le \sup_{x^* \in U_i^0} \int_0^{t_1} |\langle x^*, (k(t_2,s) - k(t_1,s)) f(s) \rangle| \, ds + \sup_{x^* \in U_i^0} \int_{t_1}^{t_2} |\langle x^*, k(t_2,s) f(s) \rangle| \, ds \\
&\le \|k(t_1,\cdot) - k(t_2,\cdot)\|_\infty \sup_{x^* \in U_i^0} \int_0^1 |\langle x^*, f(s) \rangle| \, ds + \sup_{t \in [0,1]} \|k(t,\cdot)\|_\infty \sup_{x^* \in U_i^0} \int_{t_1}^{t_2} |\langle x^*, f(s) \rangle| \, ds \\
&\le \varepsilon \sup_{x^* \in U_i^0} \max \left(\int_0^1 |\sigma(x^*, \Gamma(s))| ds, \int_0^1 |\sigma(-x^*, \Gamma(s))| ds \right) + \varepsilon \sup_{t \in [0,1]} \|k(t,\cdot)\|_\infty
\end{aligned}$$

and so, the equicontinuity is satisfied.
The condition ii) immediately comes from the fact that by Theorem 4.11 $\{z(t) : z \in \mathcal{X}\} \subset x_0 + (\mathrm{P}) \int_0^t k(t,s)\Gamma(s) ds$ and by Theorem 4.4 which yields the Pettis-integrability of $k(t,\cdot)\Gamma(\cdot)$.
As for the condition iii), take a sequence $(z_n)_n \subset \mathcal{X}$ convergent to $z \in C([0,1], X)$ and prove that $z \in \mathcal{X}$. For each z_n there exists $f_n \in S_\Gamma$ such that $z_n(t) = x_0 + (\mathrm{P}) \int_0^t k(t,s) f_n(s) ds$.
By Theorem 3.6, we can apply [9, Theorem V-13] (see also the remark at p. 147) in order to get the compactness of S_Γ with respect to the topology induced by the tensor product $L^\infty([0,1]) \otimes X^*$. This yields the compactness (and metrizability) of S_Γ for the topology of convergence on the space of measurable simple functions taking a finite number of values in X^* (described in [9, p. 176]). So, we can extract a subsequence (not re-labelled) such that for each $t \in [0,1]$, $(\mathrm{P}) \int_0^t k(t,s) f_n(s) ds$ weakly converges to $(\mathrm{P}) \int_0^t k(t,s) f(s) ds$. Now from the uniform convergence of z_n towards z it follows that for all $t \in [0,1]$, $z(t) = x_0 + (\mathrm{P}) \int_0^t k(t,s) f(s) ds$ and so, $z \in \mathcal{X}$.
Consider now the multivalued operator $\Phi : \mathcal{X} \to c(\mathcal{X})$ defined by

$$\Phi(x) = \left\{ y \in \mathcal{X} : \ y(t) = x_0 + (\mathrm{P}) \int_0^t k(t,s) f(s) ds, f \in S_{F(\cdot, x(\cdot))} \right\}.$$

By Corollary 4.7 and [30, Theorem 2.9], its values are non-empty. As in [9, Theorem VI-7] it follows that the values of Φ are closed (therefore, compact) and that Φ is upper semi-continuous. By Kakutani-Ky Fan's Theorem, the operator Φ has a fixed point, which is a solution of the integral inclusion (1). □

Taking into account the notion of pseudo-derivative, we get also:

Corollary 5.3. *Let $\Gamma : [0,1] \to ck(X)$ be Pettis-integrable and Bochner measurable and $F : [0,1] \times X \to ck(X)$ satisfy hypotheses 1)-3) of Theorem 5.2. Then* *the differential problem*

$$x'(t) \in F(t, x(t)),\ x(0) = x_0 \tag{14}$$

has at least one pseudo-differentiable solution $x \in C([0,1], X)$.

The existence of solutions can be also gathered if we impose the integrability of the multifunction Γ. In this case, the solutions are continuous and a.e. differentiable in the usual meaning since, by [44, Theorem 3.2], in Fréchet spaces, the class of indefinite Bochner-integrals coincides with the class of absolutely continuous and a.e. differentiable mappings.

Corollary 5.4. *Let* $\Gamma : [0,1] \to ck(X)$ *be integrable and* $F : [0,1] \times X \to ck(X)$ *satisfy hypotheses 1)-3) of Theorem 5.2. Then the differential problem (14) has at least one differentiable solution* $x \in C([0,1], X)$.

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