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A Laplace type problem for three lattices with non-convex cell

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Abstract

In this paper we consider three lattices with cells represented in Fig. 1, 3 and 5 and we determine the probability that a random segment of constant length intersects a side of lattice.

Keywords: Geometric probability, stochastic geometry, random sets, random convex sets and integral geometry.
2010 MSC: 60D05, 52A22.

1. Cell without obstacles.

Let $\Re_1(a)$ be the lattice with the fundamental cell $C_0^{(1)}$ rappresented in Fig. 1.

By Fig. 1 we obtain that

$$areaC_0^{(1)} = 5a^2. \tag{1.1}$$

We want to compute the probability that a random segment s of constant length l intersects a side of lattice, i.e. the probability P_{int} that the segment s intersects a side of fundamental cell $C_0^{(1)}$.

The position of segment s is determined by the center and by the angle φ that it formed with the line BC.

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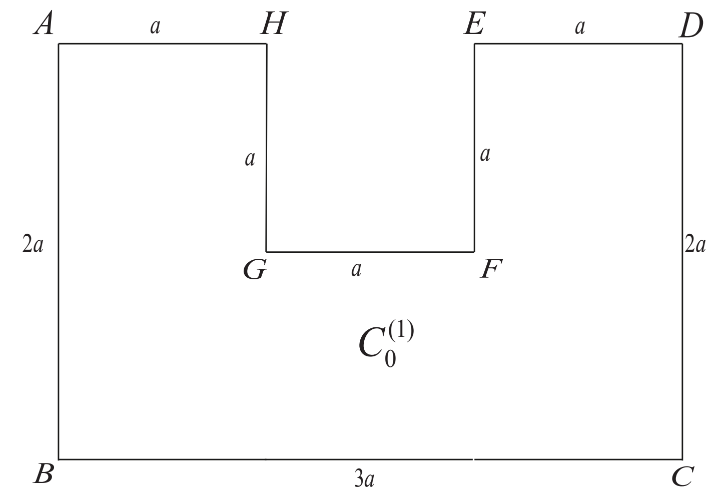

Fig. 1

To compute the probability P_{int} we consider the limiting positions, for a fixed value of φ, of segment s. We obtain the Fig.2

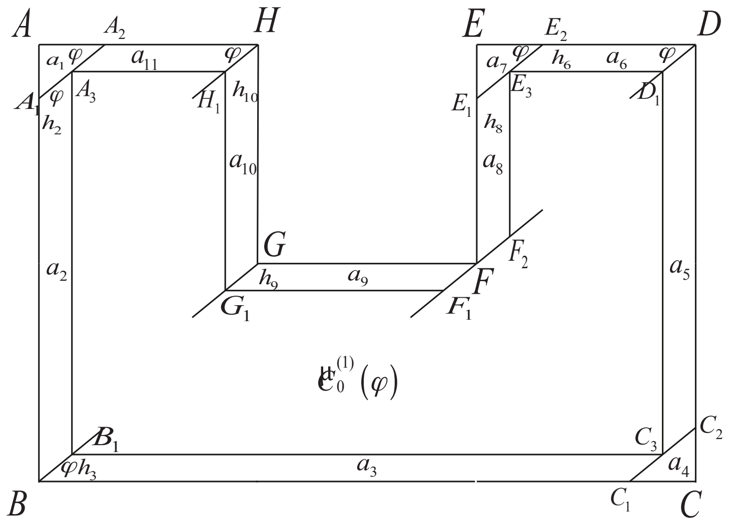

Fig. 2

and the formula

$$\widehat{C}_0^{(1)}(\varphi) = C_0^{(1)} - \sum_{i=1}^{11} areaa_i(\varphi). \tag{1.2}$$

To compute $\widehat{C}_0^{(1)}(\varphi)$ we have that

$$\begin{aligned} areaa_1(\varphi) &= areaa_4(\varphi) = areaa_7(\varphi) = \frac{l^2}{4}\sin 2\varphi, \\ areaa_2(\varphi) &= areaa_5(\varphi) = al\cos\varphi - \frac{l^2}{4}\sin 2\varphi, \\ areaa_3(\varphi) &= \frac{3al}{2}\sin\varphi - \frac{l^2}{4}\sin 2\varphi, \\ areaa_6(\varphi) &= areaa_{11}(\varphi) = \frac{al}{2}\sin\varphi - \frac{l^2}{4}\sin 2\varphi, \end{aligned}$$

$$\begin{aligned}
areaa_9(\varphi) &= \frac{al}{2}\sin\varphi, \\
areaa_8(\varphi) &= \frac{al}{2}\cos\varphi - \frac{l^2}{4}\sin\varphi, \\
areaa_{10}(\varphi) &= \frac{al}{2}\cos\varphi.
\end{aligned}$$

We can write that

$$A_1(\varphi) = \sum_{i=1}^{11} areaa_i(\varphi) = 3al(\sin\varphi + \cos\varphi) - \frac{3l^2}{4}\sin 2\varphi. \tag{1.3}$$

Replacing this formula in relation (1.2) we obtain

$$area\widehat{C}_0^{(1)}(\varphi) = areaC_0^{(1)} - A_1(\varphi). \tag{1.4}$$

Denoting with M_1, the set of all segments s that they have center in the cell $C_0^{(1)}$, and with N_1 the set of all segments s entirely conteined in the cell C_o, we have [2]:

$$P_{int} = 1 - \frac{\mu(N_1)}{\mu(M_1)}, \tag{1.5}$$

where μ is the Lebesgue measure in the euclidean plane.

To compute the measure $\mu\ (M_1)$ and $\mu\ (N_1)$ we use the kinematic measure of Poincarè [1]:

$$dk = dx \wedge dy \wedge d\varphi,$$

where x, y are the coordinate of center of s and φ the fixed angle.

For $\varphi \in \left[0, \frac{\pi}{2}\right]$ we have

$$\mu(M_1) = \int_0^{\frac{\pi}{2}} d\varphi \int\int_{\left\{(x,y)\epsilon C_0^{(1)}\right\}} dxdy = \int_0^{\frac{\pi}{2}} \left(areaC_0^{(1)}\right) d\varphi = \frac{\pi}{2} areaC_0^{(1)}. \tag{1.6}$$

Then, considering formula (1.4) we can write

$$\begin{aligned}
\mu(N_1) &= \int_0^{\frac{\pi}{2}} d\varphi \int\int_{\left\{(x,y)\epsilon \widehat{C}_0^{(1)}(\varphi)\right\}} \\
&= \int_0^{\frac{\pi}{2}} \left[area\widehat{C}_0^{(1)}(\varphi)\right] d\varphi \\
&= \int_0^{\frac{\pi}{2}} \left[areaC_0^{(1)} - A_1(\varphi)\right] d\varphi \\
&= \frac{\pi}{2} areaC_0^{(1)} - \int_0^{\frac{\pi}{2}} \left[A_1(\varphi)\right] d\varphi.
\end{aligned} \tag{1.7}$$

The formulas (1.5), (1.6) and (1.7) give us

$$P_{int}^{(1)} = \frac{2}{\pi areaC_0^{(1)}} \int_0^{\frac{\pi}{2}} \left[A_1(\varphi)\right] d\varphi. \tag{1.8}$$

By (1.3), we have that

$$\int_0^{\frac{\pi}{2}} \left[A_1(\varphi)\right] d\varphi = 6al - \frac{3l^2}{4}. \tag{1.9}$$

Replacing in (1.8) the relations (1.1) and (1.9) we obtain that

$$\widetilde{p}^{(1)} = \frac{12}{5\pi}\frac{l}{a} - \frac{3}{10\pi}\left(\frac{l}{a}\right)^2. \tag{1.10}$$

2. Cell with obstacles triangular

Let $\Re_2(a,m)$ be the lattice with the fundamental cell $C_0^{(2)}$ rappresented in Fig. 3

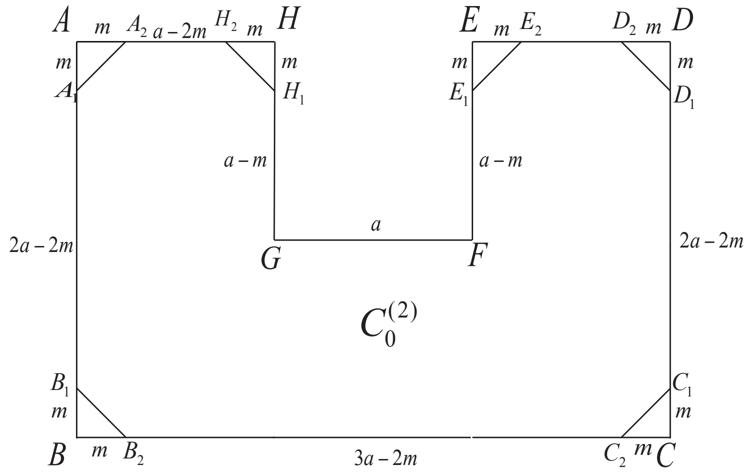

Fig. 3

where $m < \frac{a}{2}$. By Fig. 3 we have that

$$areaC_0^{(2)}(\varphi) = 5a^2 - 3m^2. \tag{2.1}$$

We want to compute the probability $P_{int}^{(2)}$that a random segment s of constant length l intersects a side of cell $C_0^{(2)}$.

As in the paragraph 1, we have Fig .4

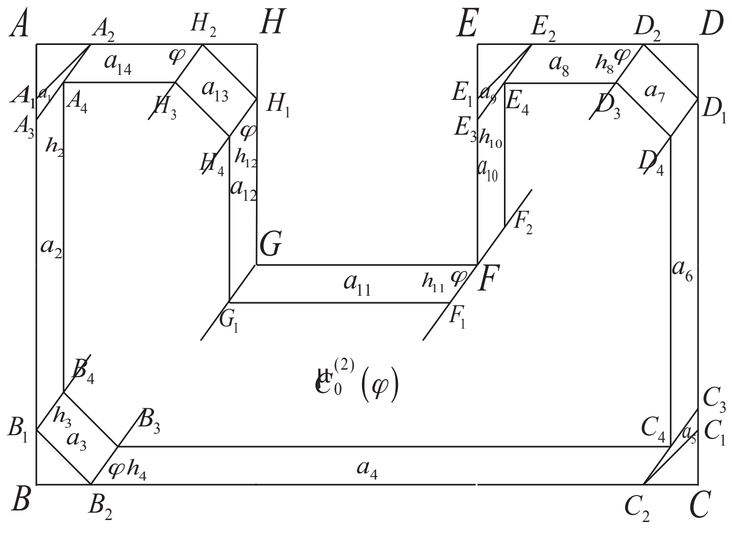

Fig. 4

and

$$area\widehat{C}_0^{(2)}(\varphi) = areaC_0^{(2)} - \sum_{i=1}^{14} areaa_i(\varphi). \tag{2.2}$$

We have that

$$areaa_1(\varphi) \quad = \quad areaa_5(\varphi) = areaa_9(\varphi) = \frac{lm}{2}(\sin\varphi - \cos\varphi),$$

$$\begin{aligned}
areaa_2(\varphi) &= areaa_6(\varphi) = al\cos\varphi - \frac{lm}{2}\cos\varphi - \frac{l^2}{4}\sin 2\varphi,\\
areaa_3(\varphi) &= areaa_7(\varphi) = areaa_{13}(\varphi) = \frac{lm}{2}(\sin\varphi + \cos\varphi),\\
areaa_4(\varphi) &= \frac{3al}{2}\sin\varphi - lm\sin\varphi,\\
areaa_{11}(\varphi) &= \frac{al}{2}\sin\varphi,\\
areaa_{10}(\varphi) &= \frac{al}{2}\cos\varphi - \frac{l^2}{4}\sin 2\varphi,\\
areaa_{12}(\varphi) &= \frac{al}{2}\cos\varphi - \frac{lm}{2}\cos\varphi,\\
areaa_8(\varphi) &= areaa_{14}(\varphi) = \frac{al}{2}\sin\varphi - lm\sin\varphi.
\end{aligned}$$

We can write,

$$A_2(\varphi) = \sum_{i=1}^{14} areaa_i(\varphi) = 3al(\sin\varphi + \cos\varphi) - \frac{3l^2}{4}\sin 2\varphi - \frac{3lm}{2}\cos\varphi. \tag{2.3}$$

As in the paragraph 1, we can write,

$$P_{int}^{(2)} = 1 - \frac{\mu(N_2)}{\mu(M_2)}. \tag{2.4}$$

We have that:

$$\mu(M_2) = \int_0^{\frac{\pi}{2}} dy \int\int_{\left\{(x,y)\epsilon C_0^{(2)}\right\}} dxdy = \int_0^{\frac{\pi}{2}} \left(areaC_0^{(2)}\right) d\varphi = \frac{\pi}{2} areaC_0^{(2)} \tag{2.5}$$

and, considering (2.2) and (2.3),

$$\begin{aligned}
\mu(N_2) &= \int_0^{\frac{\pi}{2}} d\varphi \int\int_{\left\{(x,y)\epsilon \widehat{C}_0^{(2)}\right\}} dxdy = \int_0^{\frac{\pi}{2}} \left(areaC_0^{(2)}\right) d\varphi\\
&= \int_0^{\frac{\pi}{2}} \left[areaC_0^{(2)} - A_2(\varphi)\right] d\varphi = \frac{\pi}{2} areaC_0^{(2)} - \int_0^{\frac{\pi}{2}} \left[A_2(\varphi)\right] d\varphi.
\end{aligned} \tag{2.6}$$

By (2.3) we have that

$$\int_0^{\frac{\pi}{2}} \left[A_2(\varphi)\right] d\varphi = 6al - \frac{3l^2}{4} - 2lm. \tag{2.7}$$

The relations (2.1), (2.4), (2.5), (2.6) and (2.7) give us

$$P_{int}^{(2)} = \frac{2l}{\pi(5a^2 - m^2)}\left(6a - \frac{3l}{4} - 2m\right).$$

For $m = 0$ this probability become

$$\widetilde{p}^{(2)} = \frac{12}{5\pi}\frac{l}{a} - \frac{3}{10\pi}\left(\frac{l}{a}\right)^2. \tag{2.8}$$

3. Cell with obstacles circular sectors

Let $\Re_3(a, m)$ be the lattice with the fundamental cell $C_0^{(3)}$ rappresented in Fig. 5

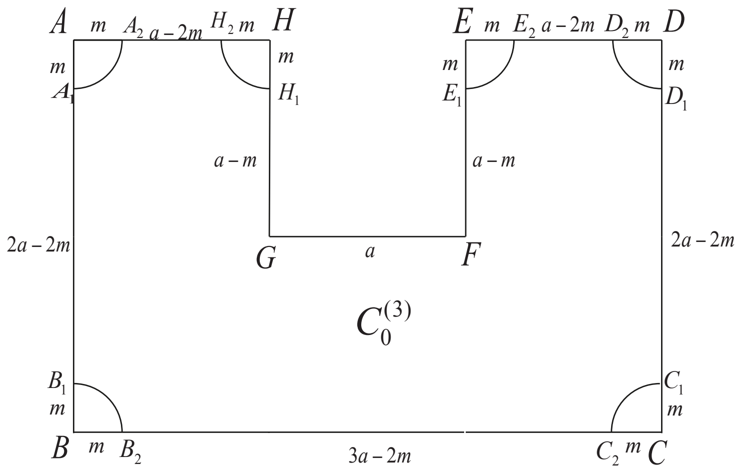

Fig. 5

where $m < \frac{a}{2}$. By Fig. 5 we have that

$$areaC_0^{(3)} = 5a^2 - \frac{3\pi m^2}{2}. \tag{3.1}$$

We want to compute the probability $P_{int}^{(3)}$ that a random segment s of constant length l intersects a side of cell $C_{0.}^{(3)}$.

As in the paragraph 1, we have the Fig. 6,

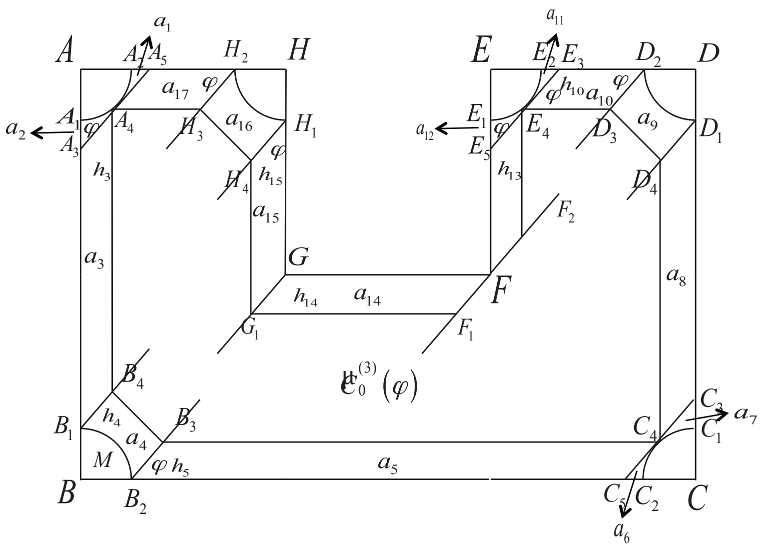

Fig. 6

and the formula

$$area\widehat{C}_0^{(3)}(\varphi) = areaC_0^{(3)} - \sum_{i=1}^{17} areaa_i(\varphi). \tag{3.2}$$

We have that

$$\begin{aligned} areaa_1(\varphi) + areaa_2(\varphi) &= areaa_6(\varphi) + areaa_7(\varphi) \\ &= areaa_{11}(\varphi) + areaa_{12}(\varphi) \\ &= \frac{l^2}{4}\sin 2\varphi - \frac{\pi m^2}{4}, \end{aligned}$$

$$\begin{aligned}
areaa_3(\varphi) &= areaa_8(\varphi) = al\cos\varphi - \frac{lm}{2}\cos\varphi - \frac{l^2}{4}\sin 2\varphi,\\
areaa_4(\varphi) &= areaa_9(\varphi) = areaa_{16}(\varphi) = \frac{lm}{2}(\sin\varphi + \cos\varphi) - \frac{m^2}{4}(\pi - 2),\\
areaa_5(\varphi) &= \frac{3al}{2}\sin\varphi - \frac{lm}{2}\sin\varphi - \frac{l^2}{4}\sin 2\varphi,\\
areaa_{13}(\varphi) &= \frac{al}{2}\cos\varphi - \frac{l^2}{4}\sin 2\varphi,\\
areaa_{15}(\varphi) &= \frac{al}{2}\cos\varphi - \frac{lm}{2}\cos\varphi,\\
areaa_{14}(\varphi) &= \frac{al}{2}\sin\varphi,\\
areaa_{10}(\varphi) + areaa_{17}(\varphi) &= \frac{al}{2}\sin\varphi - \frac{lm}{2}\sin\varphi - \frac{l^2}{4}\sin 2\varphi.
\end{aligned}$$

We can write that

$$A_3(\varphi) = \sum_{i=1}^{17} areaa_i(\varphi) = 3al(\sin\varphi + \cos\varphi) - \frac{3l^2}{4}\sin 2\varphi - \frac{3(\pi-1)m^2}{2}. \tag{3.3}$$

Replacing this relation in (3.2) we have that

$$area\widehat{C}_0^{(3)}(\varphi) = areaC_0^{(3)} - A_3(\varphi). \tag{3.4}$$

As in the paragraph 1, we can write that

$$P_{int} = 1 - \frac{\mu(N_3)}{\mu(M_3)}. \tag{3.5}$$

For $\varphi \in \left[0, \frac{\pi}{2}\right]$ we have that

$$\mu(M_3) = \int_0^{\frac{\pi}{2}} d\varphi \int\int_{\left\{(x,y)\epsilon C_0^{(3)}\right\}} dxdy = \int_0^{\frac{\pi}{2}} \left(areaC_0^{(3)}\right) d\varphi = \frac{\pi}{2} areaC_0^{(3)}$$

and considering (3.4),

$$\begin{aligned}
\mu(N_3) &= \int_0^{\frac{\pi}{2}} d\varphi \int\int_{\left\{(x,y)\epsilon \widehat{C}_0^{(3)}(\varphi)\right\}} dxdy = \int_0^{\frac{\pi}{2}} \left(areaC_0^{(3)}\right) d\varphi\\
&= \int_0^{\frac{\pi}{2}} \left[areaC_0^{(3)} - A_3(\varphi)\right] d\varphi = \frac{\pi}{2} areaC_0^{(3)} - \int_0^{\frac{\pi}{2}} \left[A_3(\varphi)\right] d\varphi.
\end{aligned} \tag{3.6}$$

From relation (3.3) follows that

$$\int_0^{\frac{\pi}{2}} \left[A_3(\varphi)\right] d\varphi = 6al - \frac{3l^2}{4} - \frac{3\pi(\pi-1)m^2}{4}. \tag{3.7}$$

The (3.1), (3.5) and (3.7) give us

$$P_{int}^{(3)} = \frac{1}{\pi\left(5a^2 - \frac{3\pi m^2}{2}\right)}\left[12al - \frac{3l^2}{2} - \frac{3\pi(\pi-1)m^2}{2}\right].$$

For $m = 0$ this probability become

$$\widetilde{p}^{(3)} = \frac{12}{5\pi}\frac{l}{a} - \frac{3}{10\pi}\left(\frac{l}{a}\right)^2. \tag{3.8}$$

The relation (1.10), (2.8) and (3.8) give us the evident equality

$$\widetilde{p}^{(1)} = \widetilde{p}^{(2)} = \widetilde{p}^{(3)}.$$

References

[1] H. Poincaré, *Calcul des probabilités,* ed.2, Gauthier Villars, Paris, (1912). 1

[2] M. Stoka, *Probabilités géométriques de type Buffon dans le plan euclidien*, Atti Acc. Sci. torino, **110** (1975-1976), 53–59. 1