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Dealing with indistinguishable particles and their entanglement

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Here, we discuss a particle-based approach to deal with systems of many identical quantum objects (particles) that never employs labels to mark them. We show that it avoids both methodological problems and drawbacks in the study of quantum correlations associated with the standard quantum mechanical treatment of identical particles. The core of this approach is represented by the multiparticle probability amplitude, whose structure in terms of single-particle amplitudes we derive here by first principles. To characterize entanglement among the identical particles, this new method uses the same notions, such as partial trace, adopted for non-identical ones. We highlight the connection between our approach and second quantization. We also define spin-exchanged multipartite states which contain a generalization of W states to identical particles. We prove that particle spatial overlap plays a role in the distributed entanglement within multipartite systems and is responsible for the appearance of non-local quantum correlations.

This article is part of a discussion meeting issue 'Foundations of quantum mechanics and their impact on contemporary society'.

1. Introduction

Identical quantum objects (e.g. qubits, atoms, quantum dots, photons, electrons and quasi-particles) typically

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are the basic 'particles' forming the building blocks of quantum-enhanced devices [1–11]. Characterizing the quantum properties of composite systems of identical particles is therefore important from both the fundamental and technological points of view. In quantum mechanics, owing to their indistinguishability, identical particles are not individually addressable and require specific treatments which differ from those used for non-identical (distinguishable) particles.

In the standard quantum mechanical approach (SA) to dealing with identical particles, the first step is to assume that they are not, marking them with *unobservable* labels [12,13]. This is the only place where non-observable quantities occur, being quantum states defined by complete sets of commuting observables. SA requires that the system is described by states whose structure is constrained to be symmetric (bosons) or antisymmetric (fermions) with respect to the labels [13–15]. So an intrinsic entanglement is present, even for independently prepared separate particles, which is attributable to the particle fundamental indistinguishability. This has given rise to different viewpoints about physical meaning and assessment of this part of entanglement [16–33]. For some, the entanglement due to indistinguishability is considered to be present but unusable in a state of independently prepared distant particles [12]; for others it is seen as a merely formal artefact even when the particles are brought to overlap [18,34]. Establishing the physical nature of identical particle entanglement is, therefore, crucial to identifying its role as a resource for quantum information and communication processing [11,17,22,35–42].

Recently, a different approach has been introduced [32] which describes the quantum states of identical particles without introducing unobservable labels. This non-standard approach (NSA), so far limited to two identical particles, exhibits peculiar advantages from both the conceptual and practical viewpoints, and is linked in particular to the treatment of their quantum correlations. In fact, from the beginning it avoids the existence of entanglement due to unobservable labels. The particle-based approach to identical particles uses, for quantifying the entanglement in these systems, the same well-established notions and tools commonly employed for non-identical particle systems, such as the basis-independent partial trace and von Neumann entropy. In this sense, the particle-based approach differs from other known approaches and is closer to the typical scenario of quantum information theory. This NSA has enabled the following to be shown: the existence of the Schmidt decomposition for identical particles [33] (showing that it is universally valid for both identical and not identical particles); a new efficient generation scheme of multipartite W entangled states [43]; and the definition of an operational framework to directly exploit entanglement due to indistinguishability for quantum information protocols [42]. Here, we reconsider the NSA from a fundamental perspective and generalize it to a system of many identical particles.

The paper is organized as follows. In §2, we briefly present, as an example, some immediate problematic implications for the use of quantum SA in the characterization of entanglement between identical particles. In §3, we give the formalism and the tools of the NSA for a system of N identical particles in a pure state. We obtain the many-particle probability amplitudes from first principles. In §4, we generalize states of non-identical particles to a system of identical ones. In §5, we demonstrate the role of spatial overlap in entanglement evaluation for three identical qubits, and the Bell inequality violation for independently prepared identical particles within an operational framework. We finally summarize our conclusion in §6.

2. Problematics of the standard quantum mechanical approach to identical particles

Ordinarily the SA treats identical particles as if they were not, and in general this works well. Here, we briefly describe how this method gives rise to unnecessary methodological and practical difficulties, especially when particle quantum correlations are involved. These difficulties arise because of the adoption of unphysical labels to mark particles and of the required (anti)symmetrization of states with respect to labels [13].

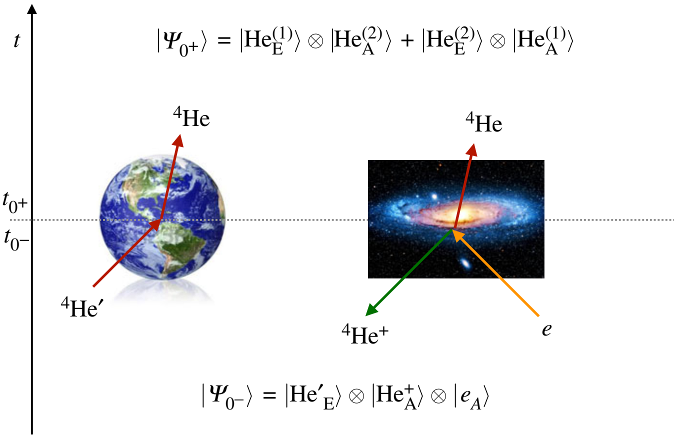

Figure 1. Simultaneous generation on the Earth and on the Andromeda galaxy of two identical helium atom states.

In the usual label approach to identical particles, the (anti)symmetrization implies that each of them has the same probability amplitude of occupying each single-particle state of the system. This justifies Peres's observation that 'only one fermion can occupy a quantum state is a not accurate statement' [12].

Other counterintuitive aspects with identical particles arise when one considers time-dependent problems. In fact, when a particle is created locally (LOCC processes) in a given location, it instantaneously appears in a far away location provided that another identical labelled particle is already present there. To make this point clear, we consider a helium atom ^{4}He in the state $|\mathrm{He}'_E\rangle$ on Earth and an ionized atom $^4\mathrm{He}^+$ in the state $|\mathrm{He}_A^+\rangle$ plus an electron in the state $|e_A\rangle$ on the Andromeda galaxy (subscripts E and A representing the spatial localization of states on Earth and Andromeda). Being at the beginning, all the involved particles are distinguishable, and the global state is the tensor product $|\Psi_{0^-}\rangle = |\mathrm{He}'_E\rangle \otimes |\mathrm{He}_A^+\rangle \otimes |e_A\rangle$. At the universal time $t=0$, $^4\mathrm{He}'$ is scattered in ^{4}He and simultaneously ion $^4\mathrm{He}^+$ on Andromeda absorbs the electron e, forming the atom ^{4}He (figure 1). At $t=0^+$ on Earth and on Andromeda, two identical bosons appear in states $|\mathrm{He}_E\rangle$ and $|\mathrm{He}_A\rangle$. In the SA, the identical atoms were distinguished with labels 1 and 2, and the global state is (except for a normalization factor) $|\Psi_{0^+}\rangle = |\mathrm{He}_E^{(1)}\rangle \otimes |\mathrm{He}_A^{(2)}\rangle + |\mathrm{He}_E^{(2)}\rangle \otimes |\mathrm{He}_A^{(1)}\rangle$. Thus, while at $t=0^-$ each particle is separately localized on either E or A, at $t=0^+$ each of the two helium atoms simultaneously occupies both states on E and A. This approach requires acceptance of the notion, for instance, that the non-relativistic helium atom generated at $t=0^+$ in A, because of the identical particle in E, instantaneously develops a non-zero amplitude of being there, although the events $(0_E^+, 0_A^+)$ are space-like separated. Moreover, the global state of the two identical particles $|\Psi_{0^+}\rangle$, when expressed in terms of labelled particles, has the form of an entangled state even if they are independently prepared far away from each other and never interact. In other words, it seems that in the standard operational framework based on local operations and classical communication (LOCC), these two particles become entangled even if they are in a separable state at the beginning. To cope with this situation, that is, to avoid observable effects of the unobservable labels, 'we must now convince ourselves that this entanglement is not matter of concern' [12].

The viewpoint of the SA thus makes a straightforward discussion of correlations (such as entanglement) in systems of identical particles problematic, because of the difficulty in formally separating the real part of correlations from the unphysical one arising from labels. Moreover, such a description hinders the use of a partial trace and the von Neumann entropy, as normally done for non-identical particles [16]. In fact, indistinguishability implies that the particles are not individually addressable and so the common reduced density matrix obtained by the partial trace is meaningless. This issue has originated different treatments for a faithful quantification of identical particle entanglement [16–33].

3. Non-standard approach to many identical particles

Here, we consider a recently introduced approach to deal with identical particles that does not adopt unphysical labels to mark them [32]. This NSA eliminates *ab ovo* the conceptual strains inherent in the SA and also allows us to directly focus on the treatment of physical quantum correlations. So far, it has been applied to the case of two identical particles [32,33]. We now re-examine this approach from a fundamental viewpoint and extend it to a system of many identical particles.

Let us take a 1-particle state $|\{\alpha\}\rangle$ characterized by a complete set of commuting observables $\{\alpha\}$. The state of two identical particles with two generic sets of commuting observables is $|\{\alpha_1\},\{\alpha_2\}\rangle$, which we write in a short hand notation as $|\{\alpha_1\},\{\alpha_2\}\rangle := |1,2\rangle$. The sets 1 and 2 may coincide, in this way giving $|\{\alpha_1\},\{\alpha_1\}\rangle := |1,1\rangle$. Generalizing to the case of N identical particles, the elementary global quantum state $|\phi^{(N)}\rangle$ is

$$|\phi^{(N)}\rangle := |1,2,\ldots,N\rangle, \tag{3.1}$$

where more indices may coincide. We remark that, in general, each index in equation (3.1) does not indicate the position of the corresponding 1-particle state in the N particle ket.

In the example illustrated in figure 1, being the single-particle states at $t=0^+$ given by $|\mathrm{He_E}\rangle$ and $|\mathrm{He_A}\rangle$, using (3.1) the global final state is $|\mathrm{He_E},\mathrm{He_A}\rangle$. This means that one helium atom is localized in E and one in A. Entanglement with respect to labels simply does not appear and the question of whether the two identical atoms are entangled does not even need to be posed.

For one particle, the relevant quantities to obtain probabilities are the transition amplitudes $\langle k'|k\rangle$ to find the particle in the exit state $|k'\rangle$ if it is prepared in the entry state $|k\rangle$. In the NSA, this must be generalized for N particles. The absence of labels implies that the state of equation (3.1) is not separable in terms of tensor products of 1-particle states and is a holistic entity.

We assume that the transition amplitude from the state $|1,\ldots,N\rangle$ to the state $|1',\ldots,N'\rangle$, namely $\langle 1',\ldots,N' \,|\, 1,\ldots,N\rangle$, can be expressed in terms of the 1-particle probability amplitudes. This is natural when each 1-particle state in the transition amplitude is localized in a region far away from the others (figure 2*a*,*b*). Under this condition, the cluster decomposition principle, stating that distant experiments provide independent outcomes [12], allows us to express the total transition amplitude as the product of the 1-particle ones as $\langle 1',2',\ldots,N' \,|\, 1,2,\ldots,N\rangle = \langle 1' \,|\, 1\rangle\langle 2' \,|\, 2\rangle\cdots\langle N'|N\rangle$ (figure 2*a*), and more generally as

$$\langle 1',2',\ldots,N' \,|\, 1,2,\ldots,N\rangle = \langle 1' \,|\, P_1\rangle\langle 2' \,|\, P_2\rangle\ldots\langle N' \,|\, P_N\rangle, \tag{3.2}$$

where the set $P_1,P_2,\ldots,P_N$ indicates one of the $N!$ permutations of the 1-particle states $|1\rangle\cdots|N\rangle$ and it represents the case in which the state $|P_k\rangle$ occupies the kth region (figure 2*b*). The transition amplitude (3.2) is linear in each of the 1-particle states.

When both the entry and exit 1-particle states are localized in overlapping spatial regions (figure 2*c*), to maintain the property of linearity, the N-particle probability amplitude can be expressed as a linear combination of $N!$ terms of the form (3.2)

$$\langle 1',2',\ldots,N' \,|\, 1,2,\ldots,N\rangle = \sum_P \alpha_P\langle 1' \,|\, P_1\rangle\langle 2' \,|\, P_2\rangle\ldots\langle N' \,|\, P_N\rangle, \tag{3.3}$$

where $P=\{P_1,P_2,\ldots,P_N\}$ runs over all the 1-particle state permutations. Taking into account that

$$\langle 1',2',\ldots,N' \,|\, 1,2,\ldots,N\rangle^* = \langle 1,2,\ldots,N \,|\, 1',2',\ldots,N'\rangle, \tag{3.4}$$

we find $\alpha_P^* = \alpha_{P^{-1}}$, where P^{-1} is the inverse permutation ($PP^{-1}=\mathbb{1}$) (see appendix A). We now consider the simplest case of two identical particles, for which any α_P is real and the 2-particle probability amplitude, using (3.3), is

$$\langle 1',2' \,|\, 1,2\rangle = a\langle 1' \,|\, 1\rangle\langle 2' \,|\, 2\rangle + b\langle 1' \,|\, 2\rangle\langle 2' \,|\, 1\rangle. \tag{3.5}$$

The above equation associates the order of the states in the 2-particle probability amplitude on the left with the order of the products of the 1-particle probability amplitudes on the right. Swapping

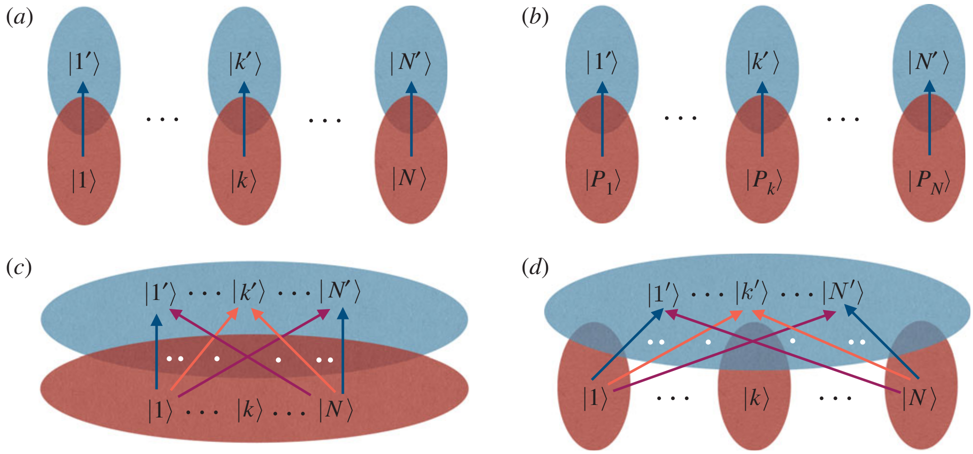

Figure 2. (*a*,*b*) Cluster decomposition principle. The set $\{P_1, \ldots, P_N\}$ represents the $N!$ permutations of the 1-particle states $|1\rangle, \ldots, |N\rangle$. (*c*) General case where there is spatial overlap among the entry 1-particle states in which the global system is prepared (red cloud) and among the exit ones on which the system is measured (blue cloud). (*d*) Particular situation where only all the entry 1-particle states do not overlap (red clouds). The coloured clouds represent the spatial regions occupied by the states written within them. The arrows show the transition of each $|k\rangle$ state ($k = 1, \ldots, N$) towards one or more $|k'\rangle$ states ($k' = 1', \ldots, N'$).

the single-particle states in the 2-particle state vector exchanges the weights of the single-particle products. However, this swapping cannot modify the 2-particle amplitude, implying that amplitudes may differ only for a global phase factor, that is,

$$\langle 1', 2'|2, 1\rangle = e^{i\zeta}\langle 1', 2'|1, 2\rangle = \mathrm{e}^{\mathrm{i}\zeta} a\langle 1'|1\rangle\langle 2'|2\rangle + \mathrm{e}^{\mathrm{i}\zeta} b\langle 1'|2\rangle\langle 2'|1\rangle$$
$$= a\langle 1'|2\rangle\langle 2'|1\rangle + b\langle 1'|1\rangle\langle 2'|2\rangle. \tag{3.6}$$

We find that $\mathrm{e}^{\mathrm{i}\zeta} b = a$ and $\mathrm{e}^{\mathrm{i}\zeta} a = b$, so $|a| = |b|$ and $(\mathrm{e}^{\mathrm{i}\zeta})^2 := \eta^2 = 1$, from which $\eta = \pm 1$. In this way, taking $a = 1$ in the linear combination of equation (3.5), $b = \eta$. So, the inner product of two 'holistic' state vectors is

$$\langle 1', 2'|1, 2\rangle_\eta := \langle 1'|1\rangle\langle 2'|2\rangle + \eta\langle 1'|2\rangle\langle 2'|1\rangle, \tag{3.7}$$

where the subscript η indicates that it is a symmetrized product. The previous expression represents the core of our approach and includes the particle spin-statistics principle. In fact, according to the Pauli exclusion principle, the probability amplitude of finding two fermions in the same state is zero, so $\langle 1', 1'|1, 2\rangle_\eta = (1 + \eta)\langle 1'|1\rangle\langle 1'|2\rangle$ requires that η is -1. The choice $\eta = +1$ gives the maximum amplitude of finding two particles in the same state and corresponds to the case of bosons.

Generalizing the symmetric and antisymmetric expression (3.7) to an arbitrary number N of identical particles, we can write

$$\langle 1', 2', \ldots, N'|1, 2, \ldots, N\rangle_\eta := \sum_P \eta^P \langle 1'|P_1\rangle\langle 2'|P_2\rangle \ldots \langle N'|P_N\rangle, \tag{3.8}$$

where, in analogy with the 2-particle case, for bosons η^P (P being the parity of the permutation) is always 1 and for fermions it is 1 (-1) for even (odd) permutations. In the following we shall omit, except where necessary, the subscript η in the bra–ket product. Equation (3.8) induces the symmetrization property of the N-system state space: $|1, 2, \ldots, j, \ldots, k, \ldots, N\rangle = \eta|1, 2, \ldots, k, \ldots, j, \ldots, N\rangle$, for $j, k = 1, \ldots, N$. Linearity of the N-system state vector with respect to each 1-particle state immediately follows from the linearity of the 1-particle amplitudes and the N-particle state vectors thus span the physical symmetric state space $\mathcal{H}_\eta^{(N)}$.

We remark that, in the situations represented by figure 2*b* and only in these cases, one can write $\langle 1',\ldots,N'|P_1,\ldots,P_N\rangle := (\langle 1'|\otimes\cdots\otimes\langle N'|)(|P_1\rangle\otimes\cdots\otimes|P_N\rangle)$. Therefore, for calculation purposes $|1,\ldots,N\rangle \simeq |1\rangle\otimes\cdots\otimes|N\rangle$. This is not true if there is overlap either among the 1-particle entry states or among the exit ones (figure 2*c*,*d*). In this sense, probability amplitudes are more fundamental than quantum states.

An arbitrary elementary normalized N-identical particle state is defined as

$$|\Phi^{(N)}\rangle := \frac{1}{\mathcal{N}}|\phi^{(N)}\rangle := \frac{1}{\mathcal{N}}|1,2,\ldots,N\rangle, \tag{3.9}$$

where $\mathcal{N}=\sqrt{\langle 1,2,\ldots,N|1,2,\ldots,N\rangle}$. $|\Phi^{(N)}\rangle$ is expressed in terms of single-particle states as a single-state vector, which is to be compared with the linear combination of the $N!$ product state vectors in the SA approach: $|\Theta^{(N)}\rangle = (1/\mathcal{N})\sum_P \eta^P |1_{P_1}\rangle\otimes|2_{P_2}\rangle\otimes\cdots\otimes|N_{P_N}\rangle$.

The next step is to define the action of operators in the NSA approach. We limit ourselves to an arbitrary 1-particle operator $A^{(1)}$ that acts on each 1-particle state at a time: $A^{(1)}|k\rangle := |A^{(1)}k\rangle$. Its action on N-particle states is naturally defined as

$$A^{(1)}|1,2,\ldots,N\rangle := \sum_k |1,\ldots,A^{(1)}k,\ldots,N\rangle. \tag{3.10}$$

(a) Partial trace

To calculate the partial trace, we have to define a product operation between bra and ket of a different number of particles. In particular, the M-particle partial trace of an N-particle state is expressed as

$$\mathrm{Tr}^{(M)}|\Phi^{(N)}\rangle\langle\Phi^{(N)}| := \sum_{\tilde{k}'}\langle\tilde{k}'|\cdot|\Phi^{(N)}\rangle\langle\Phi^{(N)}|\cdot|\tilde{k}'\rangle, \tag{3.11}$$

where $\{|\tilde{k}'\rangle := |k'_1,\ldots,k'_M\rangle/\mathcal{N}_{k'}\}$ is a collective M-particle orthonormal basis ($\mathcal{N}_{k'}$ being a normalization constant). This operation corresponds to measuring the states of M identical particles without registering the outcomes. When $M=N$, $\langle\tilde{k}'|\cdot|\Phi^{(N)}\rangle$ just coincides with the normalized probability amplitude.

We now consider the simple case in which $M=1$. Given a single-particle orthonormal basis $\{|k'\rangle\}$, we have to define the product $\langle k'|\cdot|1,2,\ldots,N\rangle$. Given the operator $A^{(1)}=|j'\rangle\langle k'|$ and using equation (3.10), its action on an N-particle state $|1,\ldots,N\rangle$ can be written as

$$\begin{aligned} A^{(1)}|1,\ldots,N\rangle &:= \sum_k |1,2,\ldots,j'\langle k'|k\rangle,\ldots,N\rangle \\ &= \sum_k \eta^{k-1}\langle k'\,|\,k\rangle|j',1,2,\ldots,\not{k},\ldots,N\rangle, \end{aligned} \tag{3.12}$$

where, in the second line, we have taken out of the N-particle ket the complex number $\langle k'\,|\,k\rangle$ and shifted the state $|j'\rangle$ from the kth site to the first one ($\not{k}$ indicates the lack of the kth state). Let us express now the ket in the second line of equation (3.12) as follows: $|j',1,2,\ldots,\not{k},\ldots,N\rangle := |j'\rangle\wedge|1,2,\ldots,\not{k},\ldots,N\rangle$, where we have introduced with the symbol $\wedge$ a non-separable symmetric external product between different kets, valid for boson and fermions, that we call the *wedge* product (for fermions this product is Penrose's wedge product defined in terms of labelled states [44]). For N identical particles, we have $|1,2,\ldots,N\rangle := |1\rangle\wedge|2\rangle\wedge\cdots\wedge|N\rangle$ and $(|1,2,\ldots,N\rangle)^\dagger := \langle N|\wedge\cdots\wedge\langle 2|\wedge\langle 1|$. Moreover, $|P_1\rangle\wedge|P_2\rangle\wedge\cdots\wedge|P_N\rangle = \eta^P|1\rangle\wedge|2\rangle\wedge\cdots\wedge|N\rangle$. The wedge product coincides with the multiplication operation of the exterior algebra associated with the symmetrized Hilbert space $\mathcal{H}_\eta^{(N)}$. Therefore, equation (3.12) can be written as

$$A^{(1)}|1,\ldots,N\rangle := |j'\rangle\wedge\sum_k \eta^{k-1}\langle k'|k\rangle|1,2,\ldots,\not{k},\ldots,N\rangle, \tag{3.13}$$

which suggests the introduction of a generalized *dot* product operation between bra and ket of different dimensionality M and N, respectively, which in the case of $M=1$ is

$$\langle k'| \cdot |1, 2, \ldots, N\rangle := \sum_{k=1}^{N} \eta^{k-1} \langle k'|k\rangle |1, \ldots, \not{k}, \ldots, N\rangle. \tag{3.14}$$

The above equation defines the projection of an N-particle state on a 1-particle state and gives an (unnormalized) $(N-1)$-particle state. We can see that if the N-particle state is explicitly expressed in terms of wedge products, the dot product is distributive with respect to the wedge.

Considering the 1-particle basis $\{|k'\rangle\}$ and taking the 1-particle projection operator $\Pi_{k'}^{(1)} = |k'\rangle\langle k'|$, the probability $p_{k'}$ of finding one of the identical particles in the state $|k'\rangle$ is given (except for a normalization factor) by $\langle \Pi_{k'}^{(1)} \rangle_{\Phi^{(N)}}$. Being the action of the 1-particle identity operator $\mathbb{I}^{(1)} = \sum_{k'} \Pi_{k'}^{(1)}$ on an N-particle state $\mathbb{I}^{(1)}|1, \ldots, N\rangle = N|1, \ldots, N\rangle$, the normalized reduced $(N-1)$-particle pure state $|\Phi_{k'}^{(N-1)}\rangle$ and the probability $p_{k'}$ are

$$|\Phi_{k'}^{(N-1)}\rangle = \frac{\langle k'| \cdot |\Phi^{(N)}\rangle}{\sqrt{\langle \Pi_{k'}^{(1)} \rangle_{\Phi^{(N)}}}} \quad \text{and} \quad p_{k'} = \frac{\langle \Pi_{k'}^{(1)} \rangle_{\Phi^{(N)}}}{\langle \mathbb{I}^{(1)} \rangle_{\Phi^{(N)}}}. \tag{3.15}$$

If all the 1-particle states in $|\Phi^{(N)}\rangle$ are orthonormal, i.e. $\langle i|j\rangle = \delta_{ij}$, one has $p_{k'} = \sum_{j=1}^{N} (|\langle k'|j\rangle|^2 / \langle \mathbb{I}^{(1)} \rangle_{\Phi^{(N)}})$, which corresponds to the sum of probabilities of incompatible outcomes.

Specifically, the normalized 1-particle partial trace of an N-particle state is

$$\rho^{(N-1)} := \frac{1}{\langle \mathbb{I}^{(1)} \rangle_{\Phi^{(N)}}} \mathrm{Tr}^{(1)} |\Phi^{(N)}\rangle\langle \Phi^{(N)}| = \sum_{k'} p_{k'} |\Phi_{k'}^{(N-1)}\rangle\langle \Phi_{k'}^{(N-1)}|. \tag{3.16}$$

We now take the case $M=2$. The relevant dot product is

$$\langle l', m'| \cdot |1, \ldots, N\rangle = \sum_{k=1}^{N} \eta^{k-1} \langle l'|k\rangle \left[\sum_{j<k} \eta^{j-1} \langle m'|j\rangle |\not{j}, \not{k}\rangle + \sum_{j>k} \eta^{j-2} \langle m'|j\rangle |\not{k}, \not{j}\rangle \right], \tag{3.17}$$

where $|\not{j}, \not{k}\rangle \equiv |1, \ldots, \not{j}, \ldots, \not{k}, \ldots, N\rangle$. If $N=2$ equation (3.17) coincides with the 2-particle probability amplitude given by (3.7).

The 2-particle unity operator is $\mathbb{I}^{(2)} = (1/2!) \sum_{\tilde{k}'} \Pi_{\tilde{k}'}^{(2)}$, where $\Pi_{\tilde{k}'}^{(2)} = |\tilde{k}'\rangle\langle \tilde{k}'|$ is the 2-particle projection operator. The normalized reduced $(N-2)$-particle pure state $|\Phi_{\tilde{k}'}^{(N-2)}\rangle$ and its probability $p_{\tilde{k}'}$ are

$$|\Phi_{\tilde{k}'}^{(N-2)}\rangle = \frac{\langle \tilde{k}'| \cdot |\Phi^{(N)}\rangle}{\sqrt{\langle \Pi_{\tilde{k}'}^{(2)} \rangle_{\Phi^{(N)}}}} \quad \text{and} \quad p_{\tilde{k}'} = \frac{1}{2!} \frac{\langle \Pi_{\tilde{k}'}^{(2)} \rangle_{\Phi^{(N)}}}{\langle \mathbb{I}^{(2)} \rangle_{\Phi^{(N)}}}. \tag{3.18}$$

The $(N-2)$-particle reduced density matrix is

$$\rho^{(N-2)} := \frac{1}{2! \langle \mathbb{I}^{(2)} \rangle_{\Phi^{(N)}}} \mathrm{Tr}^{(2)} |\Phi^{(N)}\rangle\langle \Phi^{(N)}| = \sum_{\tilde{k}'} p_{\tilde{k}'} |\Phi_{\tilde{k}'}^{(N-2)}\rangle\langle \Phi_{\tilde{k}'}^{(N-2)}|. \tag{3.19}$$

When required, the generalization of equation (3.17) to the case of any M is straightforward.

We point out that, in the NSA, once the relevant reduced density matrix is obtained, the entanglement between the bipartition of M and $(N-M)$ particles can be measured by the von Neumann entropy $S(\rho^{(N-M)}) = -\mathrm{Tr}(\rho^{(N-M)} \log_2 \rho^{(N-M)})$. Moreover, knowledge of the reduced density matrices of all the possible bipartitions of the system allows the qualitative assessment of the genuine multipartite entanglement of the pure state of N identical particles, as usually done for non-identical particles [35]. Note that, as the identical particle entanglement intrinsically depends on both global state structure and observation measurements [31,42], one has to specify the partial trace and the reduced density matrix corresponding to a desired operational framework [42]. This result differs from what is obtained in the SA, where the partial trace cannot be used [16]. We shall analyse identical particle entanglement in §5.

(b) Connection of the non-standard approach with second quantization

Above we have introduced relations between states differing in the number of particles. This suggests a relationship between the NSA to identical particles and second quantization. However, while in the second quantization particles are elementary excitations of fields, our approach applies to any system of identical quantum objects. In the second quantization approach, creation and annihilation operators connect states differing by only one particle and their commutation rules reflect the commutation rules of the fields. In the NSA, the dot products connect states differing for a generic number of particles. Finally, in the second quantization the fundamental role is played by the commutation rules, while in the NSA it is played by the symmetry of the states.

To show the connection between the NSA and the second quantization, we note that equation (3.14) suggests the introduction of a 1-particle annihilation operator

$$a(k)|1,\ldots,N\rangle := \langle k| \cdot |1,\ldots,N\rangle, \tag{3.20}$$

and its adjoint is

$$a^{\dagger}(k)|1,\ldots,N\rangle := |k\rangle \wedge |1,\ldots,N\rangle. \tag{3.21}$$

Equation (3.17) directly suggests the introduction of 2-particle annihilation and creation operators defined as

$$a(j,k)|1,\ldots,N\rangle := \langle j,k| \cdot |1,\ldots,N\rangle \tag{3.22}$$

and

$$a^{\dagger}(j,k)|1,\ldots,N\rangle := |j,k\rangle \wedge |1,\ldots,N\rangle, \tag{3.23}$$

which, in terms of the wedge product introduced in §3a, can be written as follows:

$$a(j,k)|1,\ldots,N\rangle = (\langle k| \wedge \langle j|) \cdot |1,\ldots,N\rangle = \langle k| \cdot (\langle j| \cdot |1,\ldots,N\rangle) \tag{3.24}$$

and

$$a^{\dagger}(j,k)|1,\ldots,N\rangle = |j\rangle \wedge |k\rangle \wedge |1,\ldots,N\rangle. \tag{3.25}$$

From the above equations, we find that the 2-particle annihilation and creation operators can be expressed in terms of the 1-particle ones as $a^{\dagger}(j,k) \equiv a^{\dagger}(j) \wedge a^{\dagger}(k) = \eta a^{\dagger}(k) \wedge a^{\dagger}(j)$ and $a(j,k) \equiv a(k) \wedge a(j) = \eta a(j) \wedge a(k)$.

From the symmetry properties of the states, it is simple to obtain the a's commutation rules. For the 1-particle creation and annihilation operators of equations (3.20) and (3.21), we obtain

$$\left.\begin{aligned} [a(j), a^{\dagger}(k)]_{\eta} &= \langle j|k\rangle, \\ [a^{\dagger}(j), a^{\dagger}(k)]_{\eta} &= [a(j), a(k)]_{\eta} = 0, \end{aligned}\right\} \tag{3.26}$$

where $[a(j), a^{\dagger}(k)]_{\eta} = a(j)a^{\dagger}(k) - \eta a^{\dagger}(k)a(j)$ is the commutator for bosons and the anticommutator for fermions. We show in appendix B that, using equation (3.17), equation (3.26) generalizes, for the 2-particle creation and annihilation operators, to

$$\left.\begin{aligned} [a(j,k), a^{\dagger}(m,n)] &= \langle j,k|m,n\rangle_{\eta}, \\ [a^{\dagger}(j,k), a^{\dagger}(m,n)] &= [a(j,k), a(m,n)] = 0. \end{aligned}\right\} \tag{3.27}$$

In the first equation of (3.27), the η on the right-hand side retains the memory of the bosonic or fermionic nature of the particles.

4. Properties of some multiparticle states

In this section, we show how some known structures and properties of states of non-identical particles can be generalized to the case of identical particles.

(a) Spin exchanged states

We consider the well-known W entangled state of non-identical particles, which constitutes an important resource state for several quantum information tasks [43,45]. We explicitly describe a single-particle state with the spatial mode ϕ_i ($i = 1, \ldots, N$) and the pseudospin $\uparrow, \downarrow$. The W state has the structure

$$|W\rangle = |\phi_1 \uparrow, \phi_2 \downarrow, \ldots, \phi_N \downarrow\rangle + |\phi_1 \downarrow, \phi_2 \uparrow, \ldots, \phi_N \downarrow\rangle + \cdots + |\phi_1 \downarrow, \ldots, \phi_{N-1} \downarrow, \phi_N \uparrow\rangle, \tag{4.1}$$

where for non-identical particles each term of the superposition is a tensor product of single-particle states. The very same structure is considered to hold also for identical particles in separated locations. In this case, is this structure valid under any circumstance? To analyse this aspect, we choose three fermions localized in three separated spatial modes A, B and C, so that the state of equation (4.1) is $|W_3\rangle = |A \uparrow, B \downarrow, C \downarrow\rangle + |A \downarrow, B \uparrow, C \downarrow\rangle + |A \downarrow, B \downarrow, C \uparrow\rangle$. When two fermions are in the same spatial mode, for instance $A = B$, this state becomes

$$|\bar{W}_3\rangle = |A \uparrow, A \downarrow, C \downarrow\rangle + |A \downarrow, A \uparrow, C \downarrow\rangle, \tag{4.2}$$

where the state $|A \downarrow, A \downarrow, C \uparrow\rangle$ does not appear because of the Pauli exclusion principle. Using the symmetry properties of the elementary states, $|A \downarrow, A \uparrow, C \downarrow\rangle = -|A \uparrow, A \downarrow, C \downarrow\rangle$, one gets $|\bar{W}_3\rangle = 0$, while one expects no restriction arising from particle statistics, considering that the two fermions in A have opposite spins. The same problem is thus expected to arise even when the particles are spatially non-overlapping but non-local measurements are performed. Hence, the form of the W state of equation (4.1) for identical particles does not work in general.

We define a spin-exchanged state (SPES) as a suitable linear combination of elementary states where only particle pseudospins are exchanged, that is,

$$|\mathrm{SPES}\rangle := \frac{1}{\mathcal{N}} \sum_P \frac{\eta^P}{\mathcal{N}_P} |\phi_1\, \sigma_{P_1}, \ldots, \phi_N\, \sigma_{P_N}\rangle, \tag{4.3}$$

where P are here the cyclic permutations of pseudospins σ_{P_i}, $\mathcal{N}_P$ is the normalization constant of each state entering the sum and $\mathcal{N}$ is the global normalization constant. When $\sigma_1 = \uparrow$ and $\sigma_2 = \cdots = \sigma_N = \downarrow$, the SPES represents a generalization of the W state for identical particles valid for bosons and fermions under any circumstances. Other states of interest are obtained from (4.3) for different pseudospin conditions; for instance, assuming a ring configuration of the N particles and taking $\sigma_1 = \cdots = \sigma_M = \uparrow$ and $\sigma_{(M+1)} = \cdots = \sigma_N = \downarrow$ such states can represent a linear combination of spin block systems in the ring chain.

(b) Separability of spatial and spin degrees of freedom

For any elementary state of non-identical particles, the spatial and spin degrees of freedom, being associated with the individual particle, are independent and separable. This is not the case for elementary states of identical particles.

As mentioned above, the essence of the NSA is the absence of labels and the consequent holistic form of the states that are not separable in tensor products of single-particle states. For a single particle, we have $|\phi\sigma\rangle \equiv |\phi\rangle \otimes |\sigma\rangle$, i.e. the spatial part of the state can be treated separately from the pseudospin one. Instead, for two identical particles, when $\phi_1 \neq \phi_2$ and $\sigma_1 \neq \sigma_2$, it is $|\phi_1\sigma_1, \phi_2\sigma_2\rangle \neq |\phi_1, \phi_2\rangle \otimes |\sigma_1, \sigma_2\rangle$, so measures of position and pseudospin operators may not be independent. One may ask which structure a state of identical particles must have in order that spatial and pseudospin degrees of freedom are independent from each other.

To fix our ideas, let us take the state

$$|\phi\rangle = \alpha|\varphi_1\sigma, \varphi_2\tau\rangle + \beta|\varphi_1\tau, \varphi_2\sigma\rangle, \tag{4.4}$$

with $\varphi_1 \neq \varphi_2$, $\tau \neq \sigma$, and calculate the probability amplitude on the state $|\phi'\rangle = |\varphi_1'\sigma', \varphi_2'\tau'\rangle$, that is,

$$\langle\phi' \mid \phi\rangle = \langle\varphi_1'\sigma', \varphi_2'\tau' \mid \{\alpha|\varphi_1\sigma, \varphi_2\tau\rangle + \beta|\varphi_1\tau, \varphi_2\sigma\rangle\}. \tag{4.5}$$

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Using expression (3.7), we find that only when $\alpha = 1$ and $\beta = \pm 1$ can equation (4.5) be written as

$$\langle \phi' \mid \phi \rangle = \langle \varphi_1' \varphi_2' \mid \varphi_1 \varphi_2 \rangle_{\eta\beta} \langle \sigma' \tau' \mid \sigma \tau \rangle_{\beta}, \tag{4.6}$$

where $\langle \mu' \nu' | \mu \nu \rangle_\gamma$ indicates the probability amplitude of the form of equation (3.7) with η substituted by γ. We have thus shown that, under the above conditions, in the entangled states of the form $|\Phi^{\pm}\rangle = (1/\mathcal{N})(|\varphi_1\sigma, \varphi_2\tau\rangle \pm |\varphi_1\tau, \varphi_2\sigma\rangle)$, spatial and pseudospin degrees of freedom can be separated as $|\Phi^{\pm}\rangle = |\varphi_1, \varphi_2\rangle_{\eta\pm} \otimes |\sigma, \tau\rangle_{\pm}$, where the subscripts indicate the symmetry of the state: $|\kappa, \chi\rangle_\gamma = \gamma|\chi, \kappa\rangle_\gamma$. For such a state, entanglement of pseudospins can be treated independently of the spatial degrees of freedom. When $\sigma = \uparrow$ and $\tau = \downarrow$, $|\Phi^{\pm}\rangle$ are two Bell states. It is moreover immediate to show that an entangled state of the form $|\Psi^{\pm}\rangle = (1/\mathcal{N})(|\phi_1\sigma, \phi_2\sigma\rangle \pm |\phi_1\tau, \phi_2\tau\rangle)$ is always equivalent to $|\phi_1\phi_2\rangle_\eta \otimes (|\sigma, \sigma\rangle \pm |\tau, \tau\rangle)$.

5. Applications

We now apply the NSA formalism and tools described above to examine entanglement properties of identical particle pure states by partial traces and local measurements.

(a) Effects of the spatial overlap on the entanglement in the spin-exchanged state

We consider the SPES for three qubits placed in separated spatial modes L (left), C (centre) and R (right). Following equation (4.3), it has the form

$$|\mathrm{SPES}_3\rangle = \frac{1}{\sqrt{3}}(|L\uparrow, C\downarrow, R\downarrow\rangle + \eta|L\downarrow, C\uparrow, R\downarrow\rangle + |L\downarrow, C\downarrow, R\uparrow\rangle). \tag{5.1}$$

Partially tracing this state over the 1-particle basis localized in L, $\{|L\uparrow\rangle, |L\downarrow\rangle\}$, we obtain the reduced density matrix

$$\begin{aligned}\rho_L^{(2)} = &\frac{1}{3}(|C\downarrow, R\uparrow\rangle\langle C\downarrow, R\uparrow| + |C\uparrow, R\downarrow\rangle\langle C\uparrow, R\downarrow| + |C\downarrow, R\downarrow\rangle\langle C\downarrow, R\downarrow|)\\ &+ \frac{\eta}{3}(|C\downarrow, R\uparrow\rangle\langle C\uparrow, R\downarrow| + |C\uparrow, R\downarrow\rangle\langle C\downarrow, R\uparrow|).\end{aligned} \tag{5.2}$$

The von Neumann entropy $S_L(\mathrm{SPES}_3) = -\mathrm{Tr}(\rho_L^{(2)} \log_2 \rho_L^{(2)}) = \log_2 3 - \frac{2}{3}$ measures the entanglement of pseudospins between the bipartitions L (one particle) and C–R (two particles). This entropy is less than 1 and independent of the type of particle. As expected, this result coincides with that obtained, under the same conditions, for the non-identical particle W state.

We now consider the case when the spatial wave functions of two particles completely overlap, in particular $C = L$. From equation (4.3), the corresponding (unnormalized) SPES is

$$|\mathrm{SPES}_3'\rangle = 2|L\uparrow, L\downarrow, R\downarrow\rangle + \frac{(1+\eta)}{2}\frac{|L\downarrow, L\downarrow, R\uparrow\rangle}{\sqrt{2}}. \tag{5.3}$$

Tracing again over the 1-particle basis in L, we find

$$\begin{aligned}\rho_L^{(2)} = &\frac{1}{\mathcal{N}}[4(|L\downarrow, R\downarrow\rangle\langle L\downarrow, R\downarrow| + |L\uparrow, R\downarrow\rangle\langle L\uparrow, R\downarrow|)\\ &+ 2\eta\kappa(|L\uparrow, R\downarrow\rangle\langle L\downarrow, R\uparrow| + |L\downarrow, R\uparrow\rangle\langle L\uparrow, R\downarrow|) + \kappa^2|L\downarrow, R\uparrow\rangle\langle L\downarrow, R\uparrow|],\end{aligned} \tag{5.4}$$

where $\kappa = (1+\eta)^2/(2\sqrt{2})$ and $\mathcal{N} = (8 + \kappa^2)$. The corresponding von Neumann entropy is

$$S(\mathrm{SPES}_3') = -\frac{4}{8+\kappa^2}\log_2\frac{4}{8+\kappa^2} - \frac{4+\kappa^2}{8+\kappa^2}\log_2\frac{4+\kappa^2}{8+\kappa^2}. \tag{5.5}$$

This entropy is for fermions $S_{\mathrm{f}}(\mathrm{SPES}_3') = 1$ and for bosons $S_{\mathrm{b}}(\mathrm{SPES}_3') = \log_2 5 - \frac{3}{5}\log_2 3 - \frac{2}{5}$. This result highlights the effect of spatial overlap and of the nature of the particles on the bipartite entanglement of three identical particles.

(b) Bell inequality violation for identical particles

We now apply the NSA to study a state of identical particles within a Bell test scenario [46,47], using a suitable operational framework [42].

We take two independently prepared identical qubits, one being in the spatial mode ψ with pseudospin $\uparrow$ and the other one in the spatial mode ψ' with pseudospin $\downarrow$. The global state is therefore

$$|\Psi\rangle = |\psi \uparrow, \psi' \downarrow\rangle. \tag{5.6}$$

We note that the configuration described by this state for non-identical particles does not present entanglement. To make the entanglement within this state emerge, it is natural to choose local measurements of single-particle pseudospin states performed in two separated restricted spatial regions. This modus operandi defines an operational framework founded on spatially localized operations and classical communication (sLOCC), where we do not refer 'spatially localized' to a given particle, which is individually unaddressable, but to a given spatial location [42].

In this context, it is useful to introduce the probability amplitudes of finding the two particles in the two separated spatial regions L (left) and R (right), $l = \langle L|\psi\rangle$, $l' = \langle L|\psi'\rangle$, $r = \langle R|\psi\rangle$ and $r' = \langle R|\psi'\rangle$. Following Lo Franco & Compagno [42], we choose the 2-particle basis in the subspace defined by the two separated regions, namely $B_{LR} = \{|L\uparrow, R\uparrow\rangle, |L\uparrow, R\downarrow\rangle, |L\downarrow, R\uparrow\rangle, |L\downarrow, R\downarrow\rangle\}$, and we project the state (5.6) onto this subspace by means of the projector $\Pi_{LR} = \sum_{\sigma,\tau=\uparrow,\downarrow} |L\sigma, R\tau\rangle\langle L\sigma, R\tau|$, obtaining the pure state

$$\rho_{LR} = |\Psi_{LR}\rangle\langle\Psi_{LR}|, \tag{5.7}$$

where $|\Psi_{LR}\rangle := \Pi_{LR}|\Psi\rangle/\sqrt{P_{LR}}$ is

$$|\Psi_{LR}\rangle = \frac{lr'|L\uparrow, R\downarrow\rangle + \eta l'r|L\downarrow, R\uparrow\rangle}{\sqrt{P_{LR}}}, \tag{5.8}$$

with $P_{LR} = \langle\Psi|\Pi_{LR}|\Psi\rangle = |lr'|^2 + |l'r|^2$ being the probability of obtaining it. This operational framework naturally suggests the introduction, from an experimental perspective, of a figure of merit for the exploitable entanglement of identical particles given by $E_{\exp} := E_{LR}P_{LR}$, where E_{LR} is a quantifier of the entanglement contained in the distributed resource state $|\Psi_{LR}\rangle$.

We now use the pseudospin observable $\mathcal{O}_S := \mathbf{O}_S \cdot \boldsymbol{\sigma}_S$ $(S = L, R)$ with eigenvalues ± 1, where $\mathbf{O}_S$ is the unit vector in an arbitrary direction in the spin space and $\boldsymbol{\sigma}_S = (\sigma_x^S, \sigma_y^S, \sigma_z^S)$ is the Pauli matrices vector. The Clauser–Horne–Shimony–Holt (CHSH)–Bell inequality [46] in this context is

$$\mathcal{B}(\rho_{LR}) = |\langle\mathcal{O}_L\mathcal{O}_R\rangle + \langle\mathcal{O}_L\mathcal{O}'_R\rangle + \langle\mathcal{O}'_L\mathcal{O}_R\rangle - \langle\mathcal{O}'_L\mathcal{O}'_R\rangle| \leq 2, \tag{5.9}$$

where $\mathcal{B}(\rho_{LR})$ is the Bell function expressed in terms of the correlation functions of the pseudospin observables and $\mathcal{O}'_S$ indicates the measurement in a direction different from that of $\mathcal{O}_S$ [48]. A well-known procedure [49,50] allows us to express the maximum value of the Bell function $\mathcal{B}_{\max}$ in terms of the concurrence (entanglement) [35] of the state $|\Psi_{LR}\rangle$ as

$$\mathcal{B}_{\max} = 2\sqrt{1 + C(\Psi_{LR})^2} = 2\sqrt{1 + \left(\frac{2|lr'l'r|}{P_{LR}}\right)^2}, \tag{5.10}$$

where the explicit expression of $C(\Psi_{LR})$ has been used. We have obtained $\mathcal{B}_{\max}(\rho_{LR}) > 2$ when $C(\Psi_{LR}) > 0$, i.e. whenever there is spatial overlap between ψ and ψ' and the local measurements are performed inside the overlap region (that is, all the four probability amplitudes l, l', r, r' are non-zero). The violation of the CHSH–Bell inequality identifies the presence of non-local correlations between the pseudospin outcomes in the regions L and R.

Therefore, spatial overlap between wave functions associated with independently prepared identical particles can generate non-locality effects which can be tested in quantum optical scenarios [51] and then exploited to implement quantum information or communication processes [42].

6. Conclusion

In this work, we have presented an alternative way to deal with sets of identical quantum objects. These objects, which can constitute building blocks of a complex system, can be considered as 'particles' in general, not coinciding with elementary excitations of quantum fields. This NSA to identical particles differs from the SA one in that it never makes use of labels to mark particles. The core of NSA is played by the probability amplitude between N-particle states. We have derived this multiparticle probability amplitude by first principles, that is, the *cluster decomposition principle*, stating that distant experiments provide independent outcomes [12], and *linearity* with respect to 1-particle states. The NSA is universal in that it works for any type of particle. It exhibits methodological advantages compared with the SA by avoiding the problematics arising there from the necessary symmetrization with respect to unobservable labels. This occurs, for example, when identical particles are generated in far away regions with each particle instantaneously developing a non-zero probability amplitude of being in space-like separated places.

The NSA also enables technical advantages with respect to SA by supplying essential tools which are not usable in the latter, such as projective measurements and partial trace. As by-products of such tools, one straightforwardly obtains the von Neumann entropy associated with the reduced density matrix, which estimates the entanglement of a general bipartition of N identical particles. We have moreover introduced generalized products between vector spaces of different dimensions, showing that these can be connected with generalized annihilation and creation operators. When these operators connect states differing for one excitation, they coincide with the operators in the second quantization. While in the latter the commutation rules between annihilation and creation operators derive from the field ones, here they are determined by the many-particle probability amplitude.

We have shown that some non-identical particle-entangled states, such as the W states, cannot be used *tout court* for identical particles. Within the NSA, we have introduced the SPESs which contain the extension of W states to identical particles and allow the quantitative role of spatial overlap to be demonstrated in the bipartite entanglement of multiparticle states. A new by-product of the NSA is that indistinguishability in the presence of spatial overlap of independently prepared identical particles gives rise to exploitable non-local entanglement.

Finally, the NSA provides a very convenient way for describing a system of N identical particles and their entanglement. The results of this work pave the way to further studies concerning the characterization of identical particle systems, such as multipartite coherence, correlations other than entanglement and dynamics under noisy environments.

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Appendix A

In the case of the one-dimensional representation of the permutation group, the N-particle probability amplitude can be written as

$$\langle 1', \ldots, N' | 1, \ldots, N \rangle = \sum_P \alpha_P \langle 1' | P_1 \rangle \cdots \langle N' | P_N \rangle,$$

where P runs over the $N!$ permutations of the 1-particle kets and $\{\alpha_P\}$ are complex numbers. We now take the complex conjugate of the previous equation

$$\langle 1', \ldots, N' \mid 1, \ldots, N\rangle^* = \sum_P \alpha_P^* \langle P_1|1'\rangle \cdots \langle P_N|N'\rangle.$$

From the first equation, we have

$$\langle 1, \ldots, N|1', \ldots, N'\rangle = \sum_Q \alpha_Q \langle 1|Q'_1\rangle \cdots \langle N|Q'_N\rangle = \sum_Q \alpha_Q \langle Q_1^{-1}|1'\rangle \cdots \langle Q_N^{-1}|N'\rangle,$$

where, taking into account that $\langle 1', 2'|1, 2\rangle^* = \langle 1, 2|1', 2'\rangle$, we have reordered $Q'_1, \ldots, Q'_N$ in such a way that $(Q'_1, \ldots, Q'_N) \longrightarrow (1', \ldots, N')$ and thus $(1, \ldots, N) \longrightarrow (Q_1^{-1}, \ldots, Q_N^{-1})$, in which Q^{-1} is the inverse permutation of Q. Comparing the last two equations, we find that $Q = P^{-1}$ and $\alpha_P^* = \alpha_{P^{-1}}$.

Appendix B

Here, we give the explicit derivation of equation (3.27).

— We first consider the action of $a(m', n')a^\dagger(m, n)$ on a generic N-particle state $|\phi^{(N)}\rangle$ (see equation (3.1)). Using equation (3.17), we find:

$$\begin{aligned} a(m', n')a^\dagger(m, n)|\phi^{(N)}\rangle &= a(m', n')|m, n, 1, 2, \ldots, N\rangle \\ &= (\langle m'|m\rangle\langle n'|n\rangle + \eta\langle m'|n\rangle\langle n'|n\rangle)|\phi^{(N)}\rangle \\ &\quad + \sum_{k=1}^{N} \eta^{k-1}\langle m'|k\rangle \sum_{j<k} \eta^{j-1}\langle n'|j\rangle|m, n, \not{j}, \not{k}\rangle \\ &\quad + \sum_{k=1}^{N} \eta^{k-1}\langle m'|k\rangle \sum_{j>k} \eta^{j-2}\langle n'|j\rangle, |m, n, \not{k}, \not{j}\rangle, \end{aligned}$$

where $|m, n, \not{k}, \not{j}\rangle = |m, n, 1, \ldots \not{k} \ldots, \not{j} \ldots, N\rangle$.

— The action of $a^\dagger(m, n)a(m', n')$ on the same state is

$$\begin{aligned} a^\dagger(m, n)a(m', n')|\phi^{(N)}\rangle &= a^\dagger(m, n) \sum_{k=1}^{N} \eta^{k-1}\langle m'|k\rangle \sum_{j<k} \eta^{j-1}\langle n'|j\rangle|\not{j}, \not{k}\rangle \\ &\quad + a^\dagger(m, n) \sum_{k=1}^{N} \eta^{k-1}\langle m'|k\rangle \sum_{j>k} \eta^{j-2}\langle n'|j\rangle|\not{k}, \not{j}\rangle \\ &= \sum_{k=1}^{N} \eta^{k-1}\langle m'|k\rangle \sum_{j<k} \eta^{j-1}\langle n'|j\rangle|m, n, \not{j}, \not{k}\rangle \\ &\quad + \sum_{k=1}^{N} \eta^{k-1}\langle m'|k\rangle \sum_{j>k} \eta^{j-2}\langle n'|j\rangle|m, n, \not{k}, \not{j}\rangle. \end{aligned}$$

From the above equations, we obtain

$$a(m', n')a^\dagger(m, n) - a^\dagger(m, n)a(m', n') = \langle m'|m\rangle\langle n'|n\rangle + \eta\langle m'|n\rangle\langle n'|m\rangle,$$

which is the first rule of equation (3.27). Similarly, the relations in the second line of equation (3.27) can be straightforwardly proved.

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