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Covering by discrete and closed discrete sets ☆

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ABSTRACT

Say that a cardinal number κ is *small* relative to the space X if $\kappa < \Delta(X)$, where $\Delta(X)$ is the least cardinality of a non-empty open set in X. We prove that no Baire metric space can be covered by a small number of discrete sets, and give some generalizations. We show a ZFC example of a regular Baire σ-space and a consistent example of a normal Baire Moore space which can be covered by a small number of discrete sets. We finish with some remarks on linearly ordered spaces.

1. Introduction

We will assume all spaces to be Hausdorff. *Crowded* is Eric van Douwen's apt name for a space without isolated points. All undefined notions can be found in [3,6,8]. Let $dis(X)$ be the least number of discrete sets required to cover the space X. The cardinal function $dis(X)$ is introduced by Juhász and van Mill in [9], where the authors provide some lower bounds for $dis(X)$ and ask whether $dis(X) \geqslant \mathfrak{c}$, for any crowded compact space X. Gruenhage [5] shows that this is the case, by proving that $dis(X)$ cannot be raised by perfect mappings. In [10] Juhász and Szentmiklóssy prove that if X is a compact space such that $\chi(x, X) \geqslant \kappa$ for every $x \in X$, then $dis(X) \geqslant 2^\kappa$, thus generalizing both Gruenhage's result and the classical Čech–Pospišil theorem (in which the cardinality of X takes the place of $dis(X)$). Let $\Delta(X)$ be the *dispersion character of* X, that is, the least cardinality of a non-empty open set in X. Since in a compact space where every point has character at least κ we have $\Delta(X) \geqslant 2^\kappa$, Juhász and Szentmiklóssy ask the following natural question.

Question 1.1. (See [10].) Is $dis(X) \geqslant \Delta(X)$ for any compact space X?

Our work on the above question led us to investigate for what kind of Baire spaces, other than the compact ones, Juhász and Szentmiklóssy's inequality could be true. In this note we prove that $dis(X) \geqslant \Delta(X)$ for two classes of Baire generalized metric spaces which satisfy a mild separation-type property. Moreover, we construct examples of very good Baire spaces for which $dis(X) < \Delta(X)$.

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In the last section we prove that $dis(X) = |X|$ for every locally compact Lindelöf linearly ordered space (LOTS) and show an example of a hereditarily paracompact Baire LOTS for which the gap between $dis(X)$ and $\Delta(X)$ can be arbitrarily large.

2. Generalized metric spaces

Given a collection $\mathcal{G}$ of subsets of X, set $st(x, \mathcal{G}) = \bigcup\{G \in \mathcal{G}\colon x \in G\}$ and $ord(x, \mathcal{G}) = |\{G \in \mathcal{G}\colon x \in G\}|$. Recall that a sequence $\{\mathcal{G}_n\colon n \in \omega\}$ of open covers of X is said to be a *development* if $\{st(x, \mathcal{G}_n)\colon n \in \omega\}$ is a local base at x for every $x \in X$. A space is called *developable* if it admits a development. A regular developable space is called a *Moore space*.

Definition 2.1. Let κ be a cardinal. We call a space κ*-expandable* if every closed discrete set expands to a collection of open sets $\mathcal{G}$ such that $ord(x, \mathcal{G}) \leqslant \kappa$ for every $x \in X$.

The following theorem is new even for all complete metric spaces.

Theorem 2.2. *Let X be a Baire ω_1-expandable developable space. Then $dis(X) \geqslant \Delta(X)$.*

Proof. Fix a development $\{\mathcal{G}_n\colon n \in \omega\}$ for X and suppose by contradiction that $\tau = dis(X) < \Delta(X)$. Since the inequality $dis(X) \geqslant \omega_1$ is true for every crowded Baire space X we can assume that $\tau \geqslant \omega_1$. Set $X = \bigcup_{\alpha<\tau} D_\alpha$, where each D_α is discrete. Define $D_{\alpha,n} = \{x \in D_\alpha\colon st(x, \mathcal{G}_n) \cap D_\alpha = \{x\}\}$ and set $X_n = \bigcup_{\alpha\in\tau} D_{\alpha,n}$.

Claim. *For every $x \in X_k$ there is a neighbourhood G of x such that $|G \cap X_k| \leqslant \tau$.*

Proof. Let $G \in \mathcal{G}_k$ be such that $x \in G$. Then G hits each $D_{\alpha,k}$ in at most one point: indeed, if $y, z \in G \cap D_{\alpha,k}$ with $y \neq z$, we would have both $st(y, \mathcal{G}_k) \cap D_{\alpha,k} = \{y\}$ and $z \in st(y, \mathcal{G}_k) \cap D_{\alpha,k}$, which is a contradiction. □

Now $X = \bigcup_{n\in\omega} X_n$, so, by the Baire property of X, there is $k \in \omega$ such that $U \subset \overline{X_k}$ for some non-empty open set U. By the claim we can assume that $|U \cap X_k| \leqslant \tau$. So $|U \cap (\overline{X_k} \setminus X_k) \cap D_{\alpha,j}| > \tau$ for some $\alpha < \tau$ and $j \in \omega$.

Notice that the set $D_{\alpha,j}$ is actually closed discrete: indeed suppose $y \notin D_{\alpha,j}$ were some limit point. Let $V \in \mathcal{G}_j$ be a neighbourhood of y and pick two points $z, w \in V \cap D_{\alpha,j}$. By definition of $D_{\alpha,j}$ we have $st(z, \mathcal{G}_j) \cap D_{\alpha,j} = \{z\}$. But $w \in V \subset st(z, \mathcal{G}_j)$, which leads to a contradiction.

Observe now that also $S := U \cap (\overline{X_k} \setminus X_k) \cap D_{\alpha,j}$ is closed discrete and hence we can expand it to a collection $\mathcal{U} = \{U_x\colon x \in S\}$ of open sets such that $ord(y, \mathcal{U}) \leqslant \omega_1$ for every $y \in X$. Set $V_x = U_x \cap st(x, \mathcal{G}_j) \cap U$ and observe that $V_x \neq V_y$ whenever $x \neq y$ and if we put $\mathcal{V} = \{V_x\colon x \in S\}$ then we also have that $ord(y, \mathcal{V}) \leqslant \omega_1$ for every $y \in X$. For every $x \in S$ pick $f(x) \in V_x \cap X_k$: the mapping f has domain of cardinality $> \tau$, range of cardinality $\leqslant \tau$ and fibers of cardinality $\leqslant \omega_1$, which is a contradiction. □

Corollary 2.3. *$dis(X) \geqslant \Delta(X)$, for every Baire collectionwise Hausdorff (or meta-Lindelöf) developable space X.*

Corollary 2.4. *$dis(X) \geqslant \Delta(X)$, for every Baire metric space X.*

Recall that a *network* is a collection $\mathcal{N}$ of subsets of a topological space such that for every open set $U \subset X$ and every $x \in U$ there is $N \in \mathcal{N}$ with $x \in N \subset U$. A σ*-space* is a space having a σ-discrete network.

Our next aim is proving that $dis(X) \geqslant \Delta(X)$ for every regular Baire ω_1-expandable σ-space. We could give a more direct proof, but we feel that the real explanation for that is the following probably folklore fact, a proof of which can be found in [2].

Lemma 2.5. *Every regular Baire σ-space has a dense metrizable G_δ subspace.*

Call $dis^*(X)$ the least number of *closed discrete* sets required to cover X. Clearly $dis(X) \leqslant dis^*(X)$. In a σ-space, one can use a σ-discrete network to split every discrete set into a countable union of closed discrete sets. So the following lemma is clear.

Lemma 2.6. *If X is a crowded σ-space then $dis(X) = dis^*(X)$.*

The next lemma and its proof are essentially due to the anonymous referee.

Lemma 2.7. *Let X be an ω_1-expandable crowded Baire space such that $dis^*(X) \leqslant \kappa$, and $A \subset X$ with $|A| \leqslant \kappa$. Then $|\overline{A}| \leqslant \kappa$.*

Proof. Since X is Baire crowded we can assume that $\kappa \geqslant \omega_1$. Let $X = \bigcup_{\alpha<\kappa} D_\alpha$, where each D_α is closed discrete. Let $B_\alpha = \overline{A} \cap D_\alpha$. Then B_α is closed discrete, so we may expand it to a family of open sets $\mathcal{U}_\alpha$ such that $ord(x, \mathcal{U}_\alpha) \leqslant \omega_1$ for every $x \in X$. Then $|\mathcal{U}_\alpha| = |B_\alpha|$ and for all $U \in \mathcal{U}_\alpha$, $U \cap A \neq \emptyset$. Fix some well-ordering of A and define a function $f : \mathcal{U}_\alpha \to A$ by:

$$f(U) = \min\{a \in A\colon\ a \in U\}.$$

We have that $|f^{-1}(a)| \leqslant \aleph_1$ for every $a \in A$, and therefore $|B_\alpha| = |\mathcal{U}_\alpha| \leqslant |A| \cdot \aleph_1 \leqslant \kappa$.

Since $\overline{A} = \bigcup_{\alpha\in\kappa} B_\alpha$ it follows that $|\overline{A}| \leqslant \kappa$. □

The statement of the next theorem is due to the anonymous referee, and improves our original theorem where X was assumed to be paracompact.

Theorem 2.8. *Let X be a regular ω_1-expandable Baire σ-space. Then $dis(X) \geqslant \Delta(X)$.*

Proof. Fix some dense metrizable G_δ subspace $M \subset X$ and suppose by contradiction that $dis^*(X) = dis(X) < \Delta(X)$. Then Lemma 2.7 implies that $\Delta(M) \geqslant \Delta(X)$ and, since M is Baire metric, by Corollary 2.4 we have $dis(X) \geqslant dis(M) \geqslant \Delta(M)$. So $dis(X) \geqslant \Delta(X)$, and we are done. □

Corollary 2.9. *For every paracompact Baire σ-space X (in particular, for every stratifiable Baire space), we have $dis(X) \geqslant \Delta(X)$.*

Notice that in the proofs of Theorems 2.2 and 2.8 all one needs is that X be $dis(X)$-expandable.

Also, while we did not use any separation other than Hausdorff in Theorem 2.2, regularity seems to be essential in Theorem 2.8, since one needs a σ-discrete network consisting of closed sets to prove Lemma 2.5. This suggests the following question.

Question 2.10. Is there a collectionwise Hausdorff or meta-Lindelöf (non-regular) Baire σ-space X such that $dis(X) < \Delta(X)$?

3. Good spaces with bad covers

We now offer two examples to show that ω_1-expandability is essential in Theorem 2.8. The first one is a modification of an example of Bailey and Gruenhage [1]. We will need the following combinatorial fact which slightly generalizes Lemma 9.23 of [8]. It must be well known, but we include a proof anyway since we could not find a reference to it.

Lemma 3.1. *Let κ be any infinite cardinal. There is a family $\mathcal{A} \subset [\kappa]^{cf(\kappa)}$ of cardinality κ^+ such that $|A \cap B| < cf(\kappa)$ for every $A, B \in \mathcal{A}$.*

Proof. We begin by showing that there is a family $\mathcal{F}$ of functions from $cf(\kappa)$ to κ such that $|\mathcal{F}| = \kappa^+$ and $|\{\alpha \in cf(\kappa)\colon f(\alpha) = g(\alpha)\}| < cf(\kappa)$, for any $f, g \in \mathcal{F}$. Indeed, suppose we have constructed $\{f_\alpha\colon\ \alpha < \kappa\}$ with the stated property. Let $\kappa = \sup_{\alpha<cf(\kappa)} \kappa_\alpha$. Define $f : cf(\kappa) \to \kappa$ in such a way that $f(\tau) \neq f_\alpha(\tau)$, for every $\alpha < \kappa_\tau$ and $\tau \in cf(\kappa)$. Fix $\alpha \in \kappa$: if $\tau < cf(\kappa)$ is such that $f(\tau) = f_\alpha(\tau)$ we must have $\kappa_\tau \leqslant \alpha < \kappa$. Hence $|\{\tau \in cf(\kappa)\colon\ f(\tau) = f_\alpha(\tau)\}| < cf(\kappa)$.

Now for $\mathcal{A}$ we can take (on $cf(\kappa) \times \kappa$) the family of graphs of functions in $\mathcal{F}$. □

Example 3.2. (ZFC) A regular Baire σ-space P for which $dis(P) < \Delta(P)$.

Proof. Fix an almost disjoint family $\mathcal{A} \subset [\mathfrak{c}]^{cf(\mathfrak{c})}$ such that $|\mathcal{A}| = \mathfrak{c}^+$. For every partial function $\sigma \in \mathfrak{c}^{<\omega}$ such that $dom(\sigma) = k$ let $\mathcal{L}_\sigma = \{f_{\sigma,A}\colon\ A \in \mathcal{A}\}$ where $f_{\sigma,A} : cf(\mathfrak{c}) \to \mathfrak{c}^{<\omega}$ is defined as follows: $dom(f_{\sigma,A}(\alpha)) = k + 1$, $f_{\sigma,A}(\alpha) \restriction k = \sigma$ for every $\alpha \in cf(\mathfrak{c})$ and $\{f_{\sigma,A}(\alpha)(k)\colon\ \alpha \in cf(\mathfrak{c})\}$ is a faithful enumeration of A.

When $f \in \mathcal{L}_\sigma$ we will refer to $\rho_f = \sigma$ as the *root* of f, and set $k_f = dom(\sigma)$.

Let now $L = \bigcup_{\sigma\in\mathfrak{c}^{<\omega}} \mathcal{L}_\sigma$ and $B = \mathfrak{c}^\omega$. We are going to define a topology on $P = B \cup L$ that induces on B its natural topology. For every $\sigma \in \mathfrak{c}^{<\omega}$, let $[\sigma] = \{g \in B\colon\ g \supset \sigma\}$ and

$$B(\sigma) = [\sigma] \cup \{f \in L\colon\ \rho_f \supseteq \sigma\}.$$

Let $\{A_n\colon\ n \in \omega\}$ be a partition of $\mathfrak{c}$ into sets of cardinality $\mathfrak{c}$.

For $f \in L$, $\delta \in cf(\mathfrak{c})$ and $k \in \omega$ let

$$B_{\delta,k}(f) = \{f\} \cup \bigcup_{\gamma>\delta} \left\{ B\big(f(\gamma)\big)\colon\ f(\gamma)(k_f) \in \bigcup_{n>k} A_n \right\}.$$

The set $\mathcal{B} = \{B(\sigma), B_{\delta,k}(f)\colon\ \sigma \in \mathfrak{c}^{<\omega},\ \delta \in cf(\mathfrak{c}),\ k \in \omega\}$ is a base for a topology on P, as items (2) and (3) in the following list of claims show.

(1) For $\sigma_1, \sigma_2 \in \mathfrak{c}^{<\omega}$, $B(\sigma_1) \cap B(\sigma_2) = \emptyset$ if and only if σ_1 and σ_2 are incompatible.
(2) Suppose $B(\sigma) \cap B_{\delta,k}(f) \neq \emptyset$. Then $\sigma \subseteq \rho_f$ or $\rho_f \subseteq \sigma$. If $\sigma \subseteq \rho_f$ then $B(\sigma) \cap B_{\delta,k}(f) = B_{\delta,k}(f)$. If $\sigma \supsetneq \rho_f$, then the intersection is $B(\sigma)$.
(3) If $B_{\delta,j}(f) \cap B_{\delta',k}(g) \neq \emptyset$ and $\rho_g \subsetneq \rho_f$ then the intersection is either $B_{\delta,j}(f)$ or a set of the form $B(\sigma)$, for some $\sigma \in \{f(\gamma), g(\gamma')\colon\ \gamma > \delta,\ \gamma' > \delta'\}$.
(4) If $B_{\delta,j}(f) \cap B_{\delta',k}(g) \neq \emptyset$ and $\rho_g = \rho_f$ then the intersection is a union of less than $cf(\mathfrak{c})$ sets of the form $B(\sigma)$ where $\sigma \in ran(f) \cap ran(g)$.

Proof of items (1)–(4). Item (1) is easy. For item (2), observe that $B_{\delta,k}(f) \subseteq B(\rho_f)$, so $B(\rho_f) \cap B(\sigma) \neq \emptyset$ which implies that ρ_f and σ are compatible. If $\sigma \subseteq \rho_f$ then for each $\gamma > \delta$ we have $\sigma \subseteq f(\gamma)$ and $f \in B(\sigma)$, so $B_{\delta,k}(f) \subseteq B(\sigma)$.

If $\sigma \supsetneq \rho_f$ then let $\gamma > \delta$ be the unique ordinal such that $B(\sigma) \cap B(f(\gamma)) \neq \emptyset$. Since σ and $f(\gamma)$ are compatible we must have $f(\gamma) \subset \sigma$, from which $B(\sigma) \subset B(f(\gamma))$ follows, and hence the claim.

To prove item (3) observe that if $B_{\delta,j}(f) \cap B_{\delta',k}(g) \neq \emptyset$ and $\rho_g \subsetneq \rho_f$ then $g \notin B_{\delta,j}(f)$ and, as the range of f consists of pairwise incompatible elements we have that $[g(\tau)] \cap [\rho_f] \neq \emptyset$ for at most one $\tau \in cf(\mathfrak{c})$. Therefore, $B_{\delta,j}(f) \cap B_{\delta',k}(g) = B(g(\tau)) \cap B_{\delta,j}(f)$, and the rest follows from item (2).

Item (4) follows from *almost-disjointness* of the ranges. □

Claim 1. *The base $\mathcal{B}$ consists of clopen sets.*

Proof. To see that $B_{\delta,j}(f)$ is closed pick $g \in L \setminus B_{\delta,j}(f)$ and let γ be large enough so that $f \notin B_{\gamma,j}(g)$. Suppose that $B_{\delta,j}(f) \cap B_{\gamma,j}(g) \neq \emptyset$. Then there are $\alpha > \delta$ and $\beta > \gamma$ such that $f(\alpha)$ and $g(\beta)$ are compatible. Now we must have $\rho_g = \rho_f$ or otherwise we would have either $\rho_f \supset g(\beta)$ and hence $f \in B_{\gamma,j}(g)$, or $\rho_g \supset f(\alpha)$, which would imply $g \in B_{\delta,j}(f)$. So, by item (4) we have $B_{\delta,j}(f) \cap B_{\gamma,j}(g) = \bigcup_{\tau \in C} B(g(\tau))$ where $|C| < cf(\mathfrak{c})$ and hence, if we let $\theta > \sup(C)$, then $B_{\theta,j}(g) \cap B_{\delta,j}(f) = \emptyset$.

Now, let $p \in B \setminus B_{\delta,j}(f)$ and $i = k_f + 2$. We claim that $B(p \restriction i) \cap B_{\delta,j}(f) = \emptyset$. Indeed, if that were not the case then $f(\gamma)$ and $p \restriction i$ would be compatible, for some γ. So $f(\gamma) \subset p \restriction i \subset p$, which implies $p \in B_{\delta,j}(f)$, contradicting the choice of p.

To see that $B(\sigma)$ is clopen, observe that B is dense in P and the subspace base is clopen, so we can restrict our attention to limit points of $B(\sigma)$ in L. Suppose that $f \in L \setminus B(\sigma)$ is some limit point, then, for all $\delta \in cf(\mathfrak{c})$ and all $j \in \omega$ we have $B_{\delta,j}(f) \cap B(\sigma) \neq \emptyset$. So ρ_f and σ are compatible; moreover $\rho_f \subsetneq \sigma$ or otherwise $f \in B(\sigma)$. Now there is at most one δ' such that $f(\delta')$ and σ are compatible, whence the absurd statement $B_{\delta'+1,0}(f) \cap B(\sigma) = \emptyset$. □

Claim 2. *P is a σ-space.*

Proof. For each $\sigma \in \mathfrak{c}^{<\omega}$ let $h(\sigma) \in \omega^{<\omega}$ be defined by $\sigma(i) \in A_j$ iff $h(\sigma)(i) = j$. For every $s \in \omega^{<\omega}$ put $\mathcal{B}_s = \{B(\sigma)\colon h(\sigma) = s\}$. We claim that $\mathcal{B}_s$ is a discrete collection of open sets. Notice that the elements of $\mathcal{B}_s$ are all disjoint. Now if $x \in B \setminus \bigcup \mathcal{B}_s$, let $j = dom(s)$; then either $x \restriction (j+1)$ extends (at most) one σ such that $h(\sigma) = s$ or $x \restriction (j+1)$ is incompatible with every such σ. So $B(x \restriction (j+1))$ will hit at most one element of $\mathcal{B}_s$. If $f \in L$ then let $l = \max(ran(s))$: we claim that $B_{0,l}(f)$ hits at most one element of $\mathcal{B}_s$. Indeed, for fixed α such that $f(\alpha)(k_f) \in \bigcup_{n>l} A_n$ either $f(\alpha)$ is incompatible with every σ such that $h(\sigma) = s$ or there is exactly one such σ which is compatible with $f(\alpha)$. In the latter case we cannot have $\sigma \supset \rho_f$ because $f(\alpha)(k_f) \notin ran(s)$, hence we have $\sigma \subset \rho_f$, which implies $B_{0,l}(f) \subset B(\sigma)$.

Now we claim that L is a σ-closed discrete set. Indeed, for every $s \in \omega^{<\omega}$, set $L_s = \{f \in L\colon\ h(\rho_f) = s\}$. If $g \in L_s$ then every fundamental neighbourhood of g hits L_s in the single point g. If $g \notin L_s$ then either ρ_g is incompatible with every ρ_f such that $f \in L_s$, in which case every fundamental neighbourhood of g misses L_s, or there is $f \in L_s$ such that ρ_g and ρ_f are compatible. If $\rho_g \subsetneq \rho_f$ then let $l = s(k_g)$: we have $B_{0,l}(g) \cap L_s = \emptyset$. If $\rho_f \subset \rho_g$, then the root of every function of L which is in a fundamental neighbourhood of g has domain strictly larger than $dom(s)$ and hence every fundamental neighbourhood of g misses L_s. □

Observe now that P is Baire, because $B \subset P$ is a dense Baire subset. Also, $dis(P) = \mathfrak{c} < \mathfrak{c}^+ = \Delta(P)$. □

One of the properties of Bailey and Gruenhage's example that was lost in the modification is first-countability. This suggests the following question.

Question 3.3. Is there in ZFC a first-countable regular σ-space X for which $dis(X) < \Delta(X)$?

The reason why we insist on a ZFC example, is that we already have a consistent answer to the previous question. In fact, the space we are now going to exhibit is first-countable, normal and shows that ω_1-expandability cannot be weakened to ω_2-expandability in Theorem 2.2. Our original motivation for constructing this example was showing that paracompactness could not be weakened to normality in Corollary 2.9.

Recall that a *Q-set* is an uncountable subset of a Polish space whose every subset is a relative F_σ, and a *Luzin set* is an uncountable subset of a Polish space P which meets every first category set of P in a countable set. The existence of Q-sets and Luzin sets in the reals is known to be independent of ZFC (see, for example, [12]). Fleissner and Miller [4] constructed a model of ZFC where there are a Q-set of the reals of cardinality $\aleph_2$ and a Luzin set of the reals of cardinality $\aleph_1$.

Lemma 3.4. *Let C be some Polish space having a base $\mathcal{B} = \{B_n: n \in \omega\}$ such that B_n is homeomorphic to C for every $n \in \omega$. Given a Q-set of cardinality $\aleph_2$ in C, there is one which is dense and has dispersion character $\aleph_2$. Given a Luzin set in C, there is one which is locally uncountable and dense.*

Proof. Let X be a Q-set in C. Let $\mathcal{B}' = \{B \in \mathcal{B}: |B \cap X| < \aleph_2\}$. Then $Y = X \setminus \bigcup \mathcal{B}'$ is a Q-set such that $\Delta(Y) = \aleph_2$. Set $n_0 = 0$ and let Z_0 be a homeomorphic copy of Y inside B_{n_0}. Set $Z = Z_0$ and let n_1 be the least integer such that $B_{n_1} \cap Z = \emptyset$: clearly $n_1 > n_0$. Now let $Z_1 \subset B_{n_1}$ be a homeomorphic copy of Y and set $Z = Z_0 \cup Z_1$. Now suppose you have constructed a Q-set Z such that $Z \cap B_i \neq 0$ for every $1 \leqslant i \leqslant n_{k-1}$ and let n_k be the least integer such that $Z \cap B_{n_k} = \emptyset$; let $Z_k \subset B_{n_k}$ be a homeomorphic copy of Y into B_{n_k}. At the end of the induction let $Z = \bigcup_{n \in \omega} Z_n$, then Z is a Q-set with the stated properties. The second statement is proved in a similar way. □

Example 3.5. A normal Baire Moore space X for which $dis(X) < \Delta(X)$.

Proof. Take a model of ZFC where there are a Luzin set $L' \subset \mathbb{R}$ and a Q-set $Z \subset \mathbb{R}$ with the properties stated in Lemma 3.4. Let $f: \mathbb{R} \setminus \mathbb{Q} \to (\mathbb{R} \setminus \mathbb{Q})^2$ be any homeomorphism. Then $L = f(L' \setminus \mathbb{Q})$ is a Luzin subset of $(\mathbb{R} \setminus \mathbb{Q})^2$, and by Lemma 3.4 we can assume that it is locally uncountable and dense. Let $\mathbb{Q} = \{q_n: n \in \omega\}$ be an enumeration and set $Z_n = Z \times \{q_n\}$. Set $T = \bigcup_{n \in \omega} Z_n$ and define a topology on $X = L \cup T$ as follows: points of L have neighbourhoods just as in the Euclidean topology on the plane, while a neighbourhood of a point of $x \in Z_n$ is a disk tangent at x to Z_n, and lying in the upper half plane relative to that line. Notice that L is dense in X so X is a Baire space. Moreover $\Delta(X) = \aleph_2 > \aleph_1 = dis(X)$.

To prove that X is normal let H and K be disjoint closed sets. It will be enough to show that H has a countable open cover, such that the closure of every member of it misses K (see Lemma 1.1.15 of [3]). Fix $n \in \omega$. We have $H \cap Z_n = \bigcup_{j \in \omega} H_j$, where H_j is closed in the Euclidean topology on Z_n for every $j \in \omega$. Fix $j \in \omega$. For each $x \in H_j$ let $D(x, r_x)$ be a disk tangent to Z_n at x such that $D(x, r_x) \cap K = \emptyset$ and $r_x = \frac{1}{k}$ for some $k \in \omega$. Let $U = \bigcup_{x \in H_j} D(x, r_x)$. First of all, we claim that no point of $K \cap Z_n$ is in $\overline{U}$: indeed if $x \in K \cap Z_n$ then let I_x be an interval containing x and missing H_j, then the closest that a point of H_j can come to x is one of the endpoints of I_x so there is room enough to separate x from U by a tangent disk.

Now $U = \bigcup_{n \in \omega} U_n$, where $U_n = \bigcup \{D(x, r_x): r_x = \frac{1}{n}\}$. Let $V_n = \bigcup \{D(x, \frac{r_x}{2}): r_x = \frac{1}{n}\}$. We claim that $\overline{V}_n \cap K \setminus Z_n = \emptyset$: indeed, if some point $x \in K \setminus Z_n$ were limit for V_n then we would have a sequence of disks of radius $\frac{1}{2n}$ clustering to it. But then $x \in U_n$, which contradicts $U \cap K = \emptyset$.

To separate points of $H \setminus T$ from K just choose for each such point an open set whose closure misses K and use second countability of L. That shows how to define the required countable open cover of H.

Finally, a development for X is provided by $\mathcal{G}_n = \{D(x, n): x \in X\}$ where $D(x, n) = B(x, \frac{1}{n}) \setminus \bigcup_{i<n} Z_i$ if $x \in L$, while if $x \notin L$, $D(x, n)$ is a tangent disk of radius less than $\frac{1}{n}$ which misses $\bigcup \{Z_i: i < n \text{ and } x \notin Z_i\}$. □

The cardinal $\aleph_2$ can be replaced by any cardinal not greater than $\mathfrak{c}$, under proper set theoretic assumptions (see [4]). So the previous example shows that the gap between $dis(X)$ and $\Delta(X)$ for normal Baire Moore spaces can be as big as the gap between the first uncountable cardinal and the continuum.

Since normal Moore spaces are, consistently, metrizable, there is no chance of getting in ZFC a space with all the properties of Example 3.5. Nevertheless, the following question remains open.

Question 3.6. Is there in ZFC a normal Baire σ-space X for which $dis(X) < \Delta(X)$?

Using a Q-set on a tangent disk space to get normality is an old trick (see for example [14]). Also, to get a regular Baire Moore space X for which $dis(X) < \Delta(X)$ it actually suffices to assume the negation of CH along with the existence of a Luzin set.

A potential way of weakening the set theoretic assumption in Example 3.5 would be to replace *Luzin set* with *Baire subset of cardinality* $\aleph_1$, but even such an object would be inconsistent with MA + ¬CH, while the presence of CH would make the whole construction worthless, so we have no clue even about the following.

Question 3.7. Is there, at least under MA + ¬CH or under CH, a normal Baire σ-space X for which $dis(X) < \Delta(X)$?

Also, notice that no regular Baire σ-space X for which $dis(X) < \Delta(X)$ can be separable under CH. That is because any regular separable space with points G_δ has cardinality $\leqslant \mathfrak{c}$. (Fix any dense countable set D, then, the map taking any closed neighbourhood to its intersection with D is one-to-one. So there are no more than $\mathfrak{c}$ closed neighbourhoods in the space, but every point in a regular space with G_δ points is the intersection of countably many closed neighbourhoods.) Thus $dis(X) = \aleph_1 \geqslant \Delta(X)$ if CH holds.

4. Linearly ordered spaces

Recall that a space is called a *GO space* if it embeds in a LOTS. We denote by $m(X)$ the minimum number of metrizable spaces needed to cover X. The following result is due to Ismail and Szymanski.

Lemma 4.1. *(See [7].) Let X be a locally compact Lindelöf GO space. Then $w(X) \leqslant \omega \cdot m(X)$.*

Theorem 4.2. *Let X be a locally compact Lindelöf GO space. Then $dis(X) = |X|$.*

Proof. Suppose by contradiction that there exists $\lambda < |X|$ such that $X = \bigcup\{D_\alpha\colon \alpha \in \lambda\}$ where each D_α is discrete. Then $|D_\alpha| \leqslant w(X) \leqslant \omega \cdot m(X) \leqslant \lambda$, for every $\alpha \in \lambda$. So $|X| \leqslant \sup\{|D_\alpha|\colon \alpha \in \lambda\} \cdot \lambda \leqslant \lambda < |X|$. □

In the previous theorem we cannot weaken locally compact Lindelöf to paracompact Baire, as the following example shows. Recall that a space is called *non-archimedean* if it has a base such that any two elements are either disjoint or one is contained in the other. Every non-archimedean space has a base which is a tree under reverse inclusion (see [13]), and from this it is easy to see that it is (hereditarily) paracompact.

Example 4.3. There is a Baire non-archimedean (and hence hereditarily paracompact) LOTS X such that $dis(X) < \Delta(X)$.

Proof. Let κ and λ be infinite cardinals such that $cf(\kappa) \leqslant \lambda$ but $\lambda < \kappa$. Let $\mathbb{W} = \{-1\} \cup \kappa$. Define an order on $\mathbb{W}$ by declaring -1 to be less than every ordinal. Let $X = \{f \in \mathbb{W}^{\lambda^+}\colon supp(f) < \lambda^+\}$, where $supp(f) = \min\{\gamma < \lambda^+\colon f(\alpha) = 0$ for every $\alpha \geqslant \gamma\}$. Now take the topology induced on X by the lexicographic order.

Claim 1. *X is a strong Choquet space (and hence Baire).*

Proof. We are going to describe a winning strategy for player II in the strong Choquet game (see [11]). In his first move player I chooses any open set B_1 and a point $f_1 \in B_1$. Player II then chooses points $a_1, b_1 \in X$ such that $f_1 \in (a_1, b_1) \subset B_1$. Let now $\alpha_1 = \max\{supp(f_1), supp(a_1), supp(b_1)\}$ and $\overline{f_{\alpha_1}} = (f_1(\gamma)\colon 0 \leqslant \gamma < \alpha_1)$. Define $f_1^- = \overline{f_{\alpha_1}}{}^\frown(-1, 0, \ldots, 0)$ and $f_1^+ = \overline{f_{\alpha_1}}{}^\frown(1, 0, \ldots, 0)$.

Clearly $a_1 < f_1^- < f_1 < f_1^+ < b_1$. Now in her first move player II chooses the open set $A_1 = (f_1^-, f_1^+)$.

Player I responds by choosing any open set $B_2 \subset A_1$ and a point $f_2 \in B_2$. Player II proceeds as before. Notice that f_{n+1} thus constructed agrees with f_n up to α_n and that the point $h = (\bigcup \overline{f_{\alpha_n}})^\frown(0, 0, \ldots, 0)$ is in $\bigcap_{n \geqslant 1} A_n$. So II has a winning strategy. □

Claim 2. *X is the union of λ^+ many discrete sets.*

Proof. For every $\alpha \in \lambda^+$, let $D_\alpha = \{f \in X\colon supp(f) = \alpha\}$. Then $X = \bigcup_{\alpha \in \lambda^+} D_\alpha$ and each D_α is discrete. Indeed, let $f \in D_\alpha$ and define:

$$f^-(\beta) = \begin{cases} f(\beta) & \text{if } \beta < \alpha, \\ -1 & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta > \alpha. \end{cases} \tag{1}$$

Similarly define:

$$f^+(\beta) = \begin{cases} f(\beta) & \text{if } \beta < \alpha, \\ 1 & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta > \alpha. \end{cases} \tag{2}$$

Then $(f^-, f^+) \cap D_\alpha = \{f\}$. □

Claim 3. *X is non-archimedean.*

Proof. Let $\mathcal{B} = \{[\sigma]\colon \sigma \in \mathbb{W}^\alpha$ for some $\alpha \in \lambda^+\}$, where $[\sigma] = \{f \in X\colon \sigma \subset f\}$. Then $\mathcal{B}$ is a basis for our space. Every element of $\mathcal{B}$ is open: indeed, if $f \in [\sigma]$ then let $\alpha = \max\{dom(\sigma), supp(f)\}$ and f^+ and f^- be defined as in the proof of Claim 2. Then $f \in (f^-, f^+) \subset [\sigma]$.

Now let $c \in (a, b)$. Then there are ordinals α and β such that $a(\alpha) < c(\alpha)$, $c(\beta) < b(\beta)$, while $a(\gamma) = c(\gamma)$ and $c(\tau) = b(\tau)$ for every $\gamma < \alpha$ and every $\tau < \beta$. Set $\theta = \max\{\alpha, \beta\} + 1$. We have that $[c \upharpoonright \theta] \subset (a, b)$.

Now given two elements of $\mathcal{B}$, either one is contained in the other, or they are disjoint. Therefore X is non-archimedean. □

To complete the proof observe that $\Delta(X) \geqslant \kappa^\lambda > \kappa > \lambda^+ \geqslant dis(X)$. $\square$

Since for fixed λ there are arbitrarily big cardinals κ having cofinality λ, the former example shows that the gap between $dis(X)$ and $\Delta(X)$ can be arbitrarily big for hereditarily paracompact Baire LOTS.

Notice that the Lindelöf number of the previous space is $\geqslant \kappa$, in particular X is never Lindelöf.

Question 4.4. Is $dis(X) \geqslant \Delta(X)$ true for every (Lindelöf, hereditarily paracompact) Čech complete LOTS X?

Finally, we would like to mention that we recently applied our result on metric spaces to give several partial answers to Juhász and Szentmiklóssy's original question about compact spaces. They will be the subject of another paper.

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