

P-SPACES AND THE WHYBURN PROPERTY

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ABSTRACT. We investigate the Whyburn and weakly Whyburn property in the class of P -spaces, that is spaces where every countable intersection of open sets is open. We construct examples of non-weakly Whyburn P -spaces of size continuum, thus giving a negative answer under CH to a question of Pelant, Tkachenko, Tkachuk and Wilson ([13]). In addition, we show that the weak Kurepa Hypothesis (an assumption weaker than CH) implies the existence of a non-weakly Whyburn P -space of size \aleph_2 . Finally, we consider the behavior of the above-mentioned properties under products; we show in particular that the product of a Lindelöf weakly Whyburn P -space and a Lindelöf Whyburn P -space is weakly Whyburn, and we give a consistent example of a non-Whyburn product of two Lindelöf Whyburn P -spaces.

1. INTRODUCTION

We assume all spaces to be Hausdorff. All undefined notions can be found in [6] and [8].

Whyburn and weakly Whyburn spaces have recently been considered by various authors. They provide a natural generalization of Fréchet and sequential spaces which is weak enough to offer lots of challenges, and indeed some fundamental questions are still open. They have also turned out to be useful in the $M_3 = M_1$ problem (see [7] and [19]).

Let X be a topological space. A set $F \subseteq X$ is said to be an *almost closed* converging to x if $\overline{F} \setminus F = \{x\}$; if this happens we will write $F \rightarrow x$. A space X is said to be *Whyburn* (*weakly Whyburn*) if for each non closed set $A \subseteq X$ and each (some) point $x \in \overline{A} \setminus A$ there is an almost closed set $F \subseteq A$ such that $F \rightarrow x$.

Fréchet spaces are Whyburn and sequential spaces are weakly Whyburn. The class of weakly Whyburn spaces is wide enough to include all regular scattered spaces.

Recall that a space is radial (pseudoradial) if for each $A \subseteq X$ and every (some) point $x \in \overline{A} \setminus A$ there is a transfinite sequence $\{x_\alpha : \alpha \in \kappa\} \subseteq A$ which converges to x . The radial character of a pseudoradial space X is the smallest cardinal κ such that the definition of pseudoradiality for X works by considering sequences of length at most κ .

The (weakly) Whyburn property behaves very nicely on compact spaces; namely, every countably compact Whyburn space is Fréchet and every compact weakly Whyburn space is pseudoradial.

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A space is called a *P-space* if every G_δ set is open. A Lindelöf space is one in which every open cover has a countable subcover. Lindelöf *P*-spaces exhibit a behaviour that is somewhat very close to that of compact spaces; for example, every Lindelöf subspace of a Lindelöf *P*-space is closed and every Lindelöf Hausdorff *P*-space is normal. Notice, however, that the class of Lindelöf *P*-spaces is only finitely productive.

The study of the Whyburn and weakly Whyburn property on *P*-spaces was initiated in [13], where it is proved that every *P*-space of character ω_1 is Whyburn and that not every regular Lindelöf *P*-space is Whyburn, and a bunch of interesting problems is left open.

Whyburn and weakly Whyburn spaces were formerly known as AP (WAP) spaces (see [16] and [12]). Although this terminology is still in use today (see for example [7]), we prefer to adopt the new name, because it sounds better and gives credit to a paper of Whyburn where these spaces were first studied ([20]).

2. TOWARD A SMALL NON WEAKLY WHYBURN *P*-SPACE

The main purpose of this section is to give several examples of non weakly Whyburn *P*-spaces. No example of this kind has been exhibited so far. We are particularly concerned with finding examples of minimal cardinality or with further restrictions (for example, Lindelöf examples). Problem 3.6 in [13] asks whether every regular *P*-space of cardinality \aleph_1 is (weakly) Whyburn. Let's start by settling the weak form of this problem with a very simple ZFC construction.

Example 1. *There is a zero-dimensional, non-Whyburn regular (hence, completely regular) *P*-space of cardinality ω_1 .*

Proof. For each $\alpha \in \omega_1$ let L_α be a copy of the one-point lindelöfication of a discrete space of cardinality ω_1 , p_α being the unique non isolated point of L_α (whose neighbourhoods are all co-countable subsets of L_α containing the point p_α itself). Let $Y = \bigoplus_{\alpha \in \omega_1} L_\alpha$ and choose a point $\infty \notin Y$. Topologize $X = Y \cup \{\infty\}$ in the following way. Neighbourhoods of the points of Y are as usual, while a neighbourhood of ∞ is $\{\infty\} \cup \bigcup_{\alpha \geq \beta} T_\alpha$ for some $\beta \in \omega_1$ and some choice of a neighbourhood T_α of p_α in L_α for every $\alpha \geq \beta$. It is clear that X is a zero-dimensional regular *P*-space of cardinality ω_1 . To show that it is not Whyburn consider the set $A = Y \setminus \{p_\alpha : \alpha \in \omega_1\}$ and note that $\infty \in \overline{A}$ and if $B \subseteq A$ is a set having the point ∞ in its closure then \overline{B} contains infinitely many points p_α . \square

Notice that the existence of a non-Whyburn *P*-space of cardinality \aleph_1 and character less than 2^{\aleph_1} is not provable in ZFC, see Theorem 3 in [3] and the remark at the end of the section. We now come to the main dish of this section.

Example 2. *There is a completely regular non-weakly Whyburn *P*-space of cardinality \mathfrak{c} .*

Proof. Let $Y = {}^{<\omega}\omega_1$, $Z = {}^\omega\omega_1$ and $X = Y \cup Z$. We will introduce a topology on X in the following way. Set $\Gamma = {}^Y\omega_1 = ({}^{<\omega}\omega_1)\omega_1$, and for every $g \in \Gamma$ and $\varphi \in Y$ put

$$V_{\varphi,g} = \{\psi \in X \mid \psi \text{ is an extension of } \varphi \wedge \forall n \in (\text{dom } \psi \setminus \text{dom } \varphi) : \psi(n) \geq g(\psi \upharpoonright_n)\} \quad (*)$$

(observe, in particular, that φ always belongs to $V_{\varphi,g}$).

First of all, we want to show that associating to every $\varphi \in Y$ the collection $\{V_{\varphi,g} : g \in \Gamma\}$ as a fundamental system of (open) neighbourhoods, and letting every $\varphi \in Z$ to be isolated, gives rise to a topology of P -space on X . Actually, we have to show that

- (1) $\forall \varphi \in Y : \forall g \in \Gamma : \forall \psi \in V_{\varphi,g} \cap Y : \exists g' \in \Gamma : V_{\psi,g'} \subseteq V_{\varphi,g};$
- (2) $\forall \varphi \in Y : \forall \Gamma' \in [\Gamma]^{\leq \omega} : \exists g \in \Gamma : V_{\varphi,g} \subseteq \bigcap_{g' \in \Gamma'} V_{\varphi,g'}.$

To prove (1), simply take $g' = g$; we claim that $V_{\psi,g} \subseteq V_{\varphi,g}$. Indeed, if $\eta \in V_{\psi,g}$, then first of all η extends ψ ; since ψ , in turn, extends φ (as $\psi \in V_{\varphi,g}$), we have that η extends φ . Therefore, to prove that $\eta \in V_{\varphi,g}$, we only have to show that $\eta(n) \geq g(\eta \upharpoonright_n)$ for every $n \in \text{dom } \eta \setminus \text{dom } \varphi$. Now, if $n \in \text{dom } \eta \setminus \text{dom } \varphi$, then either $n \in \text{dom } \psi \setminus \text{dom } \varphi$, in which case $\eta(n) = \psi(n) \geq g(\psi \upharpoonright_n) = g(\eta \upharpoonright_n)$ (as η extends ψ and $\psi \in V_{\varphi,g}$), or $n \in \text{dom } \eta \setminus \text{dom } \psi$, in which case we have $\eta(n) \geq g(\eta \upharpoonright_n)$ simply because $\eta \in V_{\psi,g}$.

As for property (2), let $\hat{\varphi} \in Y$ and Γ' be a countable subset of Γ , and define $g \in \Gamma$ by letting $g(\varphi) = \sup \{g'(\varphi) : g' \in \Gamma'\}$. Now, for each $\varphi \in Y$, if ψ is an arbitrary element of $V_{\hat{\varphi},g}$ and \hat{g}' is an arbitrary element of Γ' , then to prove that $\psi \in V_{\hat{\varphi},\hat{g}'}$ we just have to show that $\psi(n) \geq \hat{g}'(\psi \upharpoonright_n)$ for every $n \in \text{dom } \psi \setminus \text{dom } \hat{\varphi}$. Indeed, if $n \in \text{dom } \psi \setminus \text{dom } \hat{\varphi}$, then from $\psi \in V_{\hat{\varphi},g}$ it follows that $\psi(n) \geq g(\psi \upharpoonright_n) = \sup \{g'(\psi \upharpoonright_n) : g' \in \Gamma'\} \geq \hat{g}'(\psi \upharpoonright_n)$.

Let τ be the topology generated on X by associating to every $\varphi \in X$ the fundamental system of (open) neighbourhoods \mathcal{J}_φ , defined as

$$\mathcal{J}_\varphi = \begin{cases} \{V_{\varphi,g} : g \in \Gamma\} & \text{if } \varphi \in Y; \\ \{\{\varphi\}\} & \text{if } \varphi \in Z. \end{cases}$$

We want to prove now that (X, τ) is a completely regular space. To this end, we will first show that (X, τ) is T_0 , and then that for every $\varphi \in X$ and every $V \in \mathcal{J}_\varphi$, V is a closed (and open) set. As for the T_0 property, let φ', φ'' be distinct elements of X ; then one of them, say φ' , has domain not less than that of the other element. This implies in particular (as $\varphi' \neq \varphi''$) that φ'' cannot be an extension of φ' ; therefore, fixing any $g' \in \Gamma$, it follows from (*) that $V_{\varphi',g'}$ is a neighbourhood of φ' which does not contain φ'' . Let now show that every collection \mathcal{J}_φ consists of closed sets. If $\varphi \in Z$, then $\{\varphi\}$ is the only element of \mathcal{J}_φ ; thus, consider an arbitrary $\psi \in X \setminus \{\varphi\}$. Of course, if $\psi \in Z$ then $\{\psi\}$ is a neighbourhood of ψ disjoint from $\{\varphi\}$; therefore, we may assume $\psi \in Y$, and consider an $\hat{n} \in \omega$ with $\hat{n} \geq \text{dom } \psi$. Also, fix an element \hat{g} of Γ such that $\hat{g}(\psi \upharpoonright_{\hat{n}}) > \varphi(\hat{n})$; then it follows from (*) that $\varphi \notin V_{\psi,\hat{g}}$, i.e. $\{\varphi\} \cap V_{\psi,\hat{g}} = \emptyset$. Suppose now that $\varphi \in Y$. Let g be any element of Γ , and consider an arbitrary $\psi \in X \setminus V_{\varphi,g}$ —also, since the case $\psi \in Z$ is again trivial, we may assume that $\psi \in Y \setminus V_{\varphi,g}$. By (*), the fact that $\psi \notin V_{\varphi,g}$ means that ψ is not an extension of φ , or that there exists $\hat{n} \in \text{dom } \psi \setminus \text{dom } \varphi$ with $\psi(\hat{n}) < g(\psi \upharpoonright_{\hat{n}})$. If we are in this second case, then taking any $g^* \in \Gamma$ we see that every $\psi' \in V_{\psi,g^*}$ is in particular an extension of ψ , so that $\hat{n} \in \text{dom } \psi \setminus \text{dom } \varphi \subseteq \text{dom } \psi' \setminus \text{dom } \varphi$ and $\psi'(\hat{n}) = \psi(\hat{n}) < g(\psi \upharpoonright_{\hat{n}}) = g(\psi' \upharpoonright_{\hat{n}})$; as a consequence, $V_{\psi,g^*} \cap V_{\varphi,g} = \emptyset$. Suppose now that ψ is not an extension of φ . If φ is not an extension of ψ , either, then there exists $\hat{n} \in \text{dom } \varphi \cap \text{dom } \psi$ with $\varphi(\hat{n}) \neq \psi(\hat{n})$; of course, this entails that φ and ψ cannot have any common extension, hence in particular taking any $g^* \in \Gamma$ it will follow again that $V_{\psi,g^*} \cap V_{\varphi,g} = \emptyset$. If, on the contrary, φ is a (proper) extension of ψ , then let $\hat{n} = \text{dom } \psi$ and fix a $g^* \in \Gamma$ such that $g^*(\psi) > \varphi(\hat{n})$; since every $\eta \in V_{\varphi,g}$ is in particular an extension of φ , we will have the inequality

$\eta(\hat{n}) = \varphi(\hat{n}) < g^*(\psi) = g^*(\varphi \upharpoonright_{\hat{n}}) = g^*(\eta \upharpoonright_{\hat{n}})$ —so that, by (*), $\eta \notin V_{\psi, g^*}$. Therefore, we still see that $V_{\psi, g^*} \cap V_{\varphi, g} = \emptyset$.

Finally, we show that (X, τ) is not weakly Whyburn. First of all, notice that Z is not closed in (X, τ) ; actually, Z turns out to be even dense. Indeed, let φ be any element of Y , and $V_{\varphi, g}$ (with $g \in \Gamma$) be an arbitrary basic neighbourhood of φ . Then put $\hat{n} = \text{dom } \varphi$, and define by induction an $\alpha_n \in \omega_1$ for every $n \geq \hat{n}$ in the following way.

- $\alpha_{\hat{n}} = g(\varphi)$;
- $\alpha_{n+1} = g(\varphi \wedge \langle \alpha_{\hat{n}}, \alpha_{\hat{n}+1}, \dots, \alpha_n \rangle)$.

Now, let $\psi \in Z$ be defined by

$$\psi(n) = \begin{cases} \varphi(n) & \text{for } n < \hat{n}; \\ \alpha_n & \text{for } n \geq \hat{n}; \end{cases}$$

clearly, ψ extends φ , and for every $n \geq \hat{n}$ (i.e., for every $n \in \text{dom } \psi \setminus \text{dom } \varphi$) we have that $\psi(n) = \alpha_n = g(\psi \upharpoonright_n)$. Therefore, it follows from (*) that $\psi \in V_{\varphi, g}$, so that $V_{\varphi, g} \cap Z \neq \emptyset$.

Thus, we will conclude the proof that (X, τ) is not weakly Whyburn by showing that if $M \subseteq Z$ and $\hat{\varphi} \in Y \cap \overline{M}$, then there exists an element of $Y \setminus \{\hat{\varphi}\}$ which belongs to \overline{M} . Actually, we will show more precisely that there exists an $\alpha \in \omega_1$ such that $\hat{\varphi} \wedge \langle \alpha \rangle \in \overline{M}$. As a matter of fact, toward a contradiction, suppose that for every $\alpha \in \omega_1$ there exists $g_\alpha \in \Gamma$ such that

$$V_{\hat{\varphi} \wedge \langle \alpha \rangle, g_\alpha} \cap M = \emptyset. \quad (\#)$$

Then put $\hat{n} = \text{dom } \hat{\varphi}$ and consider a $\hat{g} \in \Gamma$ such that

$$\forall \varphi \in Y: (\text{dom } \varphi \geq \hat{n} + 1 \implies \hat{g}(\varphi) = g_{\varphi(\hat{n})}(\varphi)). \quad (\spadesuit)$$

Since $\hat{\varphi} \in \overline{M}$, there must exist an $\eta \in V_{\hat{\varphi}, \hat{g}} \cap M$, and this η will be in particular an extension of $\hat{\varphi}$ —hence also of $\hat{\varphi} \wedge \langle \eta(\hat{n}) \rangle$. However, since $V_{\hat{\varphi} \wedge \langle \eta(\hat{n}) \rangle, g_{\eta(\hat{n})}} \cap M = \emptyset$ (by (#)), we have in particular that $\eta \notin V_{\hat{\varphi} \wedge \langle \eta(\hat{n}) \rangle, g_{\eta(\hat{n})}}$. But η is an extension of $\hat{\varphi} \wedge \langle \eta(\hat{n}) \rangle$, thus by (*) there must exist an $n^* \in \omega \setminus \text{dom } (\hat{\varphi} \wedge \langle \eta(\hat{n}) \rangle) = \omega \setminus (\hat{n} + 1)$ such that $\eta(n^*) < g_{\eta(\hat{n})}(\eta \upharpoonright_{n^*})$. Thus, letting $\varphi^* = \eta \upharpoonright_{n^*}$, it follows from () (as $\text{dom } \varphi^* = n^* \geq \hat{n} + 1$) that $\eta(n^*) < g_{\eta(\hat{n})}(\eta \upharpoonright_{n^*}) = g_{\varphi^*(\hat{n})}(\varphi^*) = \hat{g}(\varphi^*) = \hat{g}(\eta \upharpoonright_{n^*})$. Clearly, this contradicts the fact that $\eta \in V_{\hat{\varphi}, \hat{g}}$ (take again (*) into account). \square

Here is another way to construct a non weakly Whyburn P -space of cardinality \mathfrak{c} . Murtinová ([11]) has come up independently with a similar example. We will exploit the following example of a Baire P -space and a trick that Gruenhage and Tamano ([7]) used to get a countable stratifiable non-weakly Whyburn space.

Example 3. *There exists a T_1 regular dense-in-itself P -space of cardinality continuum in which every meager set is nowhere dense.*

Proof. Recall that the *support* of a function is the greatest subset of its domain where the function never vanishes. Let $X = \{f: \omega_1 \rightarrow \omega : f \text{ has countable support}\}$. Set $B(f, \alpha) = \{g \in X : \forall \beta \leq \alpha: g(\beta) = f(\beta)\}$. We define a topology on X by declaring $\{B(f, \alpha) : \alpha \in \omega_1\}$ to be a fundamental system of neighbourhoods at f . It is easy to see that X is a dense-in-itself P -space and that X is T_1 and zero-dimensional. Let $\{N_n : n \in \omega\}$ be a countable family of nowhere dense subsets of X . Let $f \in X$ and $\alpha \in \omega_1$. Since N_0 is nowhere dense, there must be a function $g_0 \in B(f, \alpha)$ and an ordinal α_0 such that $B(g_0, \alpha_0) \cap N_0 = \emptyset$. Since $B(f, \alpha)$ is

an open set we can suppose $B(g_0, \alpha_0) \subset B(f, \alpha)$, and since $\{B(g_0, \gamma) : \gamma \in \omega_1\}$ is a decreasing family, we can suppose that $\alpha_0 > \alpha$. Suppose we have found points $\{g_i : i < k\}$ and increasing ordinals $\{\alpha_i : i < k\}$ in such a way that $B(g_j, \alpha_j) \subset B(g_{j-1}, \alpha_{j-1})$ and $B(g_j, \alpha_j) \cap N_j = \emptyset$ for each $j < k$. Since N_k is nowhere dense there must be a point in $g_k \in B(g_{k-1}, \alpha_{k-1})$ such that $g_k \notin \overline{N_k}$, and thus there must be α_k such that $B(g_k, \alpha_k) \cap N_k = \emptyset$. Again we can suppose $B(g_k, \alpha_k) \subset B(g_{k-1}, \alpha_{k-1})$ and $\alpha_k > \alpha_{k-1}$. At the end of the induction the set $\bigcap \{B(g_i, \alpha_i) : i \in \omega\}$ will be a nonempty open set contained in $B(f, \alpha)$, and disjoint from $\bigcup \{N_n : n \in \omega\}$ (to check that $\bigcap \{B(g_i, \alpha_i) : i \in \omega\} \neq \emptyset$, consider that such a set includes $B(\hat{g}, \hat{\alpha})$, where $\hat{\alpha} = \sup \{\alpha_i : i \in \omega\}$ and \hat{g} is any element of X such that $\hat{g}(\alpha') = g_i(\alpha')$ for every $\alpha' \in \omega_1$ and $i \in \omega$ with $\alpha' \leq \alpha_i$). Picking any point $h \in \bigcap \{B(g_i, \alpha_i) : i \in \omega\}$ we see that h is not in the closure $\overline{\bigcup_{n \in \omega} N_n}$, therefore $B(f, \alpha)$ cannot be contained in $\overline{\bigcup_{n \in \omega} N_n}$. Since the choice of f and α was arbitrary it turns out that $\bigcup_{n \in \omega} N_n$ is nowhere dense. \square

Recall that a *weak P-space* is a space in which every countable set is closed.

Example 4. *There are a non-weakly Whyburn P-space of cardinality \mathfrak{c} and a non weakly-Whyburn weak P-space of cardinality ω_1 in ZFC.*

Proof. Let X_1 and X_2 be copies of the space X in Example 3, Y_1 and Y_2 be copies of the space Y described in the above remark and f be a bijection between either X_1 and X_2 or Y_1 and Y_2 (the context will clarify which spaces we mean). Put $Z = X_1 \cup X_2$ and $W = Y_1 \cup Y_2$. Take each point of X_2 (Y_2) to be isolated and declare a neighbourhood of x in X_1 (Y_1) to be of the form $U \cup f(U) \setminus f(N)$, where U is a neighbourhood of x in X (Y), and N is a nowhere dense set in X_1 (Y_1).

Z is a P-space: Let $\mathcal{U} = \{U_n \cup (f(U_n) \setminus f(N_n)) : n \in \omega\}$ be a countable family of neighbourhoods of a point of $z \in X_1$. Then $\bigcap \mathcal{U} = \bigcap_{n \in \omega} U_n \cap \bigcap_{i \in \omega} f(U_i) \setminus \bigcup_{k \in \omega} f(N_k)$. The fact that X is a P-space where every meager set is nowhere dense implies that the last set is a neighbourhood of z in Z . Thus every point of Z is a P-point, i.e., Z is a P-space.

W is a weak P-space: Indeed, let $C \subset W$ be a countable set and $x \in Y_1 \setminus C$. Let $C_1 = C \cap Y_1$ and $C_2 = C \cap Y_2$. If $x \notin f^{-1}(C_2)$ then let U be a neighbourhood of x in Y_1 disjoint from $C_1 \cup f^{-1}(C_2)$. Then $U \cup f(U)$ will be a neighbourhood of x in W disjoint from C . If $x \in f^{-1}(C_2)$ let U be a neighbourhood of x disjoint from C_1 , and as Y is a P-space, the countable set $f^{-1}(C_2)$ will be nowhere dense in it (in fact discrete) so $U \cup (f(U) \setminus C_2)$ is a neighbourhood of x in W disjoint from C .

Z and W are not weakly Whyburn: The set X_2 is not closed, however there is no almost closed set converging outside X_2 . In fact if $A \subset X_2$ is a non-closed set in Z then $f^{-1}(A)$ cannot be nowhere dense, hence $\text{Int}(\overline{f^{-1}(A)}) \neq \emptyset$. Moreover this last set must be uncountable and dense-in-itself, or otherwise X would have isolated points. By the definition of the topology on Z , every point of $\text{Int}(\overline{f^{-1}(A)})$ will be in the closure of A . This proof also works for the space $W = Y_1 \cup Y_2$. \square

Thus, under [CH] we get a consistent negative answer to Problem 3.6 in [13].

Question 5. *Does there exist a model of ZFC + not CH in which there is a Baire P-space of cardinality ω_1 ?*

It is easy to realize that the fact that every countable union of nowhere dense sets is nowhere dense is equivalent to being a Baire space in the realm of P-spaces.

There is no hope of getting a ZFC example like in Question 5 because the non-existence of a Baire space of cardinality ω_1 is known to be consistent (see [15]).

By means of the construction of Example 4, a positive answer to the previous question would guarantee the existence of a non-weakly Whyburn P -space of cardinality ω_1 in a model where the continuum hypotheses fails. Yet, the following problem would still be open.

Question 6. *Is there a ZFC example of a non-weakly Whyburn P -space of cardinality \aleph_1 ?*

In [13] it was remarked that every Lindelof P -space of cardinality \aleph_1 is Whyburn. One may wonder if the same applies to weak P -spaces.

Theorem 7 ([CH]). *There exists a regular Lindelöf non weakly Whyburn weak P -space of cardinality and character \aleph_1 .*

Proof. Let Y be the real line with the density topology (see for example [18]). This space is known to be a Tychonoff weak P -space. Also, since it is a dense-in-itself ccc Baire space having π -weight \aleph_1 , by a well-known argument (see for example [14]) it has a Luzin subspace $Z \subset Y$, i.e. a subspace in which every nowhere dense set is countable. Let X be the Alexandroff duplicate of Z . It is clear that X is a weak P -space, it is Lindelöf because Z is, and an argument similar to the one in Example 4 shows that X is not weakly Whyburn. \square

Question 8. *Can CH be removed from the previous theorem?*

Going back to non weakly Whyburn P -spaces, we are now going to produce an example of cardinality \aleph_2 by using a set theoretic axiom weaker than CH. A tree T of cardinality ω_1 with ω_2 -many uncountable branches is called a *weak Kurepa tree*. Using a weak Kurepa tree T is non difficult to construct a regular P -space X of weight ω_1 and cardinality ω_2 ; take as points of X the uncountable branches and as a base the sets of the form $\hat{t} = \{B \in X : t \in B\}$ for each $t \in T$.

Unfortunately, the existence of a weak Kurepa tree is not provable in ZFC (see Section 8 in [21]). The *weak Kurepa Hypothesis*, briefly wKH, is the assertion that there exists a weak Kurepa tree. In [21] it is pointed out that CH implies wKH, but wKH is somehow much weaker than CH.

Example 9 ([wKH]). *There is a non-weakly Whyburn regular P -space of cardinality ω_2 .*

Proof. Let X be a regular P -space of weight ω_1 and cardinality ω_2 and let $Y = \omega_2 \cup \{p\}$, topologized in such a way that every subset of ω_2 is open and a set $U \ni p$ is open whenever $|Y \setminus U| \leq \omega_1$. Fix a one-to-one mapping $f : Y \rightarrow X$ and let $Z = Y \times X$. Obviously Z is a regular P -space of cardinality ω_2 . We claim that Z does not have the weak Whyburn property. Let $A = \{(\alpha, f(\alpha)) : \alpha \in \omega_2\} \subseteq Z$. Since the space X has weight ω_1 , the set T of all complete accumulation points of the set $\{f(\alpha) : \alpha \in \omega_2\}$ has cardinality ω_2 . It is easy to check that $\{p\} \times T \subseteq \overline{A}$ and so A is not closed in Z . Let $B = \{(\alpha, f(\alpha)) : \alpha \in C\}$ be a subset of A . A point can be in $\overline{B} \setminus A$ only if it is of the form (p, x) and this may happen only if $|B| = \omega_2$. As before, the set T' of all complete accumulation points of the set $f(C)$ has cardinality ω_2 and $\{p\} \times T' \subseteq \overline{B} \setminus A$. Therefore, no almost closed subset of A can converge to a point outside A and we conclude that Z does not have the weak Whyburn property. \square

Question 10. *Is there a ZFC example of a non-weakly Whyburn P -space of cardinality ω_2 ?*

The next result, which is of independent interest, shows that a positive answer to the previous question would require a different approach than that of our example 4. We are going to use the following simple lemma.

Lemma 11. *Every regular P -space X with $w(X) \leq \omega_1$ has a base of cardinality not greater than ω_1 , consisting of clopen sets.*

Proof. By [6, Theorem 1.1.15], it suffices to show that the clopen sets form a base for X , and this holds because every regular P -space is 0-dimensional. \square

Proposition 12. *If there exists a regular P -space X with $|X| \geq \omega_2$ and $w(X) \leq \omega_1$, then there exists a weak Kurepa tree.*

Proof. By the above lemma, there exists a base $\mathcal{L} = \{L_\alpha : \alpha \in \omega_1\}$ for X , consisting of clopen sets. Using transfinite induction, we will construct an ω_1 -sequence $\{\mathcal{A}_\alpha\}_{\alpha \in \omega_1}$ of collections of nonempty subsets of X , with the following properties:

- (1) $\forall \alpha \in \omega_1$: \mathcal{A}_α is a (cl)open partition of X not containing the empty set;
- (2) $\forall \alpha \in \omega_1$: $\forall \alpha' < \alpha$: \mathcal{A}_α is a refinement of $\mathcal{A}_{\alpha'}$.

Let $\mathcal{A}_0 = \{X\}$. If \mathcal{A}_α has been defined for a given $\alpha \in \omega_1$, then let

$$\mathcal{A}_{\alpha+1} = (\{A \cap L_\alpha : A \in \mathcal{A}_\alpha\} \cup \{A \setminus L_\alpha : A \in \mathcal{A}_\alpha\}) \setminus \{\emptyset\}.$$

Finally, if $\lambda \in \omega_1$ is limit and \mathcal{A}_α has been defined for $\alpha < \lambda$, then put

$$\mathcal{A}_\lambda = \left\{ \bigcap_{\alpha \in \lambda} A_\alpha : \forall \alpha \in \lambda: A_\alpha \in \mathcal{A}_\alpha \ \wedge \ \forall \alpha', \alpha'' \in \lambda: (\alpha' < \alpha'' \Rightarrow A_{\alpha'} \supseteq A_{\alpha''}) \right\} \setminus \{\emptyset\}.$$

We prove both property (1) and (2) by transfinite induction on $\alpha \in \omega_1$. As for (1), it is trivial when $\alpha = 0$, and if it holds for a given $\hat{\alpha} \in \omega_1$ then it is easily seen to hold also for $\hat{\alpha} + 1$. Consider now a nonzero limit $\lambda \in \omega_1$. Given an arbitrary $x \in X$, take for every $\alpha < \lambda$ an $A_\alpha \in \mathcal{A}_\alpha$ such that $x \in A_\alpha$; since (by the inductive hypothesis) (2) holds for every $\alpha < \lambda$, we have that for every $\alpha' < \alpha < \lambda$ there exists a $\hat{A} \in \mathcal{A}_{\alpha'}$ such that $A_\alpha \subseteq \hat{A}$. This implies (as $\mathcal{A}_{\alpha'}$, still by the inductive hypothesis, is a partition of X) that $A_\alpha \cap A = \emptyset$ for every $A \in \mathcal{A}_{\alpha'} \setminus \{\hat{A}\}$; thus, since $A_{\alpha'} \in \mathcal{A}_{\alpha'}$ and $A_\alpha \cap A_{\alpha'} \neq \emptyset$ (as x belongs to both sets), we must have that $A_{\alpha'} = \hat{A}$, hence $A_\alpha \subseteq A_{\alpha'}$. Therefore, since of course $\bigcap_{\alpha \in \lambda} A_\alpha \neq \emptyset$ (as such a set contains x), we have by the definition of \mathcal{A}_λ that $\bigcap_{\alpha \in \lambda} A_\alpha \in \mathcal{A}_\lambda$, so that we have found an element of \mathcal{A}_λ which contains x . Therefore, \mathcal{A}_λ covers X . Since $\emptyset \notin \mathcal{A}_\lambda$ by definition, and every element of \mathcal{A}_λ is open as a countable intersection of open sets in a P -space, it only remains to show that $A' \cap A'' = \emptyset$ for distinct $A', A'' \in \mathcal{A}_\lambda$. Thus, suppose $A' = \bigcap_{\alpha \in \lambda} A'_\alpha$ and $A'' = \bigcap_{\alpha \in \lambda} A''_\alpha$, with $A'_\alpha, A''_\alpha \in \mathcal{A}_\alpha$ for $\alpha \in \lambda$, and $\alpha \mapsto A'_\alpha$ and $\alpha \mapsto A''_\alpha$ decreasing; of course, the fact that $A' \neq A''$ implies that for at least one $\hat{\alpha} \in \lambda$ we have that $A'_{\hat{\alpha}} \neq A''_{\hat{\alpha}}$. Since $\mathcal{A}_{\hat{\alpha}}$, by the inductive hypothesis, is a partition, this implies in turn that $A'_{\hat{\alpha}} \cap A''_{\hat{\alpha}} = \emptyset$; hence $A' \cap A'' = (\bigcap_{\alpha \in \lambda} A'_\alpha) \cap (\bigcap_{\alpha \in \lambda} A''_\alpha) = \emptyset$, too.

Now we prove (2). Of course, for $\alpha = 0$ there is nothing to prove. If (2) holds for every α less than a successor ordinal $\hat{\alpha} + 1 \in \omega_1$, and $\alpha^* < \hat{\alpha} + 1$, then either $\alpha^* = \hat{\alpha}$, in which case it is straightforward to realize from the general definition of $\mathcal{A}_{\alpha+1}$ that $\mathcal{A}_{\hat{\alpha}+1}$ is a refinement of $\mathcal{A}_{\hat{\alpha}} = \mathcal{A}_{\alpha^*}$, or $\alpha^* < \hat{\alpha}$, in which case since $\mathcal{A}_{\hat{\alpha}+1}$ is a refinement of $\mathcal{A}_{\hat{\alpha}}$ and $\mathcal{A}_{\hat{\alpha}}$ (by the inductive hypothesis) is a refinement of \mathcal{A}_{α^*}

we also have that $\mathcal{A}_{\hat{\alpha}+1}$ is a refinement of \mathcal{A}_{α^*} . Finally, if $\lambda \in \omega_1$ is a limit ordinal, and A is an arbitrary element of \mathcal{A}_λ , then $A = \bigcap_{\alpha \in \lambda} A_\alpha$ with $A_\alpha \in \mathcal{A}_\alpha$ for every $\alpha < \lambda$; therefore, for every $\alpha \in \lambda$, A_α is an element of \mathcal{A}_α which include A .

Let us also point out a further property of the partitions \mathcal{A}_α which immediately follows from the general definition of $\mathcal{A}_{\alpha+1}$, and which will play a momentous rôle below:

$$\forall \alpha \in \omega_1 : \forall A \in \mathcal{A}_{\alpha+1} : (A \cap L_\alpha = \emptyset \vee A \subseteq L_\alpha). \quad (*)$$

Now, set $T = \{(A, \alpha) : \alpha \in \omega_1 \wedge A \in \mathcal{A}_\alpha\}$, and let \sqsubseteq be the binary relation on T defined by

$$(A', \alpha') \sqsubseteq (A'', \alpha'') \iff (\alpha' \leq \alpha'' \wedge A' \supseteq A'').$$

Observe that, since \sqsubseteq is the restriction to T of the product order of the two ordered sets $(\wp(X), \supseteq)$ and (ω_1, \leq) , we have by general reasons that \sqsubseteq is a (partial) order on T . We will further prove the following:

Fact. (T, \sqsubseteq) is a tree of height ω_1 , and for every $\alpha \in \omega_1$ we have that $\text{Lev}_\alpha(T) = \{(A, \alpha) : A \in \mathcal{A}_\alpha\}$.

Proof. Let $(\hat{A}, \hat{\alpha})$ be an arbitrary element of T (so that $\hat{A} \in \mathcal{A}_{\hat{\alpha}}$). By property (2), we have that for every $\alpha < \hat{\alpha}$ there exists an $A_\alpha \in \mathcal{A}_\alpha$ with $\hat{A} \subseteq A_\alpha$; since each \mathcal{A}_α is a partition and \hat{A} , as an element of $\mathcal{A}_{\hat{\alpha}}$, is nonempty, it is apparent that for every $\alpha < \hat{\alpha}$ the set A_α is the only element of \mathcal{A}_α including \hat{A} . Then it easily follows that $\{t \in T : t \sqsubset (\hat{A}, \hat{\alpha})\} = \{(A_\alpha, \alpha) : \alpha < \hat{\alpha}\}$; moreover, by an argument similar to one of those used to prove property (1) when α is a limit ordinal λ , we may show that for every α', α'' with $\alpha' < \alpha'' < \hat{\alpha}$ we have that $A_{\alpha''} \subseteq A_{\alpha'}$ (actually, we know that $A_{\alpha''}$ must be included in some element A' of $\mathcal{A}_{\alpha'}$, and since $\mathcal{A}_{\alpha'}$ is a partition and $A_{\alpha''} \cap A_{\alpha'} \supseteq \hat{A} \neq \emptyset$, we conclude that A' necessarily coincides with $A_{\alpha'}$). Therefore, the above-considered set $\{t \in T : t \sqsubset (\hat{A}, \hat{\alpha})\} = \{(A_\alpha, \alpha) : \alpha < \hat{\alpha}\}$, endowed with the order induced by \sqsubseteq , is similar to the ordinal $\hat{\alpha}$, and this proves at the same time that (T, \sqsubseteq) is a tree and that $(\hat{A}, \hat{\alpha}) \in \text{Lev}_{\hat{\alpha}}(T)$. On the other hand, every element of $\text{Lev}_{\hat{\alpha}}(T)$ must have the second component equal to $\hat{\alpha}$ —as for each $(A, \alpha) \in T$ with $\alpha \neq \hat{\alpha}$ we may prove as above that it belongs to the set $\text{Lev}_\alpha(T)$, which is disjoint from $\text{Lev}_{\hat{\alpha}}(T)$. Therefore $\text{Lev}_{\hat{\alpha}}(T) = \{(A, \hat{\alpha}) : A \in \mathcal{A}_{\hat{\alpha}}\}$, and this holds for each $\hat{\alpha} \in \omega_1$. Of course, ω_1 turns out to be the height of (T, \sqsubseteq) .

Now we finish the proof of the proposition. First of all, observe that $|\mathcal{A}_\alpha| \leq \omega_1$ for every $\alpha \in \omega_1$; this is an elementary consequence of the fact that each \mathcal{A}_α is an open partition of X consisting of nonempty sets, and that $d(X) \leq w(X) \leq \omega_1$. Since $|\text{Lev}_\alpha(T)| = |\{(A, \alpha) : A \in \mathcal{A}_\alpha\}| = |\mathcal{A}_\alpha| \leq \omega_1$ for every $\alpha \in \omega_1$, to show that (T, \sqsubseteq) is a weak Kurepa tree we only have to prove that it has at least ω_2 branches.

For every $x \in X$ and $\alpha \in \omega_1$, denote by $A_{x,\alpha}$ the only element of \mathcal{A}_α containing x ; notice that, since $x \in A_{x,\alpha'} \cap A_{x,\alpha''} \neq \emptyset$ for every $\alpha', \alpha'' \in \omega_1$, we may prove as before that $A_{x,\alpha''} \subseteq A_{x,\alpha'}$ for $\alpha' < \alpha'' < \omega_1$. Let, for every $x \in X$, $\Pi_x = \{(A_{x,\alpha}, \alpha) : \alpha \in \omega_1\}$. Then $(A_{x,\alpha'}, \alpha') \sqsubset (A_{x,\alpha''}, \alpha'')$ for $\alpha' < \alpha'' < \omega_1$, so that Π_x is a chain in T intersecting all levels, i.e., Π_x is a branch. Therefore, if we can prove that the association $x \mapsto \Pi_x$ is one-to-one, it will follow that in T there are at least ω_2 branches. Indeed, suppose that \bar{x}, \bar{y} are arbitrary distinct elements in X ; since \mathcal{L} is a base for X , which is T_1 , we may consider an $\hat{\alpha} \in \omega_1$ such that $\bar{x} \in L_{\hat{\alpha}}$ and $\bar{y} \notin L_{\hat{\alpha}}$. Observe that, from $\bar{x} \in A_{\bar{x},\hat{\alpha}+1}$, we have the inequality $L_{\hat{\alpha}} \cap A_{\bar{x},\hat{\alpha}+1} \neq \emptyset$, and this implies by (*) that $A_{\bar{x},\hat{\alpha}+1} \subseteq L_{\hat{\alpha}}$; thus $\bar{y} \notin A_{\bar{x},\hat{\alpha}+1}$ (as $\bar{y} \notin L_{\hat{\alpha}}$), while of course $\bar{y} \in A_{\bar{y},\hat{\alpha}+1}$. Then it follows that $A_{\bar{x},\hat{\alpha}+1} \neq A_{\bar{y},\hat{\alpha}+1}$, and since $(A_{\bar{y},\hat{\alpha}+1}, \hat{\alpha}+1)$

is the only element of $\Pi_{\bar{y}}$ having the second component equal to $\hat{\alpha} + 1$, we conclude that $(A_{\bar{x}, \hat{\alpha}+1}, \hat{\alpha} + 1) \notin \Pi_{\bar{y}}$. Therefore $\Pi_{\bar{x}} \neq \Pi_{\bar{y}}$ (as the former set contains $A_{\bar{x}, \hat{\alpha}+1}$ while the latter does not). \square

We have looked hard for *small* non weakly Whyburn P -spaces in ZFC, but all we have are partial results. Here is another way to construct such an example, provided that the following innocent-looking question can be answered in the positive.

Question 13. *Does there exist in ZFC a dense-in-itself P -space of cardinality \aleph_1 every relatively discrete subset of which is closed?*

Let X be such a space and $Y = \omega_1 \cup \{p\}$ be the one-point Lindelöfication of the discrete space ω_1 and put $Z = X \times Y$. Fix an injective mapping $f : X \rightarrow \omega_1$ and let $A = \{(x, f(x)) : x \in X\} \subseteq Z$. It is easy to realize that A is not closed in Z . If for some $B \subseteq A$ we have $\overline{B} \setminus A \neq \emptyset$, then $(x, p) \in \overline{B}$ for some $x \in X$. Without any loss of generality, we may assume $x \notin \pi_X(B)$. Since the set $\pi_X(B)$ is not closed, it cannot be discrete so we may fix some $z \in \pi_X(B)$ which is not isolated in $\pi_X(B)$. It is not difficult to see that $(z, p) \in \overline{B}$ so Z is not weakly Whyburn.

Before going on to discuss Lindelöf spaces we would like to recall another problem from [13] about which we know very little. Consider the space $\beta\omega$; under CH this space has character ω_1 so its ω -modification is even Whyburn, but what happens in a model where CH fails?

Question 14. ([13]) *Is the ω -modification of $\beta\omega$ weakly Whyburn?*

Problem 3.5 in [13] asks whether every (regular Lindelöf) P -space is weakly Whyburn. Koszmider and Tall constructed ([10]) a model of ZFC+CH where there exists a regular Lindelöf P -space of cardinality \aleph_2 without Lindelöf subspaces of size ω_1 . Such a space cannot be weakly Whyburn, because of the following easy fact.

Proposition 15. *Every Lindelöf weakly Whyburn P -space X of uncountable cardinality has a Lindelöf subspace of cardinality ω_1 .*

Proof. Pick a set $A \subseteq X$ with $|A| = \omega_1$. If A is closed, then we are done. Otherwise, there is some almost closed $B \subseteq A$ converging outside it. Since in a P -space countable sets are closed, the set B must have cardinality ω_1 . Thus \overline{B} is a Lindelöf subspace of X of cardinality ω_1 . \square

At least consistently, the assumption that X is a P -space can be dropped in the above proposition.

Proposition 16 ([CH]). *Every Lindelöf weakly Whyburn space X of uncountable cardinality has a Lindelöf subspace of cardinality ω_1 .*

Proof. We will show that X has a closed subset of size ω_1 . Suppose that it is not so and let $A \subseteq X$ be a set of size ω_1 . Then every almost closed set converging outside A is countable, so that the Whyburn-closure of A has cardinality $\omega_1^\omega = \omega_1$. Let A^α be the α -iterate of the Whyburn closure of A . Then $B = \bigcup_{\alpha \in \omega_1} A^\alpha$ has cardinality ω_1 ; if this last set were non-closed then we could find some almost closed P converging outside it. Thus, P would be countable and, by the regularity of ω_1 , there would be some $\beta \in \omega_1$ such that $P \subseteq \bigcup_{\alpha \in \beta} A^\alpha$. Hence the limit of P would be inside B , which is a contradiction. \square

As a corollary we get a proposition established in [4] with the aid of elementary submodels:

Theorem 17 ([CH]). ([4]) *Every Hausdorff Lindelöf sequential space of uncountable cardinality has a Lindelöf subspace of cardinality ω_1 .*

Question 18. *Is it true in ZFC that every Lindelöf weakly Whyburn space of uncountable cardinality has a Lindelöf subspace of cardinality ω_1 ?*

This is known to be true in the special case of regular Lindelöf scattered spaces by a result of Ofelia T. Alas (see [4, Theorem 10]).

It is known (see [2]) that compact weakly Whyburn spaces are pseudoradial. We don't know whether the same holds for Lindelöf weakly Whyburn P -spaces, but we prove the following partial result.

Theorem 19. *Every regular Lindelöf weakly Whyburn P -space of pseudocharacter less than \aleph_ω is pseudoradial.*

Proof. Let $A \subseteq X$ be a non closed set and $B \subseteq A$ such that $\overline{B} \setminus A = \{x\}$ for some $x \in X$. Every regular P -space is zero-dimensional, so let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be a family of clopen sets in \overline{B} , such that $\bigcap_{\alpha \in \kappa} U_\alpha = \{x\}$, and having the minimal cardinality among all families with the previous properties. Since we are in a P -space and x is an accumulation point of B , the cardinal κ must be uncountable. Use minimality of \mathcal{U} to pick $x_\gamma \in \bigcap_{\alpha < \gamma} U_\alpha \setminus \{x\}$ for each $\gamma \in \kappa$. Since \overline{B} is Lindelöf and κ is a regular uncountable cardinal, the sequence $\{x_\gamma : \gamma < \kappa\} \subseteq A$ converges to x . \square

Question 20. *Does there exist a Lindelöf weakly Whyburn non-pseudoradial P -space?*

A negative answer to the above question would exhibit an interesting pathology in the class of Lindelöf P -spaces. We now suggest a possible way to look for a counterexample. Kunen has constructed a compact space X and a point $p \in X$ with the following properties:

- (1) $\chi(p, X) > \omega$.
- (2) No sequence of uncountable regular length converges to p in X .

Suppose that a compact *scattered* space Y having the above properties exists, then it is easy to construct a weakly Whyburn non pseudoradial Lindelöf P -space. Indeed, since Y is compact scattered its omega-modification Y_δ is a Lindelöf P -space (see [1], Lemma II.7.14). Since Y_δ is regular and scattered it is weakly Whyburn. By 2) no sequence will converge to p in Y_δ . (Since character equals pseudocharacter in compact spaces 1) will guarantee that p is not isolated in Y_δ).

3. PRODUCT OF WEAKLY WHYBURN P -SPACES

The behaviour of the weakly Whyburn property under the product operation is not very clear. The first listed author has proved that the product of a compact semiradial space and a compact weakly Whyburn space is weakly Whyburn but we are not aware of a non weakly Whyburn product of compact weakly Whyburn spaces. We don't know whether the same holds replacing compact space with Lindelöf P -space but at least we prove that the product of Lindelöf weakly Whyburn P -space and a Lindelöf Whyburn P -space is weakly Whyburn (see Corollary 24).

Recall that a space has countable extent if each of its closed discrete subsets is countable. Every Lindelöf space has countable extent. So the next lemma is a strengthening of Theorem 2.5 in [13].

Lemma 21. *If X is a Whyburn P -space of countable extent, then X is a radial space of radial character \aleph_1 .*

Proof. Assume that $A \subseteq X$ and $x \in \overline{A} \setminus A$. Fix an almost closed $P \subseteq A$ such that $x \in \overline{P}$. Take a maximal disjoint family γ of open subsets of P whose closures do not contain x . Then $x \in \overline{\bigcup \gamma}$ and hence there is an almost closed $Q \subseteq \bigcup \gamma$ such that $x \in \overline{Q}$. Let $\gamma' = \{U \in \gamma : U \cap Q \neq \emptyset\}$. Since X is a P -space, it is easy to see that γ' is uncountable. For each $U \in \gamma'$ take an $x_U \in U \cap Q$. The set $B = \{x_U : U \in \gamma'\}$ is discrete and $(\overline{B} \setminus B) \cap (\bigcup \gamma) = \emptyset$. This means B is closed in Q and therefore x has to be the only accumulation point of B in the space $Q \cup \{x\}$. $B \cup \{x\}$ is an uncountable space with the unique non-isolated point x . Being of countable extent, such a space must be Lindelöf and of cardinality \aleph_1 . Hence B is a sequence of length \aleph_1 which converges to x . \square

Proposition 22. *A P -space X of cardinality \aleph_1 and countable extent is Lindelöf and radial.*

Proof. We first prove that X is Lindelöf. Toward a contradiction, suppose there exists an open cover \mathcal{A} of X with no countable subcover. Of course, since $|X| = \omega_1$, we may assume that $|\mathcal{A}| = \omega_1$. Also, since \mathcal{A} has no countable subcover, we can further assume (up to passing to a suitable subcover) that \mathcal{A} is indexed as $\{A_\alpha : \alpha \in \omega_1\}$, in such a way that

$$\forall \alpha \in \omega_1 : A_\alpha \setminus \bigcup_{\alpha' < \alpha} A_{\alpha'} \neq \emptyset.$$

For every $\alpha \in \omega_1$, pick an $x_\alpha \in A_\alpha \setminus \bigcup_{\alpha' < \alpha} A_{\alpha'}$; we claim that the set $D = \{x_\alpha : \alpha \in \omega_1\}$ is closed and discrete, and this will contradict the fact that X has countable extent—as clearly $\alpha \mapsto x_\alpha$ is one-to-one.

To prove the discreteness of D , note that for every $\alpha \in \omega_1$ the set A_α is a neighbourhood of x_α missing all points $x_{\alpha'}$ with $\alpha' > \alpha$; also, since the set $C_\alpha = \{x_{\alpha'} : \alpha' < \alpha\}$ is closed (because X is a P -space), we see that $A_\alpha \setminus C_\alpha$ is an open neighbourhood of x_α whose intersection with D gives $\{x_\alpha\}$. Suppose now $\hat{x} \in X \setminus D$; then $\hat{x} \in A_{\hat{\alpha}}$ for some $\hat{\alpha} \in \omega_1$. Arguing essentially as before, we see that $A_{\hat{\alpha}} \setminus \{x_{\alpha'} : \alpha' \leq \hat{\alpha}\}$ is an open neighbourhood of \hat{x} missing D ; thus D is closed.

Notice that X is Hausdorff and, being a Lindelöf P -space, it is also regular. To prove that X is radial, due to Lemma 21 it will suffice to show that X has Whyburn property. Let $a \in X$ and $M \subseteq X$ be such that $a \in \overline{M} \setminus M$; index $X \setminus \{a\}$ as $\{x_\alpha : \alpha \in \omega_1\}$ and consider a decreasing ω_1 -sequence $\{V_\alpha\}_{\alpha \in \omega_1}$ of open neighbourhoods of a such that

$$(3;1) \quad \forall \alpha \in \omega_1 : \overline{V_\alpha} \cap \{x_{\alpha'} : \alpha' \leq \alpha\} = \emptyset;$$

let also y_α be an element of $V_\alpha \cap M$ for every $\alpha \in \omega_1$, and set $L = \{y_\alpha : \alpha \in \omega_1\}$. We claim that L is an almost-closed set converging to a . Indeed, on the one hand if z is an arbitrary element of $X \setminus (L \cup \{a\})$ then $z = x_{\hat{\alpha}}$ for some $\hat{\alpha} \in \omega_1$, so that $z \in X \setminus \overline{V_{\hat{\alpha}}}$; since $z \notin L$, it also follows that $U = X \setminus (\{y_\alpha : \alpha < \hat{\alpha}\} \cup \overline{V_{\hat{\alpha}}})$ is an open neighbourhood of z , and U is disjoint from L as $L = \{y_\alpha : \alpha \in \omega_1\} \subseteq \{y_\alpha : \alpha < \hat{\alpha}\} \cup \overline{V_{\hat{\alpha}}}$. On the other hand, to show that $a \in \overline{L}$, consider an arbitrary

neighbourhood W of a ; clearly, if we can prove that $V_\alpha \subseteq W$ for some $\alpha \in \omega_1$, then the inequality $W \cap L \neq \emptyset$ will follow. Actually, consider the closed subset $T = X \setminus W$ of X , and let $\mathcal{A} = \{X \setminus \overline{V_\alpha} : \alpha \in \omega_1\}$. We see that \mathcal{A} covers T , as every element of T must be x_α for some $\alpha \in \omega_1$, and $x_\alpha \in X \setminus \overline{V_\alpha}$ by (3; 1). Since T is Lindelöf, there must exist a countable subset A of ω_1 such that $X \setminus W \subseteq \bigcup_{\alpha \in A} (X \setminus \overline{V_\alpha}) = X \setminus \bigcap_{\alpha \in A} \overline{V_\alpha}$; using the decreasing characters of the neighbourhoods V_α , it also follows that $X \setminus W \subseteq X \setminus \overline{V_{\hat{\alpha}}}$ —where $\hat{\alpha} = \sup A$ —whence $V_{\hat{\alpha}} \subseteq \overline{V_{\hat{\alpha}}} \subseteq W$. \square

Theorem 23. *The product of a pseudoradial P -space X of radial character ω_1 with a weakly Whyburn P -space Y of countable extent is weakly Whyburn.*

Proof. Let $A \subseteq X \times Y$ be a non-closed subset. Take $(x, y) \in \overline{A} \setminus A$. If $A \cap (\{x\} \times Y)$ were non-closed, we could immediately apply the weak Whyburn property of Y to find a set $B \subseteq A$ satisfying $\overline{B} \setminus A = \{(x, y)\}$. So, we may assume that $A \cap (\{x\} \times Y)$ is closed. By passing to a suitable subset, we may further assume that $A \cap (\{x\} \times Y) = \emptyset$. Since $x \in \overline{\pi_X(A)}$, it follows that $\pi_X(A)$ is not closed in X . So, there is a convergent sequence $\{x_\alpha : \alpha \in \omega_1\} \subseteq \pi_X(A)$ with limit outside $\pi_X(A)$. For each $\alpha \in \omega_1$ take a point $y_\alpha \in Y$ in such a way that $(x_\alpha, y_\alpha) \in A$.

Case 1: The set $\{y_\alpha : \alpha \in \omega_1\}$ is countable. There exists an uncountable subset $S \subseteq \omega_1$ such that $y_\alpha = y_\beta$ for any $\alpha, \beta \in S$. Then, we put $B = \{(x_\alpha, y_\alpha) : \alpha \in S\}$.

Case 2: The set $\{y_\alpha : \alpha \in \omega_1\}$ is uncountable. Since Y has countable extent, we may assume—up to removing one point—that the set $\{y_\alpha : \alpha \in \omega_1\}$ is not closed. Therefore, there exists an uncountable set $S \subseteq \omega_1$ such that the set $C = \{y_\alpha : \alpha \in S\}$ is almost closed. As the set $d_Y(C)$ has still countable extent, by Lemma 21 there exists an (uncountable) set $T \subseteq S$ such that the set $\{y_\alpha : \alpha \in T\}$ is a convergent sequence. Then put $B = \{(x_\alpha, y_\alpha) : \alpha \in T\}$.

In both of the previous cases, the set B is a convergent sequence which is also an almost closed subset of A . \square

Corollary 24. *The product of a Whyburn Lindelöf P -space with a weakly Whyburn Lindelöf P -space is weakly Whyburn.*

Question 25. *Is the product of two Lindelöf weakly Whyburn P -spaces always weakly Whyburn?*

Constructing a non-Whyburn product of two Lindelöf P -spaces turned out not to be so easy. Since every compact Whyburn space is Fréchet, the analogous problem for compact spaces is the construction of two compact Fréchet spaces whose product is not Fréchet. This is usually tackled using the one-point compactification of Mrowka spaces on suitable almost disjoint families on ω (see [5] and [17]), so it was natural to try and extend that approach.

We first recall that a family \mathcal{A} of uncountable subsets of ω_1 is said to be *almost disjoint* if $|a \cap b| < \omega_1$ for all distinct $a, b \in \mathcal{A}$; it is said to be a MAD family if it is maximal with respect to this property. Define now a topology on $X = \omega_1 \cup \mathcal{A}$ in the following way. Each point of ω_1 is isolated and a neighbourhood of $p \in \mathcal{A}$ is $\{p\} \cup p$ minus a countable set. The space X is locally Lindelöf so we may take its one-point Lindelöfication $\mathcal{L}(\mathcal{A})$; in concrete terms, $\mathcal{L}(\mathcal{A})$ is obtained by adding a point ∞ to \mathcal{A} , and is endowed with a topology whose restriction to X coincides with the original topology, while the basic neighbourhoods of ∞ are all sets of the form $\{\infty\} \cup (\mathcal{A} \setminus \mathcal{F}) \cup (\bigcup (\mathcal{A} \setminus \mathcal{F}) \setminus F)$, where $\mathcal{F} \in [\mathcal{A}]^{<\omega_1}$ and $F \in [\omega_1]^{<\omega_1}$. Notice that, this way, $\mathcal{L}(\mathcal{A})$ turns out to be a Lindelöf (regular) P -space.

Let \mathcal{A} be an almost disjoint family on ω_1 and let $\mathcal{J}(\mathcal{A})$ be the ideal of those uncountable subsets of ω_1 which can be *almost-covered* by a countable subfamily of \mathcal{A} (which means that the set-theoretic difference between the former and the union of the latter one is countable); let also $\mathcal{J}^+(\mathcal{A}) = [\omega_1]^{\omega_1} \setminus \mathcal{J}(\mathcal{A})$. Furthermore, let $\mathcal{M}(\mathcal{A})$ be the set of those $X \in [\omega_1]^{\omega_1}$ such that $\mathcal{A}_X = \{X \cap p : p \in \mathcal{A}, X \cap p \text{ is uncountable}\}$ is a MAD family on X . We introduce the following:

Definition 26. *An almost disjoint family \mathcal{A} on ω_1 is said to be nowhere MAD if $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{J}(\mathcal{A})$.*

Now we will link some topological features of $\mathcal{L}(\mathcal{A})$ to the combinatorial structure of \mathcal{A} via the previous definition.

Theorem 27. *Let \mathcal{A} be an almost disjoint family on ω_1 . The space $\mathcal{L}(\mathcal{A})$ is Whyburn if and only if \mathcal{A} is nowhere MAD.*

Proof. Suppose first that there is some $X \in \mathcal{M}(\mathcal{A}) \cap \mathcal{J}^+(\mathcal{A})$, then $\infty \in \overline{X}$ (where ∞ is the point added to \mathcal{A} , to obtain $\mathcal{L}(\mathcal{A})$). We will show that X contains no almost closed set converging to ∞ . Indeed let $Y \subseteq X$ be such that $\infty \in \overline{Y}$; then Y is clearly uncountable, hence—since the family \mathcal{A}_X is MAD on X —there must be some $p \in \mathcal{A}$ such that $p \cap Y$ is uncountable. So $p \in \overline{Y}$ and we are done. To prove the converse note that at each $q \in \mathcal{A}$ the Whyburn property is clearly satisfied, therefore we only need to check it at ∞ . Suppose that \mathcal{A} is nowhere MAD and pick some $X \subseteq \omega_1 \cup \mathcal{A}$ such that $\infty \in \overline{X}$. If $\infty \in \overline{X \cap \mathcal{A}}$ then clearly $\overline{X \cap \mathcal{A}} = (X \cap \mathcal{A}) \cup \{\infty\}$, so we may restrict ourselves to the case where $X \subseteq \omega_1$. Now, $\infty \in \overline{X}$ implies that $X \notin \mathcal{J}(\mathcal{A})$ (in particular, of course, X is uncountable), so by nowhere MADness of \mathcal{A} we can find some uncountable $Y \subseteq X$ such that $Y \cap p$ is countable for each $p \in \mathcal{A}$. Thus Y is an almost closed set such that $Y \rightarrow \infty$. \square

Lemma 28. *Let \mathcal{A} be a MAD family on a set X of cardinality ω_1 , and $\{\mathcal{A}_1, \mathcal{A}_2\}$ be a partition of \mathcal{A} , with \mathcal{A}_1 nowhere MAD on X and $|\mathcal{A}_1| \geq \omega_1$. Suppose also to have associated to every $A \in \mathcal{A}_1$ an element $M(A)$ of $[A]^{\omega_1}$. Then there exists $\tilde{A} \in \mathcal{A}_2$ such that $\tilde{A} \cap (\bigcup_{A \in \mathcal{A}_1} M(A)) \neq \emptyset$.*

Proof. Let, for the sake of simplicity, $M^* = \bigcup_{A \in \mathcal{A}_1} M(A)$, and notice that $M^* \in \mathcal{J}^+(\mathcal{A}_1)$. Indeed, if \mathcal{F} is a countable subfamily of \mathcal{A}_1 , then there exists an $\hat{A} \in \mathcal{A}_1 \setminus \mathcal{F}$. Since $\mathcal{A}_1 \subseteq \mathcal{A}$ and \mathcal{A} is almost disjoint, it follows that $|\hat{A} \cap (\bigcup \mathcal{F})| < \omega_1$. Therefore, in particular, $M(\hat{A}) \setminus \bigcup \mathcal{F} \neq \emptyset$, whence also $M^* \setminus \bigcup \mathcal{F} \neq \emptyset$.

Now, since \mathcal{A}_1 is nowhere MAD, there must exist $S \in [M^*]^{\omega_1}$ such that the set $S \cap (A \cap M^*) = S \cap A$ is countable for every $A \in \mathcal{A}_1$. On the other hand, since $S \in [M^*]^{\omega_1} \subseteq [X]^{\omega_1}$ and \mathcal{A} is MAD on X , there will be $\tilde{A} \in \mathcal{A}$ with $|\tilde{A} \cap S| = \omega_1$; therefore, such \tilde{A} must belong to \mathcal{A}_2 , and of course we have in particular $\tilde{A} \cap M^* \supseteq \tilde{A} \cap S \neq \emptyset$. \square

Corollary 29. *Let, as in the previous lemma, \mathcal{A} be a MAD family on a set X of cardinality ω_1 , and $\{\mathcal{A}_1, \mathcal{A}_2\}$ be a partition of \mathcal{A} , with \mathcal{A}_1 nowhere MAD on X and of cardinality ω_1 . Then there exists $A \in \mathcal{A}_1$ such that $|A \cap (\bigcup \mathcal{A}_2)| = \omega_1$.*

Proof. Toward a contradiction, assume $|A \cap (\bigcup \mathcal{A}_2)| < \omega_1$ for every $A \in \mathcal{A}_1$; letting, for every $A \in \mathcal{A}_1$, $M(A) = A \setminus (\bigcup \mathcal{A}_2)$, we see that each $M(A)$ is uncountable, hence by Lemma 28 it follows that $\tilde{A} \cap (\bigcup_{A \in \mathcal{A}_1} M(A)) \neq \emptyset$ for some $\tilde{A} \in \mathcal{A}_2$. Clearly, this is a contradiction, as each $M(A)$ is disjoint from $\bigcup \mathcal{A}_2$. \square

Theorem 30. *Let \mathcal{A} be an almost disjoint family on ω_1 , and $\{\mathcal{A}_1, \mathcal{A}_2\}$ be a partition of \mathcal{A} into two uncountable subfamilies. If \mathcal{A} is not nowhere MAD while \mathcal{A}_1 is nowhere MAD, then $\mathcal{L}(\mathcal{A}_1) \times \mathcal{L}(\mathcal{A}_2)$ is not a Whyburn space.*

Proof. Let $X \in \mathcal{M}(\mathcal{A}) \cap \mathcal{J}^+(\mathcal{A})$. Consider the diagonal $\Delta = \{(\alpha, \alpha) : \alpha \in X\}$; then $(\infty, \infty) \in \overline{\Delta}$. To verify this fact, consider a basic neighbourhood W of (∞, ∞) in $\mathcal{L}(\mathcal{A}_1) \times \mathcal{L}(\mathcal{A}_2)$ of the form

$$W = \left(\{\infty\} \cup (\mathcal{A}_1 \setminus \mathcal{F}_1) \cup \left(\bigcup (\mathcal{A}_1 \setminus \mathcal{F}_1) \setminus F_1 \right) \right) \times \left(\{\infty\} \cup (\mathcal{A}_2 \setminus \mathcal{F}_2) \cup \left(\bigcup (\mathcal{A}_2 \setminus \mathcal{F}_2) \setminus F_2 \right) \right),$$

where $\mathcal{F}_i \in [\mathcal{A}_i]^{<\omega_1}$ and $F_i \in [\omega_1]^{<\omega_1}$ for $i = 1, 2$; since $X \notin \mathcal{J}(\mathcal{A})$, the set $X \setminus \left(\left(\bigcup \mathcal{F}_1 \right) \cup \left(\bigcup \mathcal{F}_2 \right) \cup F_1 \cup F_2 \right)$ must be nonempty, and taking any α in it we easily see that $(\alpha, \alpha) \in \Delta \cap W$.

Now, let $B = \{(\alpha, \alpha) : \alpha \in Y\} \subseteq \Delta$ be such that $(\infty, \infty) \in \overline{B}$, with $Y \subseteq X$. Then Y must be uncountable, and the family $\mathcal{A}_Y = \{A \cap Y : A \in \mathcal{A}, |A \cap Y| = \omega_1\}$ is MAD on Y (as the restriction of \mathcal{A} to X is MAD on X), while the family $\mathcal{A}'_1 = \{A \cap Y : A \in \mathcal{A}_1, |A \cap Y| = \omega_1\}$ is nowhere MAD on Y (as \mathcal{A}_1 is nowhere MAD on ω_1). Letting also $\mathcal{A}'_2 = \{A \cap Y : A \in \mathcal{A}_2, |A \cap Y| = \omega_1\}$, we see that $\{\mathcal{A}'_1, \mathcal{A}'_2\}$ is a partition of \mathcal{A}_Y , so that by Corollary 29 there exists $A^* \in \mathcal{A}'_1$ such that $|A^* \cap \left(\bigcup \mathcal{A}'_2 \right)| = \omega_1$. Let $p \in \mathcal{A}$ be such that $A^* = p \cap Y$; we claim that $(p, \infty) \in \overline{B}$.

Indeed, consider a basic neighbourhood of (p, ∞) of the form

$$\left(\{p\} \cup (p \setminus F_1) \right) \times \left(\{\infty\} \cup (\mathcal{A}_2 \setminus \mathcal{F}_2) \cup \left(\bigcup (\mathcal{A}_2 \setminus \mathcal{F}_2) \setminus F_2 \right) \right),$$

with $\mathcal{F}_2 \in [\mathcal{A}_2]^{<\omega_1}$ and $F_1, F_2 \in [\omega_1]^{<\omega_1}$. Then from $|A^* \cap \left(\bigcup \mathcal{A}'_2 \right)| = \omega_1$ it follows that $|p \cap \left(\bigcup \mathcal{A}_2 \right) \cap Y| = \omega_1$; moreover, since $|A \cap p| < \omega_1$ for every $A \in \mathcal{F}_2$, it also follows that $|p \cap \left(\bigcup (\mathcal{A}_2 \setminus \mathcal{F}_2) \right) \cap Y| = \omega_1$, and finally that $|(p \setminus F_1) \cap \left(\bigcup (\mathcal{A}_2 \setminus \mathcal{F}_2) \setminus F_2 \right) \cap Y| = \omega_1$. Then taking $\hat{\alpha} \in (p \setminus F_1) \cap \left(\bigcup (\mathcal{A}_2 \setminus \mathcal{F}_2) \setminus F_2 \right) \cap Y$, we see that

$$(\hat{\alpha}, \hat{\alpha}) \in \left(\left(\{p\} \cup (p \setminus F_1) \right) \times \left(\{\infty\} \cup (\mathcal{A}_2 \setminus \mathcal{F}_2) \cup \left(\bigcup (\mathcal{A}_2 \setminus \mathcal{F}_2) \setminus F_2 \right) \right) \right) \cap B.$$

□

Thus to find a non-Whyburn product of Lindelöf Whyburn P -spaces it suffices to construct a MAD family on ω_1 which can be split into two nowhere MAD families. This can be done under a proper set-theoretic assumption, by a well-known line of reasoning (see [5]).

Theorem 31 ($[2^{\aleph_1} = \aleph_2]$). *There is a MAD family on ω_1 which can be split into two nowhere MAD subfamilies.*

Proof. Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_2\}$ be an enumeration of $[\omega_1]^{\omega_1}$. We will construct almost disjoint families $\{\mathcal{S}_\alpha : \alpha < \omega_2\}$ of uncountable subsets of ω_1 with the property that for each $\beta \leq \alpha$ either A_β is covered by a countable subcollection of \mathcal{S}_α or there is some $S \in \mathcal{S}_\alpha$ such that $S \subseteq A_\beta$. Let \mathcal{S}'_0 be any partition of ω_1 into ω_1 many uncountable sets. If A_0 is covered by a countable subfamily of \mathcal{S}'_0 , let $\mathcal{S}_0 = \mathcal{S}'_0$, else let $\mathcal{S}'_0 = \{T_\alpha : \alpha \in \omega_1\}$ be an enumeration; then for each $\beta \in \omega_1$ the set $A_0 \setminus \bigcup_{\alpha \leq \beta} T_\alpha$ is uncountable. Pick $a_0 \in A_0 \setminus T_0$ and for each $\beta \neq 0$ pick

$$a_\beta \in A_0 \setminus \left(\bigcup_{\alpha \leq \beta} T_\alpha \cup \{a_\alpha : \alpha < \beta\} \right).$$

Let $S_0 = \{a_\alpha : \alpha \in \omega_1\}$ and set $\mathcal{S}_0 = S'_0 \cup \{S_0\}$. Clearly, $S_0 \subseteq A_0$ and \mathcal{S}_0 is almost disjoint.

If \mathcal{S}_β is chosen for each $\beta < \alpha$, $\mathcal{S}'_\alpha = \bigcup_{\beta < \alpha} \mathcal{S}_\beta$. If A_α is not covered by a countable subfamily of \mathcal{S}'_α let $\mathcal{S}'_\alpha = \{T_\gamma^\alpha : \gamma \in \omega_1\}$ be an enumeration; then, for each $\beta \in \omega_1$ the set $A_\alpha \setminus \bigcup_{\gamma \leq \beta} T_\gamma^\alpha$ is uncountable. Pick $a_0^\alpha \in A_\alpha \setminus T_0^\alpha$ and for each $\beta \neq 0$ pick

$$a_\beta^\alpha \in A_\alpha \setminus \left(\bigcup_{\gamma \leq \beta} T_\beta^\alpha \cup \{a_\gamma^\alpha : \gamma < \beta\} \right).$$

Let $S_\alpha = \{a_\gamma^\alpha : \gamma \in \omega_1\}$ and let $\mathcal{S}_\alpha = \mathcal{S}'_\alpha \cup \{S_\alpha\}$. It is clear that each \mathcal{S}_α has the property stated at the beginning and is almost disjoint; note also that $\mathcal{S}_\beta \subseteq \mathcal{S}_\alpha$ whenever $\beta < \alpha$. Put $\mathcal{S}' = \bigcup_{\alpha \in \omega_2} \mathcal{S}_\alpha$; since \mathcal{S}' is almost disjoint there is some MAD family \mathcal{S} such that $\mathcal{S}' \subseteq \mathcal{S}$. The family \mathcal{S} inherits from \mathcal{S}' the property that if $X \subseteq \omega_1$ then either X is covered by a countable subfamily of \mathcal{S} or there is some $S \in \mathcal{S}$ such that $S \subseteq X$. For each $S \in \mathcal{S}$ let $\{S^+, S^*\}$ be a partition of S into two sets of cardinality ω_1 ; then set $\mathcal{S}^* = \{S^* : S \in \mathcal{S}\}$ and let $\mathcal{S}^+ = \{S^+ : S \in \mathcal{S}\}$. We see that $\mathcal{T} = \mathcal{S}^* \cup \mathcal{S}^+$ is a MAD family on ω_1 with the property that if $X \subseteq \omega_1$ is not covered by a countable subfamily of \mathcal{T} then there are disjoint $S_1 \in \mathcal{S}^*$ and $S_2 \in \mathcal{S}^+$ with $S_1 \cup S_2 \subseteq X$. This clearly implies that \mathcal{S}^* and \mathcal{S}^+ are nowhere MAD. \square

Corollary 32 ($[2^{\aleph_1} = \aleph_2]$). *There exist two Lindelöf Whyburn P-spaces whose product is not Whyburn.*

Question 33. *Does there exist in ZFC a non-Whyburn product of Lindelöf Whyburn P-spaces?*

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