A generalized Degn–Harrison reaction–diffusion system: asymptotic stability and non-existence results

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Abstract

In this paper we study the Degn-Harrison system with a generalized reaction term. Once proved the global existence and boundedness of a unique solution, we address the asymptotic behaviour of the system. The conditions for the global asymptotic stability of the steady state solution are derived using the appropriate techniques based on the eigen–analysis, the Poincaré–Bendixson theorem and the direct Lyapunov method. Numerical simulations are also shown to corroborate the asymptotic stability predictions.

Moreover, we determine the constraints on the size of the reactor and the diffusion coefficient such that the system does not admit non-constant positive steady state solutions.

Keywords: Generalized Degn–Harrison system; existence of solutions; steady states; asymptotic stability; non-constant steady state solutions.

1. Introduction

Reaction–diffusion systems provide accurate models of different natural and physical phenomena across a spectrum of disciplines including biology, ecology, chemistry, life sciences and engineering [1, 2, 3, 4, 5, 6]. The Degn–Harrison model is a reaction–diffusion system proposed in 1969 to describe the experimentally observed oscillatory behavior of the respiration rate in continuous cultures of the bacteria Klebsiella [7].

In recent years, several studies have been dedicated to the investigation of the Degn–Harrison dynamics. The diffusion–driven instability, the existence of nonconstant solutions and Turing patterns have been investigated in [8, 9, 10]. The stability analysis of the constant steady state solution, both in the ODE and the PDE scenario, has been studied in [11], where the authors also explain the mechanism leading to pattern formation.

More recently, sufficient conditions for the global asymptotic stability of the unique constant steady state have been obtained in [12] and more relaxed sufficient conditions have been then derived in [13]. The existence of Hopf bifurcation and the corresponding normal form have been also determined in [14, 15].

The Degn-Harrison system describes the reaction scheme between the oxygen u and the nutrient v, taking into account that the excess of oxygen inhibits the respiration according to a nonlinear rate of the type $u/(1+u^2)$, see for details [7]. In [16] the author, having in mind that other phenomena can be described by similar reaction schemes, generalizes the inhibitory law using an arbitrary function $\varphi(u)$. For the resulting generalized Degn–Harrison reaction–diffusion system, the existence of periodic solutions has been established in [16], using the Hopf bifurcation theory.

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$$\begin{cases} u_t - \Delta u = \gamma\left(a - u - \lambda\varphi\left(u\right)v\right) =: F\left(u,v\right), & x\in\Omega,\ t>0, \\ v_t - d\Delta v = \gamma\left(b - \lambda\varphi\left(u\right)v\right) =: G\left(u,v\right), & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

where u and v represent the dimensionless concentrations of the reactants. The parameters a, b, λ, γ and d are positive constants and the inhibitory function $\varphi \in C^1(0,\infty) \cap C[0,\infty)$ satisfies the following conditions:

$$\varphi(0) = 0, \tag{1.2}$$

and for $u \in [\delta, a]$:

$$\varphi(u) > 0, \tag{1.3}$$

with:

$$0 < \delta < a - b. \tag{1.4}$$

The system (1.1), defined in the bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, is supplemented with the initial data:

$$u(x,0) = u_0(x) \geq 0, \qquad v(x,0) = v_0(x) \geq 0, \qquad x \in \Omega \tag{1.5}$$

where $u_0(x)$ and $v_0(x)$ are smooth functions, and the following Neumann boundary conditions:

$$\frac{\partial u}{\partial \upsilon} = \frac{\partial v}{\partial \upsilon} = 0, \quad x \in \partial\Omega, \qquad t > 0, \tag{1.6}$$

where υ is the outward unit normal vector of the boundary $\partial\Omega$.

In what follows we shall assume $a > b$, therefore the system (1.1) admits a unique constant positive steady state:

$$(u^*, v^*) = \left(a - b, \frac{b}{\lambda\varphi(a-b)}\right). \tag{1.7}$$

The aim of the paper is twofold: prove the global existence of a unique bounded solution of the system (1.1) and find suitable conditions which prevent stationary pattern formation. In particular, the global attractivity of the steady state solution (1.7) and the non–existence of non–constant positive solutions will be addressed.

The plan of the paper is the following: in Section 2, once identified an invariant rectangle of the system (1.1), we will prove the existence of a unique solution for all $t > 0$ and we will establish its boundedness; in Section 3, the eigenfunction expansion method is used to settle the local asymptotic stability of the steady state solution (1.7). Then, the direct Lyapunov method is employed to obtain the conditions, involving both the system parameters and the arbitrary function $\varphi(u)$, assuring the global convergence to the homogeneous equilibrium solution (1.7); in Section 4, we will discuss the elliptic boundary value problem obtaining a priori estimates for the nonconstant steady state solutions. Moreover, the nonexistence of non-constant positive solutions will be proved when the size of the reactor are large enough or when the diffusion coefficient is below a threshold depending on the size of the reactor. Finally, in Section 5, numerical simulations are performed in order to corroborate the analytical findings of Section 3.

Throughout the paper the following notation will be used:

- the sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ denotes the eigenvalues of the elliptic Laplacian operator $-\Delta$ under the imposed Neumann boundary conditions on Ω.
- The algebraic multiplicity of the eigenvalue λ_i is denoted by $m_i \geq 1$.
- The normalized eigenfunctions associated with the eigenvalue λ_i are denoted by Φ_{ij}, $1 \leq j \leq m_i$.
- We let $E(\lambda_i) = \{\Phi_{ij} : i \geq 0, 1 \leq j \leq m_i\}$.

Recall that Φ_{0j} are constant, the eigenvalue λ_i tends to ∞ as $i \to \infty$ and $\int_\Omega \Phi_{ij}^2(x)dx = 1$, therefore the set $E(\lambda_i)$ is a complete orthonormal basis in $L^2(\Omega)$.

2. Global existence of a unique bounded solution

In this section, we shall show that the system (1.1) has a unique solution $(u(x,t), v(x,t))$, defined for all $t > 0$, which is bounded by some positive constants depending on the system parameters, the arbitrary function $\varphi(u)$ and the initial conditions u_0 and v_0. The existence of a unique bounded global solution will be proved applying the theory of invariant regions as was developed in [17, 18, 19].

Lemma 1. *For any $d > 0$, the system (1.1) admits a unique solution $(u,v) = (u(x,t), v(x,t))$ defined for all $x \in \Omega$ and $t > 0$. Moreover, there exist two positive constants C_1 and C_2, depending on the initial conditions (u_0, v_0), the system parameters a, b, λ and the arbitrary function φ, such that:*

$$C_1 < u(x,t), v(x,t) < C_2. \tag{2.1}$$

Proof The local existence and uniqueness of the solution for the system (1.1) are classical [20].

In order to prove the global existence and the boundedness, we construct the following rectangular region:

$$\Re = (u_1, u_2) \times (v_1, v_2),$$

where:

$$u_1 = \min\left\{\frac{a}{1+\lambda \sup\limits_{u\in[u_1,u_2]} \varphi_1(u) v_2}, \min_{x\in\overline{\Omega}} u_0(x)\right\}, \quad u_2 = \max\left\{a, \max_{x\in\overline{\Omega}} u_0(x)\right\},$$

$$v_1 = \min\left\{\frac{b}{\lambda u_2 \sup\limits_{u\in[u_1,u_2]} \varphi_1(u)}, \min_{x\in\overline{\Omega}} v_0(x)\right\}, \quad v_2 = \max\left\{\frac{b}{\lambda \min\limits_{u\in[u_1,u_2]} \varphi(u)}, \max_{x\in\overline{\Omega}} v_0(x)\right\},$$

and $\varphi_1(u)$ is such that $\varphi(u) = u\varphi_1(u)$ (it exists by condition (1.2)).

Along the edge $u = u_1$ and $v_1 \le v \le v_2$ of the rectangle $\Re$ the following inequality holds:

$$F(u,v) = \gamma(a - u_1 - \lambda\varphi(u_1)v) \ge \gamma a - \gamma u_1\left[1 + \lambda \sup_{u\in[u_1,u_2]} \varphi_1(u) v_2\right] \ge 0. \tag{2.2}$$

Analogously, evaluating $F(u,v)$ at the boundary $u = u_2$ and $v_1 \le v \le v_2$ of the rectangle $\Re$, we get:

$$F(u,v) = \gamma(a - u_2 - \lambda\varphi(u_2)v) \le \gamma(a - u_2) \le 0. \tag{2.3}$$

From (2.2) and (2.3), it follows that $F(u,v)$ points inside the rectangle $\Re$.

Evaluating the function $G(u,v)$ at the edge $v = v_1$ and $u_1 \le u \le u_2$ of the rectangle $\Re$, we obtain:

$$\begin{aligned} G(u,v) = \gamma[b - \lambda\varphi(u)v_1] &\ge \gamma\left[b - \lambda u v_1 \sup_{u\in[u_1,u_2]} \varphi_1(u)\right] \\ &\ge \gamma\left[b - \lambda u_2 v_1 \sup_{u\in[u_1,u_2]} \varphi_1(u)\right] \ge 0. \end{aligned} \tag{2.4}$$

At the last boundary of the rectangle $\Re$ defined as $v = v_2$ and $u_1 \le u \le u_2$, the following inequality holds for the function G:

$$G(u,v) = \gamma[b - \lambda\varphi(u)v_2] \le \gamma\left[b - \lambda \min_{u\in[u_1,u_2]} \varphi(u) v_2\right] \le 0. \tag{2.5}$$

By (2.4) and (2.5), it follows that $G(u,v)$ points inside the rectangle $\Re$. Therefore, the rectangle $\Re$ is an invariant rectangle for the system (1.1) [21, 18]. Finally, defining the constants C_1 and C_2 in (2.1) as follows:

$$C_1 = \min\{u_1, v_1\} > 0 \quad \text{and} \quad C_2 = \max\{u_2, v_2\} > 0, \tag{2.6}$$

we complete the proof.

□

Let us now prove the boundedness of the solutions.

Lemma 2. *Let $(u, v) = (u(x,t), v(x,t))$ be the unique solution of (1.1). Then, for all $x \in \overline{\Omega}$:*

$$\lim_{t\to\infty} \sup u < a, \quad \lim_{t\to\infty} \sup v < \frac{a-\delta}{\lambda\varphi(\delta)}. \tag{2.7}$$

Proof Let ε be a constant such that:

$$\varepsilon < \lambda\varphi(u)\, v, \tag{2.8}$$

and $\tilde{u} = \tilde{u}(t)$ be the unique solution of the following Cauchy problem:

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = \gamma(\tilde{a} - \tilde{u}), \\ \tilde{u}(0) = 2\max\limits_{x\in\overline{\Omega}} u_0(x), \end{cases} \tag{2.9}$$

with:

$$\tilde{a} = a - \frac{\varepsilon}{2}.$$

Let us also define the variable $\hat{u} = u - \tilde{u}$. From (2.9) and (1.1), we obtain:

$$\begin{cases} -\hat{u}_t + \Delta\hat{u} - \gamma\hat{u} = \gamma\left[\lambda\varphi(u)\, v - a + \tilde{a}\right] > 0, \\ \tilde{u}(x,0) < 0. \end{cases}$$

Using the maximum principle for parabolic equations and the Neumann boundary conditions (1.6), we get:

$$\hat{u}(x,t) < 0 \quad \Rightarrow \quad u(x,t) < \tilde{u}(t) \qquad \text{for all } t > 0 \text{ and } x \in \overline{\Omega}. \tag{2.10}$$

The maximum principle for parabolic equations cannot be directly used for the solution $v = v(x,t)$, therefore we define $\tilde{v}(t)$ as the solution of the following Cauchy problem:

$$\begin{cases} \dfrac{d\tilde{v}}{dt} = \gamma\tilde{g}(\tilde{u}, \tilde{v}), \\ \tilde{v}(x,0) = 2\max\limits_{x\in\overline{\Omega}} v_0(x), \end{cases} \tag{2.11}$$

where:

$$\tilde{g}(\tilde{u}, \tilde{v}) = \sup_{C_1<\xi<\tilde{u}} \left[\tilde{b} - \lambda(\tilde{v} - \varepsilon_0)\right]\varphi(\xi), \tag{2.12}$$

with $\varepsilon_0 > 0$, $\tilde{b} > b$ and:

$$\frac{\tilde{b}}{\lambda\varphi(\tilde{a})} + \varepsilon_0 < \frac{a-\delta}{\lambda\varphi(\delta)}.$$

Let $\hat{v} = v - \tilde{v}$. It follows straightforwardly that $\hat{v}(x,0) < 0$. Hence, we may prove by contradiction that for all $x \in \overline{\Omega}$ and $t > 0$:

$$\hat{v}(x,t) < 0. \tag{2.13}$$

If we let $\hat{v}(x,t) < 0$, then there exists $T > 0$ such that $\hat{v}(x,t) < 0$ for $(x,t) \in \overline{\Omega} \times (0,T)$ and $\hat{v}(x,t) = 0$ for some $x \in \overline{\Omega}$, which leads to:

$$\max_{x \in \overline{\Omega}} \hat{v}(x,t) = 0.$$

If there exists $x_1 \in \Omega$ such that $\hat{v}(x_1,T) = 0$, then $\hat{v}_t(x_1,T) \geq 0$ and $\Delta \hat{v}(x_1,T) \leq 0$ and thus we have:

$$-\hat{v}_t(x_1,T) + d\Delta \hat{v}(x_1,T) \leq 0. \tag{2.14}$$

However, if we combine (1.1) and (2.11) for point (x_1,T), we end up with:

$$-\hat{v}_t + d\Delta \hat{v} = \gamma \left[\tilde{g}\left(\tilde{u},\tilde{v}\right) - \left[b - \lambda \varphi\left(u\right) v\right]\right]. \tag{2.15}$$

Setting $\tilde{v} = v$ and $\tilde{u} > u$ yields:

$$\begin{aligned}
\tilde{g}\left(\tilde{u},\tilde{v}\right) &= \sup_{C_1 < \xi < \tilde{u}} \left[\tilde{b} - \lambda\left(\tilde{v} - \varepsilon_0\right)\right] \varphi\left(\xi\right), \\
&= \sup_{C_1 < \xi < \tilde{u}} \left[\tilde{b} - \lambda\left(v - \varepsilon_0\right)\right] \varphi\left(\xi\right), \\
&> \sup_{C_1 < \xi < \tilde{u}} \left[b - \lambda v\right] \varphi\left(\xi\right), \\
&\geq \sup_{C1 < \xi < u} \left[b - \lambda v\right] \varphi\left(\xi\right), \\
&\geq \left[b - \lambda v\right] \varphi\left(u\right).
\end{aligned}$$

Therefore:

$$\tilde{g}\left(\tilde{u},\tilde{v}\right) - \left[b - \lambda v\right] \varphi\left(u\right) \geq 0,$$

and consequently:

$$-\hat{v}_t(x_1,T) + d\Delta \hat{v}(x_1,T) > 0,$$

which contradicts the result in (2.14). Hence, (2.13) holds and we conclude that there exists some $x_1 \in \partial\Omega$ such that $\hat{v}(x_1,T) = 0$ leading to a positive right–hand side of (2.15) at (x_1,T). By continuity, we know that it remains positive in $\Omega' \times \{T\}$ for any Ω' being a sub–domain of Ω and $x_1 \in \Omega'$. Hence, we get:

$$-\hat{v}_t(x_1,T) + d\Delta \hat{v}(x_1,T) > 0,$$

on $\Omega' \times \{T\}$. Up to this point, we cannot state whether or not this inequality holds for $\Omega \times (0,T]$. Using Hopf's boundary lemma on (2.15) in $\overline{\Omega}' \times \{T\}$, we get $\partial\hat{v} = \partial\hat{v}(x_1,T) > 0$, which contradicts the Neumann boundary conditions and thus:

$$v(x,T) < \tilde{v}(t) \qquad \text{for all } x \in \overline{\Omega} \text{ and } t > 0. \tag{2.16}$$

Finally, we consider the ODEs system:

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = \gamma\left(\tilde{a} - \tilde{u}\right), \\ \\ \dfrac{d\tilde{v}}{dt} = \gamma \tilde{g}\left(\tilde{u},\tilde{v}\right), \end{cases}$$

in $\Re$. From (2.12), we find that:

$$\begin{cases} \tilde{g}\left(\tilde{u},\tilde{v}\right) < 0 & \text{for } \tilde{v} > \frac{\tilde{b}}{\lambda\varphi(\tilde{u})} + \varepsilon_0, \\ \tilde{g}\left(\tilde{u},\tilde{v}\right) > 0 & \text{for } \tilde{v} < \frac{\tilde{b}}{\lambda\varphi(\tilde{u})} + \varepsilon_0. \end{cases}$$

Hence, $\tilde{v} = \dfrac{\tilde{b}}{\lambda\varphi(\tilde{u})} + \varepsilon_0$ constitutes the nullcline of $\tilde{g}$ and the system admits the unique equilibrium:

$$(\tilde{u}, \tilde{v}) = \left(\tilde{a}, \frac{\tilde{b}}{\lambda\varphi(\tilde{a})} + \varepsilon_0\right).$$

Since $\lim_{t\to\infty} \tilde{u}(t) = \tilde{a}$, it follows that $(\tilde{u}, \tilde{v})$ is globally asymptotically stable in $\Re$, which implies that:

$$\lim_{t\to\infty} \tilde{v}(t) = \frac{\tilde{b}}{\lambda\varphi(\tilde{a})} + \varepsilon_0.$$

By (2.10) and (2.16), being $\tilde{a} < a$ and $\frac{\tilde{b}}{\lambda\varphi(\tilde{a})} + \varepsilon_0 < \frac{a-\delta}{\lambda\varphi(\delta)}$, the Lemma is proved.

□

3. Asymptotic Stability

In this Section we shall study the asymptotic behaviour of the generalized Degn–Harrison system (1.1). In particular, we will find the conditions on the system parameters and the arbitrary function $\varphi(u)$ which guarantee the attractivity of the unique homogeneous steady state solution (1.7) and therefore prevent pattern formation. The asymptotic analysis shall be performed at first for the local dynamics using the eigenfunction expansion method, then we will derive suitable conditions for the global asymptotic stability using also the direct Lyapunov method.

3.1. Local Asymptotic Stability

At first let us perform the linear stability analysis of the equilibrium (u^*, v^*) in (1.7).

Proposition 1. *Given the following ODEs system associated to the generalized Degn–Harrison system (1.1):*

$$\begin{cases} \dfrac{du}{dt} = \gamma\left[a - u - \lambda\varphi(u)v\right], & t > 0 \\[2ex] \dfrac{dv}{dt} = \gamma\left[b - \lambda\varphi(u)v\right], \end{cases} \tag{3.1}$$

the solution (u^*, v^*) is locally asymptotically stable as an equilibrium of (3.1) if:

$$-\left[\varphi(a-b) + b\varphi'(a-b)\right] < \lambda\varphi^2(a-b). \tag{3.2}$$

Proof The Jacobian matrix associated to the system (3.1) and evaluated in the equilibrium (u^*, v^*) is computed as:

$$J(u^*, v^*) = \gamma\begin{pmatrix} F_0 & -G_0 \\ 1 + F_0 & -G_0 \end{pmatrix}, \tag{3.3}$$

where:

$$F_0 = -1 - b\frac{\varphi'(a-b)}{\varphi(a-b)} \quad \text{and} \quad G_0 = \lambda\varphi(a-b). \tag{3.4}$$

The equilibrium (u^*, v^*) is locally asymptotically stable if the eigenvalues of the jacobian matrix $J(u^*, v^*)$ are both with negative real parts. The following characteristic equation associated to $J(u^*, v^*)$:

$$\sigma^2 - \mathrm{tr}(J(u^*, v^*))\sigma + \det(J(u^*, v^*)) = 0$$

admits roots with negative real parts if $\det(J(u^*, v^*)) > 0$ and $\mathrm{tr}(J(u^*, v^*)) < 0$. From the assumption (1.3) it follows that:

$$\det(J(u^*, v^*)) = \gamma^2 G_0 = \gamma^2\lambda\varphi(a-b) > 0. \tag{3.5}$$

Being:

$$\operatorname{tr}(J(u^*, v^*)) = \gamma(F_0 - G_0) = -\gamma\left[1 + b\frac{\varphi'(a-b)}{\varphi(a-b)} + \lambda\varphi(a-b)\right], \tag{3.6}$$

under the hypothesis (3.2) we have $\operatorname{tr}(J(u^*, v^*)) < 0$, therefore (u^*, v^*) is locally asymptotically stable. □

Using the eigenvalue/eigenfunction notation defined at the end of the Introduction, if

$$\lambda_1 < \gamma F_0, \tag{3.7}$$

then i_α is defined as the largest positive integer such that:

$$\lambda_i < \gamma F_0 \quad \text{for} \quad i \leq i_\alpha. \tag{3.8}$$

Clearly, if (3.7) holds, then $1 \leq i_\alpha < \infty$. In this case, we define the constant:

$$\tilde{d} = \min_{1 \leq i \leq i_\alpha} d_i, \qquad \text{with } d_i = \frac{\gamma^2 G_0(\lambda_i + 1)}{\lambda_i(\gamma F_0 - \lambda_i)}. \tag{3.9}$$

The following two theorems can now be formulated for the local stability of (u^*, v^*) as a steady state of (1.1).

Theorem 1. *Let us assume that condition (3.2) holds. The constant steady state (u^*, v^*) is locally asymptotically stable for the system* (1.1) *if:*

$$\begin{cases} \lambda_i \geq \gamma F_0 & \text{or} \\ \lambda_i < \gamma F_0 & \text{and} \quad 0 < d < \tilde{d}. \end{cases} \tag{3.10}$$

If:

$$\lambda_i < \gamma F_0 \qquad \text{and} \qquad d > \tilde{d},$$

then (u^*, v^*) is locally asymptotically unstable.

Proof Let L be the linearized operator associated to the system (1.1) in (u^*, v^*):

$$L = \begin{pmatrix} \Delta + \gamma F_0 & -\gamma G_0 \\ 1 + \gamma F_0 & d\Delta - \gamma G_0 \end{pmatrix}.$$

The constant steady state (u^*, v^*) is said to be locally asymptotically stable for the system (1.1) if and only if all the eigenvalues of L have negative real parts. Denoting $(\phi_1(x), \phi_2(x))$ the eigenfunction associated with the eigenvalue ξ, we get:

$$[L - \xi I](\phi_1(x), \phi_2(x))^t = (0, 0)^t,$$

which explicitly reads:

$$\begin{pmatrix} \Delta + \gamma F_0 - \xi & -\gamma G_0 \\ \gamma(1 + F_0) & d\Delta - \gamma G_0 - \xi \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Defining $(\phi_1(x), \phi_2(x))$ in sequence form as follows:

$$\phi_1 = \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_i} a_{ij}\Phi_{ij} \quad \text{and} \quad \phi_2 = \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_i} b_{ij}\Phi_{ij},$$

we obtain:

$$\sum_{0 \leq i \leq \infty, 1 \leq j \leq m_i} \begin{pmatrix} \gamma F_0 - \lambda_i - \xi & -\gamma G_0 \\ \gamma(1 + F_0) & -\gamma G_0 - d\lambda_i - \xi \end{pmatrix} \begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \Phi_{ij} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then, ξ is an eigenvalue of L if for some $i \geq 0$ the following equation is satisfied:

$$\xi^2 + P_i \xi + Q_i = 0,$$

where:

$$P_i = \lambda_i (d+1) + \gamma (G_0 - F_0),$$

and:

$$Q_i = \lambda_i d (\lambda_i - \gamma F_0) + \gamma^2 G_0 (\lambda_i + 1).$$

Since the condition (3.2) holds, then $P_i > 0$. Moreover, being $G_0 = \frac{1}{\gamma^2} \det(J(u^*, v^*))$, it is clear that $Q_0 > 0$ for $\lambda_0 = 0$. Let us now check the sign of Q_i if the conditions (3.10) of the Theorem 1 are satisfied:

- If $\lambda_i \geq \gamma F_0$, then $Q_i > 0$ for $i \geq 1$.
- If $\lambda_i < \gamma F_0$ and $0 < d < \tilde{d}$, then:

$$\lambda_i < \gamma F_0 \text{ and } 0 < d < d_i, \text{ for } i \in [1, i_\alpha].$$

Hence, $Q_i > 0$ for $i \in [1, i_\alpha]$. Furthermore, if $i \geq i_\alpha$, then $\lambda_i \geq \gamma F_0$ and $Q_i > 0$.

Therefore, when (3.2) and (3.10) hold, we get $P_i > 0$ and $Q_i > 0$ for all $i \geq 0$, which implies that all the eigenvalues ξ have negative real part, and the steady–state (u^*, v^*) is locally asymptotically stable.

Finally, if $\lambda_i < \gamma F_0$ and $d > \tilde{d}$, we assume that the minimum in (3.9) is obtained for some $k \in [1, i_\alpha]$:

$$d > d_k, \tag{3.11}$$

therefore $Q_k < 0$ and (u^*, v^*) is locally asymptotic unstable.

□

Theorem 2. *The homogeneous steady state (u^*, v^*) is locally asymptotically stable for the system (1.1) if $F_0 \leq 0$ or:*

$$0 < F_0 < G_0, \tag{3.12}$$

and:

$$\begin{cases} \lambda_1 \geq \gamma F_0 \quad or \\ \lambda_1 < \gamma F_0 \; and \; \begin{cases} d \leq \frac{G_0}{F_0} \quad or \\ \frac{G_0}{F_0} < d < \wp, \end{cases} \end{cases} \tag{3.13}$$

where $\wp$ is the solution of the following equation:

$$(F_0 x + G_0)^2 = 4(1 + F_0) G_0 x. \tag{3.14}$$

Proof First of all, let us rewrite (1.1) in vector form as follows:

$$\frac{\partial \mathbf{z}}{\partial t} = D \Delta \mathbf{z} + \mathbf{F}(\mathbf{z}), \tag{3.15}$$

where:

$$\mathbf{z} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \text{ and } \mathbf{F}(\mathbf{z}) = \gamma \begin{pmatrix} a - u - \lambda \varphi(u) v \\ b - \lambda \varphi(u) v \end{pmatrix}.$$

To prove the local asymptotic stability of (u^*, v^*) as the steady–state solution of (3.15) is equivalent to show that $\mathbf{z}^* = (0,0)^T$ is asymptotically stable as a steady state solution of the linearized system:

$$\frac{\partial \mathbf{z}}{\partial t} = D \Delta \mathbf{z} + A, \tag{3.16}$$

where $A = J(u^*, v^*)$.

The steady state $\mathbf{z}^*$ is locally asymptotic stable for the system (3.16), if the eigenvalues of $A - \lambda_n d$ have negative real parts for all $n \geq 0$. Since the system is 2×2, it suffices that the trace of the matrix $A - \lambda_n d$ is negative and its determinant is positive.

Let:

$$A - \lambda_n D = \begin{pmatrix} \gamma F_0 - \lambda_n & -\gamma G_0 \\ \gamma(1 + F_0) & -\gamma G_0 - d\lambda_n \end{pmatrix},$$

therefore:

$$\det(A - \lambda_n d) = \lambda_n d(\lambda_n - \gamma F_0) + \gamma \lambda_n G_0 + \gamma^2 G_0, \tag{3.17}$$
$$\operatorname{tr}(A - \lambda_n d) = (\gamma F_0 - \lambda_n) - \gamma G_0 - \lambda_n d \tag{3.18}$$

Let us recall that $G_0 > 0$ due to the assumption (1.3) and check the signs of the above expressions in (3.17)-(3.18) under the hypotheses of the Theorem 2.

- Let $F_0 \leq 0$.

  It is straightforward to check that the determinant in (3.17) is positive and the trace in (3.18) is negative.Hence, all the eigenvalues of $A - \lambda_n d$ have negative real parts and the steady–state $\mathbf{z}^*$ is locally asymptotic stable.

- Let (3.12) and the first condition in (3.13) be satisfied.

  For the first eigenvalue $\lambda_0 = 0$, the matrix $A - \lambda_0 d = A$ and:

  $$\det A = \gamma^2 G_0 > 0,$$

  and

  $$\operatorname{tr} A = \gamma(F_0 - G_0) < 0.$$

  Being $\lambda_1 \geq \gamma F_0$, then $\lambda_n \geq \gamma F_0$. Hence, the determinant in (3.17) is positive, the trace in (3.18) is negative and the steady state $\mathbf{z}^*$ is locally asymptotically stable.

- Let the second condition in (3.13) be satisfied.

  For the eigenvalues $\lambda_n,\ n > 1$ such that $\lambda_n \geq \gamma F_0$, with the same arguments as above we can conclude that $A - \lambda_n D$ has eigenvalues with negative real parts. Let θ one of the remaining eigenvalues such that $\theta < \gamma F_0$. Rewriting the trace as follows:

  $$\operatorname{tr}(A - \theta D) = \gamma(F_0 - G_0) - \theta(d + 1), \tag{3.19}$$

  it is straightforward to check it is negative under condition (3.12). We should now check the sign of the determinant:

  $$\det(A - \theta D) = \theta^2 d - \gamma\theta(dF_0 - G_0) + \gamma^2 G_0. \tag{3.20}$$

  If the condition $d \leq \frac{G_0}{F_0}$ also holds, then the determinant in (3.20) is positive and the steady state $\mathbf{z}^*$ is locally asymptotically stable.

  Generally, if the following discriminant:

  $$(\gamma(dF_0 - G_0))^2 - 4d\gamma^2 G_0 \tag{3.21}$$

  is negative, then the determinant in (3.20) is positive for all θ, which is equivalent to require that the following inequality holds:

  $$(F_0 d + G_0)^2 < 4(1 + F_0) G_0 d.$$

In the interval $[0, +\infty)$, between the parabola $y = (F_0 x + G_0)^2$ and the line $y = 4(1 + F_0)G_0 x$, it is easy to see that, at the point $\bar{x} = \frac{G_0}{F_0}$, we have:

$$(F_0 \bar{x} + G_0)^2 < 4(1 + F_0) G_0 \bar{x}.$$

The line intersects the parabola at two points x_1 and x_2 such that $0 < x_1 < \overline{x} < x_2$. Setting $\wp = x_2$, we obtain that $\wp$ is the solution of (3.14) satisfying:

$$\wp > \frac{G_0}{F_0}.$$

In addition, the inequality:

$$(F_0 x + G_0)^2 < 4(1 + F_0) G_0 x,$$

holds for:

$$\frac{G_0}{F_0} < x < \wp.$$

We can again conclude that the steady state $\mathbf{z}^*$ is locally asymptotically stable.

□

3.2. Global Asymptotic Stability

In this Section, we shall obtain sufficient conditions to achieve global asymptotic stability of the steady state solution (1.7). At first, we will apply the Poincaré–Bendixson theorem [22] at the ODEs system associated with (1.1) in order to obtain global stability for the local dynamics. Then, in Theorem 3, we shall find suitable conditions to guarantee the global stability of the steady state for the PDEs system (1.1).

The global stability of the equilibrium solution (1.7) will be also discussed performing the well–known direct Lyapunov method. Further conditions ensuring that the steady state solution is globally asymptotically stable for the system (1.1) are obtained in Theorem 6.

Let us first find the invariant rectangle $\Re_\delta$ defined as in (3.23).

Proposition 2. *Subject to conditions 1.4 and*

$$\frac{a - \delta}{\varphi(\delta)} > \frac{b}{\inf\limits_{u \in [\delta, a]} \varphi(u)}, \tag{3.22}$$

the rectangle:

$$\Re_\delta = [\delta, a] \times \left[\frac{b}{\lambda \sup\limits_{u \in [\delta, a]} \varphi(u)}, \frac{a - \delta}{\lambda \varphi(\delta)} \right], \tag{3.23}$$

is an invariant rectangle for system (1.1).

Proof Let us evaluate the vector field (F, G) given in (1.1) at the boundaries of the rectangle $\Re_\delta$. Let:

$$\frac{b}{\lambda \sup\limits_{u \in [\delta, a]} \varphi(u)} < v < \frac{a - \delta}{\lambda \varphi(\delta)},$$

then it straightforwardly results:

$$F(\delta, v) = \gamma(a - \delta - \lambda \varphi(\delta) v) > 0,$$

and

$$F(a, v) = \gamma(a - a - \lambda \varphi(a) v) = -\gamma \lambda \varphi(a) v < 0.$$

Similarly, assuming $\delta < u < a$ leads to:

$$G\left(u, \frac{b}{\lambda \sup\limits_{u\in[\delta,a]} \varphi(u)}\right) = \gamma\varphi(u)\left(\frac{b}{\varphi(u)} - \frac{b}{\sup\limits_{u\in[\delta,a]} \varphi(u)}\right) > 0,$$

and

$$G\left(u, \frac{a-\delta}{\lambda\varphi(\delta)}\right) = \gamma\left(b - \varphi(u)\frac{a-\delta}{\varphi(\delta)}\right) \leq \gamma\left(b - \inf_{u\in[\delta,a]} \varphi(u)\frac{a-\delta}{\varphi(\delta)}\right) < 0,$$

where the last inequality follows by condition (3.22). Therefore the vector field (F, G) points inside on the boundary $\partial\Re$, which implies the rectangle $\Re$ is an invariant set for the system (1.1).

□

The following Theorem will give the conditions for the global asymptotic stability of (u^*, v^*) as a solution of the reduced ODEs system associated to (1.1).

Theorem 3. *Given the ODEs system* (3.1)*, let us define* $f(u) = \dfrac{a-u}{\varphi(u)}$ *and* $u_i, i = 1, \ldots, N$*, be the inflection points of the function* $f(u)$*. If the following condition holds:*

$$\max\left\{\max_{i=1,\ldots,N} f'(u_i), f'(\delta), f'(a)\right\} < \lambda, \tag{3.24}$$

then the equilibrium (u^*, v^*) *given in* (1.7) *is globally asymptotically stable for the system (3.1).*

Proof Let us rewrite the system (3.1) in terms of the function $f(u)$:

$$\begin{cases} \dfrac{du}{dt} = F(u,v) = \gamma\varphi(u)\left(\dfrac{a-u}{\varphi(u)} - \lambda v\right) = \gamma\varphi(u)(f(u) - \lambda v), \\ \\ \dfrac{dv}{dt} = G(u,v) = \gamma\varphi(u)\left(\dfrac{b}{\varphi(u)} - \lambda v\right). \end{cases} \tag{3.25}$$

We would like to apply the Dulac criterion to the plane system (3.25) in the invariant region $\Re_\delta$ defined in (3.23).

Let $\psi = \frac{1}{\gamma\varphi(u)}$ be the Dulac function candidate. We shall check the sign of the following divergence:

$$\frac{\partial(\psi F)}{\partial u} + \frac{\partial(\psi G)}{\partial v} = \frac{-\varphi(u) - (a-u)\varphi'(u)}{(\varphi(u))^2} - \lambda = f'(u) - \lambda. \tag{3.26}$$

If $f(u)$ is decreasing, then $f'(u) < 0$ and the sign of the divergence in (3.26) is negative. If $f(u)$ is not decreasing, then:

$$f'(u) \leq \max\left\{\max_{i=1,\ldots,N} f'(u_i), f'(\delta), f'(a)\right\} \quad \text{in} \quad [\delta, a],$$

which implies:

$$f'(u) - \lambda \leq \max\left\{\max_{i=1,\ldots,N} f'(u_i), f'(\delta), f'(a)\right\} - \lambda < 0,$$

where the last inequality holds under the hypothesis (3.24) of the Theorem. Therefore, the divergence in (3.26) has the same negative sign in $\Re_\delta$ and, according to the Dulac criterion, there are no closed orbits lying entirely in $\Re_\delta$.

To complete the proof it suffices to show that (u^*, v^*) is locally asymptotic stable. Since $f'(u^*) <$

$\max\left\{\max_{i=1,\ldots,N} f'(u_i), f'(\delta), f'(a)\right\}$, using the assumption (3.2), it follows that:

$$f'(u^*) < \lambda. \tag{3.27}$$

The condition in (3.27) is equivalent to the assumption (3.2) which guarantees the local asymptotic stability of the equilibrium (u^*, v^*). Therefore, using the absence of periodic solutions and the Poincaré- Bendixson theorem, we complete the proof.

□

In order to achieve the global stability of the steady state solution for the system (1.1), we first prove the following preliminary Theorem 4.

Let α denote the following quantity:

$$\alpha = \max_{(u,v)\in\Re_\delta} \varsigma(u,v), \tag{3.28}$$

where $\varsigma(u,v)$ is the greatest real eigenvalue of the symmetric matrix J^H:

$$J^H = \frac{1}{2}\left(J + J^T\right),$$

with J the Jacobian matrix associated to the system (3.1) and J^T its transpose matrix.

Theorem 4. *Assume that:*

$$f'(u^*) > 0 \quad \textit{and} \quad \lambda_1 > \frac{\alpha}{\beta}, \tag{3.29}$$

where $f(u) = \dfrac{a-u}{\varphi(u)}$ *as in the previous theorem,* α *is defined by (3.28) and* $\beta = \min\{1, d\}$*. Let* $\mathbf{z}(x,t)$ *be a solution of the Neumann boundary value problem associated with the linearized system (3.16). Then:*

$$\lim_{t\to\infty} \|\nabla \mathbf{z}(.,t)\|_{L^2(\Omega)} = 0. \tag{3.30}$$

Proof In order to prove (3.30), we show that there exist two constants T and C such that:

$$\|\nabla \mathbf{z}(.,t)\|_{L^2(\Omega)} \leq Ce^{-(\beta\lambda_1-\alpha)t}, \quad \text{for } t > T. \tag{3.31}$$

In fact, the inequality (3.31) together with the assumption $\lambda_1 > \frac{\alpha}{\beta}$ in (3.29) will directly imply (3.30).

At first, we observe that the assumption $f'(u^*) > 0$ is equivalent to $F_0 > 0$.

Let us evaluate the matrix J^H at the steady state (u^*, v^*):

$$J^H(u^*, v^*) = \gamma \begin{pmatrix} F_0 & \frac{1}{2}(1 + F_0 - G_0) \\ \frac{1}{2}(1 + F_0 - G_0) & -G_0 \end{pmatrix}.$$

Being $F_0 > 0$, it follows that:

$$\det J^H(u^*, v^*) = -F_0 G_0 - \frac{1}{4}(1 + F_0 - G_0)^2 < 0,$$

therefore the constant α in (3.28) is positive:

$$\alpha \geq \varsigma(u^*, v^*) > 0.$$

For the linearized system (3.16), there exist $T > 0$ such that:

$$\mathbf{z}(x,t) = (u(x,t), v(x,t)) \in \Re_\delta, \qquad t > T.$$

Let us define the following function:

$$\begin{aligned} \Phi(t) &= \frac{1}{2} \|\nabla \mathbf{z}(.,t)\|^2_{L^2(\Omega)} \\ &= \frac{1}{2} \int_\Omega \langle \nabla \mathbf{z}(x,t), \nabla \mathbf{z}(x,t) \rangle \, dx, \qquad \text{for } t > T, \end{aligned} \tag{3.32}$$

where $\langle .,. \rangle$ denotes the inner product in $\mathbb{R}^2$. The derivative of $\Phi(t)$ is thus given by:

$$\begin{aligned} \frac{d\Phi(t)}{dt} &= \int_\Omega \langle \nabla \mathbf{z}, \nabla \mathbf{z}_t \rangle \, dx \\ &= -\int_\Omega \langle \Delta \mathbf{z}, D\Delta \mathbf{z} \rangle \, dx + \int_\Omega \left\langle \nabla \mathbf{z}, J^H(\mathbf{z}) \nabla \mathbf{z} \right\rangle dx. \end{aligned} \tag{3.33}$$

Using Lemma A.1 of [23], we deduce the following inequality:

$$\int_\Omega \langle \Delta \mathbf{z}, D\Delta \mathbf{z} \rangle \, dx \geq \beta \lambda_1 \int_\Omega |\nabla \mathbf{z}|^2 \, dx. \tag{3.34}$$

Using the definition (3.28) and the properties of the symmetric matrix J^H, the inequality in (3.34) can be rearranged as follows:

$$\left\langle \nabla \mathbf{z}, J^H(\mathbf{z}) \nabla \mathbf{z} \right\rangle \leq \varsigma(\mathbf{z}) |\nabla \mathbf{z}|^2 \leq \alpha |\nabla \mathbf{z}|^2. \tag{3.35}$$

Using (3.35) into (3.33), we obtain:

$$\frac{d\Phi(t)}{dt} \leq -(\beta \lambda_1 - \alpha) \int_\Omega |\nabla \mathbf{z}|^2 \, dx, \qquad t > T.$$

Hence, the function Φ satisfies the following differential inequality:

$$\Phi'(t) \leq -2(\beta \lambda_1 - \alpha) \Phi(t), \qquad \text{for } \ t > T. \tag{3.36}$$

From (3.36) we can state that there exists a constant $c_1 > 0$ such that:

$$\Phi(t) \leq c_1 e^{-2(\beta \lambda_1 - \alpha)t},$$

and by the definition in (3.32), the (3.31) trivially follows with $C = 2c_1$.

□

Let us now state the following Theorem on the global asymptotic stability of the steady–state solution (u^*, v^*) for the system (1.1).

Theorem 5. *Under the same assumptions of the Theorems 3 and 4, we have:*

$$\lim_{t \to \infty} \|u(x,t) - u^*\|_{L^2(\Omega)} = \lim_{t \to \infty} \|v(x,t) - v^*\|_{L^2(\Omega)} = 0. \tag{3.37}$$

Proof Let $\mathbf{z} = (u(x,t), v(x,t))$ be a solution of the system (1.1). As demonstrated in Lemma A.2 of [23], we may use the Poincaré inequality to obtain:

$$\|\mathbf{z}(.,t) - \overline{\mathbf{z}}(.,t)\|^2_{L^2(\Omega)} \leq \frac{1}{\lambda_1} \|\nabla \mathbf{z}(.,t)\|^2_{L^2(\Omega)}, \tag{3.38}$$

where:

$$\overline{\mathbf{z}}(t) = \frac{1}{|\Omega|}\int_\Omega \mathbf{z}(x,t)\,dx.$$

Using the inequality (3.38) and (3.30) we obtain:

$$\lim_{t\to\infty} \|u(x,t) - \overline{u}(t)\|_{L^2(\Omega)} = \lim_{t\to\infty} \|v(x,t) - \overline{v}(t)\|_{L^2(\Omega)} = 0, \tag{3.39}$$

where $\overline{u}(t)$ and $\overline{v}(t)$ denote, respectively, the averages on Ω of $u(x,t)$ and $v(x,t)$. Now, using Theorem 3.1 in [23] again, we deduce that the pair $(\overline{u}(t), \overline{v}(t))$ satisfies the following ODEs system:

$$\begin{cases} u' = F(u,v) + q_1(t) \\ v' = G(u,v) + q_2(t) \\ u(0) = \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx, \quad v(0) = \frac{1}{|\Omega|}\int_\Omega v_0(x)\,dx, \end{cases} \tag{3.40}$$

where for some $k > 0, t > T$, and $i = 1,2$, we have:

$$|q_i(t)| \le k e^{-(\beta\lambda_1 - \alpha)t}. \tag{3.41}$$

From (3.41) it follows that as $t \to \infty$,

$$\int_t^{t+1} q_i(s)\,ds \to 0, \qquad \text{for } i = 1,2.$$

Moreover, Theorem 3 guarantees that the constant steady state solution is globally asymptotically stable for the ODE system. At this stage, we apply Theorem 5.5.7 of [24] to show that every solution of (3.40) converges to (u^*, v^*), thus:

$$\lim_{t\to\infty} |\overline{u}(t) - u^*| = \lim_{t\to\infty} |\overline{v}(t) - v^*| = 0. \tag{3.42}$$

Since the following inequalities hold:

$$\|u(.,t) - u^*\|_{L^2(\Omega)} \le \|u(.,t) - \overline{u}(t)\|_{L^2(\Omega)} + |\Omega|^{\frac{1}{2}} |\overline{u}(t) - u^*|,$$

and

$$\|v(.,t) - v^*\|_{L^2(\Omega)} \le \|v(.,t) - \overline{v}(t)\|_{L^2(\Omega)} + |\Omega|^{\frac{1}{2}} |\overline{v}(t) - v^*|,$$

using (3.39) and (3.42) we end up the proof of the Theorem.

□

In what follows we will discuss the global asymptotic stability of the equilibrium (1.7) using the direct Lyapunov method. Once given some preliminary results, we will obtain the suitable conditions for global stability of the equilibrium in the Theorem 6.

Lemma 3. *If $u \in [\delta, a]$, then there exists a constant μ between u and u^* such that*

$$\frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)} = (u - u^*)\left(\frac{b}{\varphi(u)}\right)'_{u=\mu}. \tag{3.43}$$

Lemma 4. *The derivative of the function:*

$$H(u(x,t)) = \int_\alpha^u \left(\frac{b}{\varphi(r)} - \frac{b}{\varphi(u^*)}\right) dr \ge 0, \tag{3.44}$$

is given by:

$$\frac{d}{du}H(u) = \frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)}. \tag{3.45}$$

Proposition 3. *Let:*

$$V(t) = \int_\Omega E(u(x,t), v(x,t))\,dx, \tag{3.46}$$

where

$$E(u,v) = H(u) + \frac{\lambda}{2}(v - v^*) \tag{3.47}$$

and $(u(x,t), v(x,t))$ *is a solution of the system* (1.1). *If* $\varphi(u)$ *is a decreasing function and:*

$$(u^* - u)\left(\frac{a-u}{\varphi(u)} - \frac{a-u^*}{\varphi(u^*)}\right) > 0 \quad \text{for} \quad u \in [\delta, u^*) \cup (u^*, a], \tag{3.48}$$

then $V(t)$ *is a Lyapunov functional.*

Proof Let us rewrite the system (1.1) in the following convenient form:

$$\begin{cases} u_t - \Delta u = \gamma\varphi(u)\left[\left(\frac{a-u}{\varphi(u)} - \frac{a-u^*}{\varphi(u^*)}\right) - \lambda\left(v - \frac{b}{\lambda\varphi(u^*)}\right)\right], \\ v_t - d\Delta v = \gamma\varphi(u)\left[\left(\frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)}\right) - \lambda\left(v - \frac{b}{\lambda\varphi(u^*)}\right)\right], \end{cases} \tag{3.49}$$

with $u^* = a - b$ and $x \in \Omega,\ t > 0$.

Differentiating the functional $V(t)$ with respect to t yields:

$$\begin{aligned}\dot{V}(t) &= \lambda\int_\Omega\left[(v-v^*)\left(d\Delta v + \gamma\varphi(u)\left(\left(\frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)}\right) - \lambda(v-v^*)\right)\right)\right]dx \\ &+ \int_\Omega\left[\left(\frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)}\right)\left(\Delta u + \gamma\varphi(u)\left(\left(\frac{a-u}{\varphi(u)} - \frac{a-u^*}{\varphi(u^*)}\right) - \lambda(v-v^*)\right)\right)\right]dx\,,\end{aligned}$$

which we rewrite as follow:

$$\dot{V}(t) = I + J, \tag{3.50}$$

where:

$$I = \int_\Omega\left(\frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)}\right)\Delta u\ dx + d\lambda\int_\Omega(v-v^*)\,\Delta v\ dx,$$

and:

$$J = \int_\Omega\gamma\varphi(u)\left[\left(\frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)}\right)\left(\frac{a-u}{\varphi(u)} - \frac{a-u^*}{\varphi(u^*)}\right) - \lambda^2(v-v^*)^2\right]dx.$$

We now check the sign of I and J:

$$\begin{aligned} I &= \int_\Omega\left(\frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)}\right)\Delta u\ dx + d\lambda\int_\Omega(v-v^*)\,\Delta v\ dx \\ &= -\int_\Omega\nabla\left(\frac{b}{\varphi(u)} - \frac{b}{\varphi(u^*)}\right)\nabla u\ dx - d\lambda\int_\Omega\nabla(v-v^*)\,\nabla v\ dx \\ &= -\int_\Omega\left(\frac{b}{\varphi(u)}\right)'|\nabla u|^2\,dx - d\lambda\int_\Omega|\nabla v|^2\,dx \le 0.\end{aligned}$$

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The condition (3.48) leads to:

$$u \leq u^* \Longrightarrow (u-u^*)\left(\frac{a-u}{\varphi(u)} - \frac{a-u^*}{\varphi(u^*)}\right) \leq 0, \tag{3.51}$$

$$u \geq u^* \Longrightarrow (u-u^*)\left(\frac{a-u}{\varphi(u)} - \frac{a-u^*}{\varphi(u^*)}\right) \leq 0. \tag{3.52}$$

Using (3.51)-(3.52), it is straightforward to show that $J \leq 0$. Therefore:

$$\dot{V}(t) \leq 0$$

and V is a Lyapunov functional.

□

Theorem 6. *Let $\varphi(u)$ be a decreasing function and assume that (3.48) holds. Then, for any solution (u, v) of (1.1) in $\Re_\delta$ we have:*

$$\lim_{t\to\infty} \|u(x,t) - u^*\|_{L^2(\Omega)} = \lim_{t\to\infty} \|v(x,t) - v^*\|_{L^2(\Omega)} = 0. \tag{3.53}$$

Proof If $(u, v) \in \Re_\delta$ is a solution of (1.1) for which $\frac{d}{dt}V(t) = 0$, where $V(t)$ is the Lyapunov functional defined in (3.46), then u and v must be spatially homogeneous. Therefore, (u, v) satisfies the ODE system (3.1). Noting that $\{(u^*, v^*\})$ is the largest invariant subset of the system (3.1):

$$\left\{(u,v) \in \Re_\delta \mid \frac{d}{dt}V(t) = 0\right\},$$

we can employ the La Salle's invariance theorem [25, 26] to obtain:

$$\lim_{t\to\infty} |u(x,t) - u^*| = \lim_{t\to\infty} |v(x,t) - v^*| = 0,$$

uniformly in x. Hence:

$$\lim_{t\to\infty} \int_\Omega (u(x,t) - u^*)^2\, dx = \lim_{t\to\infty} \int_\Omega (v(x,t) - v^*)^2\, dx = 0, \tag{3.54}$$

which implies (3.53).

□

4. Nonconstant positive solutions

Let us now analyze the following elliptic boundary value problem:

$$\begin{cases} \Delta u + \gamma\left[a - u - \lambda\varphi(u)v\right] = 0, & x \in \Omega, \\ d\Delta v + \gamma\left[b - \lambda\varphi(u)v\right] = 0, & x \in \Omega, \end{cases} \tag{4.1}$$

supplemented with the following Neumann boundary conditions:

$$\frac{\partial u}{\partial \upsilon} = \frac{\partial v}{\partial \upsilon} = 0 \text{ for all } x \in \partial\Omega, \tag{4.2}$$

in such a way to determine a priori estimates for the nonconstant steady state solution and to find conditions for the nonexistence of nonconstant positive solutions.

4.1. A priori estimates of the nonconstant steady state solution

Let us preliminarily state the following useful Proposition whose complete proof can be found in [27].

Proposition 4. *Given the functions $g \in C\left(\overline{\Omega} \times \mathbb{R}\right)$ and $w \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$, it follows that:*
(i) *if:*

$$\Delta w(x) + g(x, w(x)) \geq 0 \quad in \ \ \Omega,$$

with $\dfrac{\partial w}{\partial \nu} \leq 0$ *on* $\partial\Omega$ *and* $w(x_0) = \max_{\overline{\Omega}}(w(x))$*, then:*

$$g(x_0, w(x_0)) \geq 0.$$

(ii) *Alternatively, if:*

$$\Delta w(x) + g(x, w(x)) \leq 0 \quad in \ \Omega,$$

with $\dfrac{\partial w}{\partial \nu} \geq 0$ *on* $\partial\Omega$ *and* $w(x_0) = \min_{\overline{\Omega}}(w(x))$*, then:*

$$g(x_0, w(x_0)) \leq 0.$$

Proposition 5. *(A priori estimates) Let $(u, v) = (u\,(x)\,, v\,(x))$ be a positive solution to the elliptic boundary value problem (4.1). Assuming:*

$$\min_{u \in [\delta, a]} \varphi\,(u) > b, \tag{4.3}$$

the following estimates hold for all $x \in \Omega$:

$$\begin{cases} \dfrac{a}{\sup\limits_{u \in [\delta, a]} \varphi_1\,(u)} \left(1 - \dfrac{b}{\min\limits_{u \in [\delta, a]} \varphi\,(u)}\right) < u\,(x) < a, \\[2ex] \dfrac{b}{\lambda \max\limits_{u \in [\delta, a]} \varphi\,(u)} < v\,(x) < \dfrac{b}{\lambda \left(\min\limits_{u \in [\delta, a]} \varphi\,(u) - b\right)}. \end{cases} \tag{4.4}$$

Proof If the function u has a maximum over $\overline{\Omega}$ at some point in space, then by applying Proposition 4 to the boundary value problem (4.1), we obtain:

$$a - u - \lambda \varphi\,(u)\, v \geq 0,$$

then:

$$a - u > a - u - \lambda \varphi\,(u)\, v \geq 0,$$

which implies the following upper bound for the solution u:

$$u < a. \tag{4.5}$$

Similarly, if v has a maximum over $\overline{\Omega}$ at some point, then by Proposition 4 it follows:

$$b - \lambda \varphi\,(u)\, v \geq 0.$$

Being:

$$b - \lambda \min \varphi\,(u)\, v + \lambda b v > b - \lambda \varphi\,(u)\, v \geq 0,$$

by condition (4.3) we get:

$$b - \lambda v \left(\min_{u \in [\delta, a]} \varphi\,(u) - b\right) > 0,$$

leading to the following upper bound for the function v:

$$v < \frac{b}{\lambda \left(\min\limits_{u\in[\delta,a]} \varphi(u) - b \right)}. \tag{4.6}$$

In order to find the lower bounds in (4.4), we consider the case in which u has a minimum over $\overline{\Omega}$ at some point, then by Proposition 4 it follows:

$$a \le u + \lambda\varphi(u)\, v < u \sup_{u\in[\delta,a]} \varphi_1(u)\,(1+\lambda v)\,.$$

Then, taking into account the bound (4.6), we get:

$$a < u \sup_{u\in[\delta,a]} \varphi_1(u) \left(1 + \lambda \frac{b}{\lambda \left(\min\limits_{u\in[\delta,a]} \varphi(u) - b \right)} \right),$$

which implies:

$$a \left(\min_{u\in[\delta,a]} \varphi(u) - b \right) < u \sup_{u\in[\delta,a]} \varphi_1(u) \left(\min_{u\in[\delta,a]} \varphi(u) \right),$$

and thus the following lower bound for u is obtained:

$$u > \frac{a}{\sup\limits_{u\in[\delta,a]} \varphi_1(u)} \left(1 - \frac{b}{\min\limits_{u\in[\delta,a]} \varphi(u)} \right). \tag{4.7}$$

Assuming that v admits a minimum at some point over $\overline{\Omega}$ leads to:

$$b - \lambda\varphi(u)\, v \le 0,$$

which implies:

$$b - \lambda \max_{u\in[\delta,a]} \varphi(u)\, v \le b - \lambda\varphi(u)\, v \le 0,$$

then the lower bound for v is given by:

$$\frac{b}{\lambda \max\limits_{u\in[\delta,a]} \varphi(u)} \le v. \tag{4.8}$$

□

Notice that the estimates in (4.4) guarantee that there exist two positive constants $c_1 \equiv c_1(b,\gamma)$ and $c_2 \equiv c_2(a,\gamma)$ such that:

$$|G(u,v)| = |\gamma\,[b - \lambda\varphi(u)\, v]| \le c_1, \tag{4.9}$$

and

$$|F(u,v)| = |\gamma\,[a - u - \lambda\varphi(u)\, v]| \le c_2. \tag{4.10}$$

Let us now define the averages of a given pair of solutions $(u,v) = (u(x), v(x))$ to the elliptic problem (4.1) over Ω as follows:

$$\overline{u} = \frac{1}{|\Omega|} \int_\Omega u(x)\, dx \quad \text{and} \quad \overline{v} = \frac{1}{|\Omega|} \int_\Omega v(x)\, dx,$$

where $|\Omega|$ is the volume of Ω.

Lemma 5. *The average of $u(x)$ over Ω is given by:*

$$\overline{u} = a - b. \tag{4.11}$$

Proof Let us define the following change of variable:

$$w(x) = dv(x) - u(x). \tag{4.12}$$

From (4.1), we get:

$$\Delta w(x) = \gamma [a - b - u]. \tag{4.13}$$

Integrating (4.13) over Ω yields:

$$\gamma \int_\Omega [a - b - u]\, dx = \int_\Omega \Delta w(x)\, dx = \int_\Omega \frac{\partial w}{\partial \nu} ds = 0,$$

therefore:

$$\overline{u} = a - b.$$

□

Let us denote:

$$\phi = u - \overline{u} \quad \text{and} \quad \psi = v - \overline{v}, \tag{4.14}$$

then:

$$\int_\Omega \phi\ dx = \int_\Omega \psi\ dx = 0. \tag{4.15}$$

If (u, v) is not a constant solution, then ϕ and ψ must not be trivial and their signs should alternate in Ω. The following Lemma shows that the product $\phi\psi$ has a positive average over Ω.

Lemma 6. *Let (u, v) be a noncostant solution of (4.1) and (ϕ, ψ) defined as in (4.14). Then:*

$$\int_\Omega \phi\psi\ dx > 0 \quad \textit{and} \quad \int_\Omega \nabla\phi\nabla\psi\ dx > 0. \tag{4.16}$$

Proof Equation (4.13) can be rewritten as:

$$-\Delta w = \gamma\phi. \tag{4.17}$$

Multiplying (4.17) by $w = dv - u$ and integrating by parts lead to:

$$\int_\Omega |\nabla w|^2\, dx = \gamma d \int_\Omega \phi v\ dx - \gamma \int_\Omega \phi u\ dx,$$

which implies:

$$\int_\Omega |\nabla w|^2\, dx = \gamma d \int_\Omega \phi\psi\ dx - \gamma \int_\Omega \phi^2 dx.$$

Therefore:

$$\int_\Omega \phi\psi\ dx = \frac{1}{\gamma d} \int_\Omega |\nabla w|^2\, dx + \frac{1}{d} \int_\Omega \phi^2 dx > 0 \tag{4.18}$$

and the first inequality in (4.16) is proved.

Multiplying (4.17) by ϕ and integrating by parts yields:

$$\gamma \int_\Omega \phi^2 dx = d \int_\Omega \nabla\phi\nabla\psi\ dx - \int_\Omega \nabla\phi^2 dx,$$

which implies the second inequality in (4.16):

$$\int_\Omega \nabla\phi\nabla\psi\ dx = \frac{\gamma}{d}\int_\Omega \phi^2 dx + \frac{1}{d}\int_\Omega \nabla\phi^2 dx > 0. \tag{4.19}$$

□

Lemma 7. *There exists a constant $C_G \equiv C_G(b,\gamma,\Omega)$ such that:*

$$\int_\Omega \psi^2 dx + \int_\Omega |\nabla\psi|^2\, dx \le C_G d^{-2}. \tag{4.20}$$

Proof Using (4.1), the Cauchy–Schwarz inequality and condition (4.9), we obtain:

$$d\int_\Omega |\nabla\psi|^2\, dx = \int_\Omega G(u,v)\,\psi\ dx \le c_1\sqrt{|\Omega|}\left(\int_\Omega |\psi|^2\, dx\right)^{1/2}. \tag{4.21}$$

The Poincaré inequality yields:

$$\int_\Omega \psi^2 dx \le \frac{1}{\lambda_1}\int_\Omega |\nabla\psi|^2\, dx, \tag{4.22}$$

where $\lambda_1 > 0$ is the first positive eigenvalue of $(-\Delta)$. Therefore, under the Neumann boundary conditions, from (4.21) it follows:

$$d\int_\Omega |\nabla\psi|^2\, dx \le c_1\sqrt{\frac{|\Omega|}{\lambda_1}}\left(\int_\Omega |\nabla\psi|^2\, dx\right)^{1/2},$$

and consequently:

$$\int_\Omega |\nabla\psi|^2\, dx \le \frac{|\Omega|\, c_1^2}{\lambda_1 d^2}. \tag{4.23}$$

Adding up (4.22) and (4.23) and using once again the inequality in (4.23) leads to:

$$\int_\Omega \psi^2 dx + \int_\Omega |\nabla\psi|^2\, dx \le C_G d^{-2},$$

where:

$$C_G = c_1^2\,|\Omega|\left(\frac{1+\lambda_1}{\lambda_1^2}\right).$$

□

Lemma 8. *There exists a constant $C_F \equiv C_F(a,\gamma,\Omega)$ such that:*

$$\int_\Omega \phi^2 dx + \int_\Omega |\nabla\phi|^2\, dx \le C_F. \tag{4.24}$$

Proof The proof follows the same lines of the previous Lemma. Applying the Cauchy–Schwarz inequality to (4.1) and using (4.10) yields:

$$\int_\Omega |\nabla\phi|^2\, dx = \int_\Omega F(u,v)\,\phi\ dx \le c_2\sqrt{|\Omega|}\left(\int_\Omega |\phi|^2\, dx\right)^{1/2}. \tag{4.25}$$

The Poincaré inequality assures that:

$$\int_\Omega \phi^2 dx \le \frac{1}{\lambda_1}\int_\Omega |\nabla\phi|^2\, dx, \tag{4.26}$$

where $\lambda_1 > 0$ is the first positive eigenvalue of $(-\Delta)$. Hence from (4.25) it follows:

$$\int_\Omega |\nabla\phi|^2\, dx \le c_2\sqrt{\frac{|\Omega|}{\lambda_1}}\left(\int_\Omega |\nabla\phi|^2\, dx\right)^{1/2},$$

implying that:

$$\int_\Omega |\nabla\phi|^2\, dx \le \frac{|\Omega|\, c_2^2}{\lambda_1}. \tag{4.27}$$

Adding up (4.26) and (4.27) and using once again (4.27) leads to:

$$\int_\Omega \phi^2 dx + \int_\Omega |\nabla\phi|^2\, dx \le C_F,$$

where:

$$C_F = c_2^2\, |\Omega| \left(\frac{1+\lambda_1}{\lambda_1^2}\right).$$

□

Lemma 9. *Let (u, v) be a nonconstant solution of the problem (4.1). Then, the following inequalities hold:*

$$\frac{\lambda_1^2}{\gamma^2 + 2\lambda_1(\lambda_1+\gamma)} \le \frac{\int_\Omega |\nabla\phi|^2\, dx}{d^2\int_\Omega |\nabla\psi|^2\, dx} \le 1, \tag{4.28}$$

$$\frac{\lambda_1^3}{(\lambda_1+1)(2\lambda_1(\lambda_1+\gamma)+\gamma^2)} < \frac{\int_\Omega \left(|\nabla\phi|^2 + 2\gamma\phi^2\right) dx}{d^2\int_\Omega \left(|\nabla\psi|^2+\psi^2\right) dx} < 1, \tag{4.29}$$

where ϕ and ψ are defined in (4.14) *and λ_1 is the first positive eigenvalue of $-\Delta$.*

Proof Let $w = dv - u$. Using the definitions in (4.14), we get:

$$\begin{aligned}\int_\Omega |\nabla w|^2\, dx &= \int_\Omega |\nabla(dv-u)|^2\, dx \\ &= d^2\int_\Omega |\nabla\psi|^2\, dx + \int_\Omega |\nabla\phi|^2\, dx - 2d\int_\Omega \nabla\phi\nabla\psi\; dx.\end{aligned}$$

Using (4.19) leads to:

$$\int_\Omega |\nabla w|^2\, dx = d^2\int_\Omega |\nabla\psi|^2\, dx - \int_\Omega |\nabla\phi|^2\, dx - 2\gamma\int_\Omega \phi^2 dx, \tag{4.30}$$

which implies:

$$d^2\int_\Omega |\nabla\psi|^2\, dx \ge \int_\Omega |\nabla\phi|^2\, dx + 2\gamma\int_\Omega \phi^2 dx \ge \int_\Omega |\nabla\phi|^2\, dx. \tag{4.31}$$

Therefore, the second inequality in (4.28) is obtained, i.e.:

$$\frac{\int_\Omega |\nabla\phi|^2\, dx}{d^2\int_\Omega |\nabla\psi|^2\, dx} \le 1. \tag{4.32}$$

Next, we use (4.18) and (4.30) to compute:

$$\frac{d^2}{\gamma}\int_\Omega |\nabla\psi|^2\, dx = \frac{1}{\gamma}\int_\Omega |\nabla\phi|^2\, dx + \int_\Omega \phi^2 dx + d\int_\Omega \phi\psi\; dx.$$

Using the ϵ–Young inequality $ab \leq \dfrac{1}{4\epsilon}a^2 + \epsilon b^2$ leads to:

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Let us now prove the inequalities in (4.29). The Poincaré inequality leads to:

$$\int_\Omega \left(|\nabla \psi|^2 + \psi^2 \right) dx \leq \left(\frac{(\lambda_1 + 1)}{\lambda_1} \right) \int_\Omega |\nabla \psi|^2 \, dx.$$

Therefore, we compute:

$$\frac{\int_\Omega \left(|\nabla \phi|^2 + 2\gamma\phi^2 \right) dx}{d^2 \int_\Omega \left(|\nabla \psi|^2 + \psi^2 \right) dx} \geq \left(\frac{\lambda_1}{\lambda_1 + 1} \right) \frac{\int_\Omega \left(|\nabla \phi|^2 + 2\gamma\phi^2 \right) dx}{d^2 \int_\Omega \left(|\nabla \psi|^2 \right) dx} > \left(\frac{\lambda_1}{\lambda_1 + 1} \right) \frac{\int_\Omega |\nabla \phi|^2 \, dx}{d^2 \int_\Omega |\nabla \psi|^2 \, dx},$$

and the left hand side of inequality (4.29) follows from (4.28). Moreover, we have:

$$\frac{\int_\Omega |\nabla \phi|^2 \, dx + 2\gamma \int_\Omega \phi^2 dx}{d^2 \int_\Omega \left(|\nabla \psi|^2 + \psi^2 \right) dx} < \frac{\int_\Omega |\nabla \phi|^2 \, dx + 2\gamma \int_\Omega \phi^2 dx}{d^2 \int_\Omega |\nabla \psi|^2 \, dx},$$

and using (4.31) we obtain the right hand side of the inequality in (4.29).

□

4.2. Nonexistence of nonconstant positive solutions

In this section, we shall concern the nonexistence of nonconstant positive solutions of (4.1).

Our results show that the size of the reactor (reflected by its first eigenvalue λ_1), and the diffusion coefficient d play a critical role in obtaining the nonexistence of nonconstant positive solutions. In particular, in Theorem 7 the nonexistence of non-constant positive solutions will be proved when the diffusion coefficient is below a threshold proportional to the size of the reactor; in Theorem 8 the nonexistence of nonconstant positive solutions will be achieved when the size of the reactor is large enough.

Theorem 7. *If the diffusion coefficient d satisfies the following condition:*

$$0 < d < d_0, \qquad \text{where} \;\; d_0 = \frac{4\lambda_1 C_2 (a, b, \gamma, \lambda)}{C_1^2 (a, b, \gamma, \lambda)}, \tag{4.33}$$

then the problem (4.1) does not admit nonconstant solutions.

Proof Multiplying by ψ the second equation of (4.1) and integrating by parts yields:

$$d \int_\Omega |\nabla \psi|^2 \, dx = \gamma b \int_\Omega \psi \; dx - \gamma\lambda \int_\Omega \varphi(u) \, v\psi \; dx. \tag{4.34}$$

Being $\varphi(u) = u\varphi_1(u)$ and using (4.15), we get:

$$
\begin{aligned}
d\int_{\Omega}|\nabla\psi|^{2}\,dx = & -\gamma\lambda\int_{\Omega}\varphi\left(u\right)v\psi\;dx\\
= & -\gamma\lambda\int_{\Omega}\left[\left(u\varphi_{1}\left(u\right)v-\overline{u}\varphi_{1}\left(u\right)v\right)+\left(\overline{u}\varphi_{1}\left(u\right)v-\overline{u}\varphi_{1}\left(u\right)\overline{v}\right)\right.\\
& \left.+\left(\overline{u}\varphi_{1}\left(u\right)\overline{v}-u\varphi_{1}\left(u\right)\overline{v}\right)+u\varphi_{1}\left(u\right)\overline{v}\right]\psi\;dx\\
= & -\gamma\lambda\int_{\Omega}v\varphi_{1}\left(u\right)\left(u-\overline{u}\right)\psi\;dx-\gamma\lambda\int_{\Omega}\overline{u}\varphi_{1}\left(u\right)\left(v-\overline{v}\right)\psi\;dx\\
& +\gamma\lambda\int_{\Omega}\left(u-\overline{u}\right)\varphi_{1}\left(u\right)\overline{v}\psi\;dx-\gamma\lambda\int_{\Omega}u\varphi_{1}\left(u\right)\overline{v}\psi\;dx\\
\leq & -\gamma\lambda\int_{\Omega}\overline{u}\varphi_{1}\left(u\right)\left(v-\overline{v}\right)\psi\;dx+\gamma\lambda\int_{\Omega}\left(\overline{u}-u\right)\varphi_{1}\left(u\right)\overline{v}\psi\;dx\\
= & \;\gamma\lambda\int_{\Omega}\varphi_{1}\left(u\right)\overline{v}\phi\psi\;dx-\gamma\lambda\int_{\Omega}\overline{u}\varphi_{1}\left(u\right)\psi^{2}dx.
\end{aligned}
$$

Using the estimates in Proposition 5 it follows that:

$$
d\int_{\Omega}|\nabla\psi|^{2}\,dx\leq C_{1}\int_{\Omega}\phi\psi\;dx-C_{2}\int_{\Omega}\psi^{2}dx, \tag{4.35}
$$

where C_1 and C_2 are constants depending on a, b, γ and λ.

By the Cauchy-Schwarz inequality and the ϵ–Young inequality, we have:

$$
\begin{aligned}
C_{1}\int_{\Omega}\phi\psi\;dx \leq & \; C_{1}\left(\int_{\Omega}|\phi|^{2}\,dx\right)^{1/2}\left(\int_{\Omega}|\psi|^{2}\,dx\right)^{1/2}\\
\leq & \; \frac{C_{1}^{2}}{4\epsilon}\int_{\Omega}|\phi|^{2}\,dx+\epsilon\int_{\Omega}|\psi|^{2}\,dx.
\end{aligned}
$$

Putting $C_2 = \epsilon$ and substituting in (4.35), we get:

$$
d\int_{\Omega}|\nabla\psi|^{2}\,dx\leq\frac{C_{1}^{2}}{4C_{2}}\int_{\Omega}|\phi|^{2}\,dx.
$$

By the Poincaré inequality, we have:

$$
d\int_{\Omega}|\nabla\psi|^{2}\,dx\leq\frac{C_{1}^{2}}{4\lambda_{1}C_{2}}\int_{\Omega}|\nabla\phi|^{2}\,dx. \tag{4.36}
$$

It follows from (4.28) and (4.36) that:

$$
\int_{\Omega}|\nabla\psi|^{2}\,dx\leq\frac{d}{d_{0}}\int_{\Omega}|\nabla\psi|^{2}\,dx, \tag{4.37}
$$

where $d_0 \equiv d_0\left(a, b, \gamma, \lambda, \lambda_1\right) = \frac{4\lambda_1 C_2}{C_1^2}$. Therefore, if $d < d_0$ the inequality in (4.37) implies:

$$
\int_{\Omega}|\nabla\psi|^{2}\,dx=0.
$$

Moreover, by the estimates in (4.28) it follows:

$$
\int_{\Omega}|\nabla\phi|^{2}\,dx=0.
$$

Hence, $|\nabla\phi| = |\nabla\psi| \equiv 0$ over Ω and nonconstant solutions are not admitted.

$\square$

Theorem 8. *There exists a positive constant $\Lambda \equiv \Lambda(a, b, \gamma, \lambda)$ such that the problem (4.1) does not admit nonconstant positive solutions when $\lambda_1 > \Lambda$.*

Proof Multiplying equation (4.1) by ϕ and integrating by parts, we obtain:

$$\int_\Omega |\nabla\phi|^2 \, dx = \gamma a \int_\Omega \phi \, dx - \gamma \int_\Omega \phi^2 dx - \gamma\lambda \int_\Omega \varphi(u) v\phi \, dx.$$

Being $\varphi(u) = u\varphi_1(u)$ and using (4.15), it follows:

$$\begin{aligned}
\int_\Omega |\nabla\phi|^2 \, dx &= -\gamma \int_\Omega \phi^2 dx - \gamma\lambda \int_\Omega \varphi(u) v\phi \, dx \\
&= -\gamma \int_\Omega \phi^2 dx - \gamma\lambda \int_\Omega [(u\varphi_1(u) v - \overline{u}\varphi_1(u) v) + (\overline{u}\varphi_1(u) v - \overline{u}\varphi_1(u)\overline{v}) \\
&\qquad + (\overline{u}\varphi_1(u)\overline{v} - u\varphi_1(u)\overline{v}) + u\varphi_1(u)\overline{v}]\, \phi \, dx \\
&= -\gamma \int_\Omega \phi^2 dx - \gamma\lambda \int_\Omega [\varphi_1(u) v (u - \overline{u}) + \overline{u}\varphi_1(u)(v - \overline{v}) \\
&\qquad -\varphi_1(u)\overline{v}(u - \overline{u}) + u\varphi_1(u)\overline{v}]\, \phi \, dx \\
&\leq -\gamma \int_\Omega \phi^2 (1 + \lambda\varphi_1(u) v) \, dx - \gamma\lambda \int_\Omega \overline{u}\varphi_1(u)\psi\phi \, dx + \gamma\lambda \int_\Omega \varphi_1(u)\overline{v}\phi^2 dx.
\end{aligned}$$

Therefore:

$$\int_\Omega |\nabla\phi|^2 \, dx \leq \gamma\lambda \int_\Omega \varphi_1(u)\overline{v}\phi^2 dx - \gamma\lambda \int_\Omega \overline{u}\varphi_1(u)\phi\psi \, dx.$$

Applying the a priori estimates in Proposition 5, we obtain:

$$\int_\Omega |\nabla\phi|^2 \, dx \leq C_3 \int_\Omega \phi^2 dx + C_3 \int_\Omega |\phi\psi| \, dx, \tag{4.38}$$

where C_3 is a constant depending on (a, b, γ, λ).

By the Cauchy-Schwartz inequality and the Poincaré inequality, we have:

$$\int_\Omega \phi\psi \, dx \leq \left(\int_\Omega |\phi|^2 \, dx\right)^{1/2} \left(\int_\Omega |\psi|^2 \, dx\right)^{1/2} \leq \frac{1}{\lambda_1} \left(\int_\Omega |\nabla\phi|^2 \, dx\right)^{1/2} \left(\int_\Omega |\nabla\psi|^2 \, dx\right)^{1/2}.$$

Thus, by (4.36) it follows that:

$$\int_\Omega \phi\psi \, dx \leq C_4 \lambda^{-3/2} d^{-1/2} \int_\Omega |\nabla\phi|^2 \, dx, \qquad \text{where} \quad C_4 = \frac{C_1}{2\sqrt{C_2}}. \tag{4.39}$$

Combining the condition in (4.39) with the inequality in (4.38), we obtain:

$$\int_\Omega |\nabla\phi|^2 \, dx \leq \frac{C}{\lambda_1}\left(1 + \frac{1}{(\lambda_1 d)^{1/2}}\right) \int_\Omega |\nabla\phi|^2 \, dx, \tag{4.40}$$

where $C(a, b, \gamma, \lambda) = \max\{C_3, C_3 \times C_4\}$.

If $d \geq 1$, there exists $\Lambda > 0$ such that, once chosen $\lambda_1 > \Lambda$ we have:

$$\frac{C}{\lambda_1}\left(1 + \frac{1}{(\lambda_1 d)^{1/2}}\right) \leq \frac{C}{\lambda_1}\left(1 + \frac{1}{(\lambda_1)^{1/2}}\right) < 1.$$

Therefore, the inequality in (4.40) implies $\int_\Omega |\nabla \phi|^2 = 0$. The estimates in (4.28) also implies $\int_\Omega |\nabla \psi|^2 = 0$ and nonconstant solutions cannot exist.

On the other hand, if $d < 1$ there exists $\Lambda > 0$ such that for $\lambda_1 > \Lambda$ we have $d_0 > 1$, where the expression of d_0 is given in (4.33). Hence, the nonexistence of nonconstant solutions follows by the Theorem 7.

□

5. Numerical Examples

In this Section, we aim to validate the analytical findings regarding the asymptotic stability of the equilibrium (1.7). We choose the following form of the arbitrary function $\varphi(u)$:

$$\varphi(u) = \frac{u^p}{k + u^q} =: \varphi_k(u), \tag{5.1}$$

with $p, q \geq 0$ and $k \geq 0$. We also assume that $\lambda = \gamma = 1$, $p = \frac{1}{2}$, and $q = 1$. Substituting these parameters into (1.1), we get:

$$\begin{cases} u_t - \Delta u = a - u - \dfrac{\sqrt{u}}{k+u} v, \\ v_t - d\Delta v = b - \dfrac{\sqrt{u}}{k+u} v, \end{cases} \tag{5.2}$$

which admits the following unique equilibrium:

$$(u^*, v^*) = \left(a - b, \frac{b\,(k + a - b)}{\sqrt{a-b}} \right). \tag{5.3}$$

The invariant region for the system (5.2) is:

$$\Re_\delta = [\delta, a] \times \left[\frac{b\,(k+\delta)}{\sqrt{\delta}}, \frac{(a-\delta)\,(k+\delta)}{\sqrt{\delta}} \right].$$

Letting $b = \frac{a}{2} = \frac{1}{8}$, since condition (1.4) must hold, then it should be:

$$\delta < \frac{a}{2}. \tag{5.4}$$

We choose the value $\delta = \frac{1}{10}$, which clearly satisfies the condition (5.4). With the above choices for the system parameters, the steady state is given by $(u^*, v^*) = \left(\frac{1}{8}, 2\sqrt{2}\left(\frac{1}{8}k + \frac{1}{64}\right)\right)$. Since the chosen function $\varphi(u)$ is decreasing over $[\delta, a]$, then (3.22) is satisfied.

The equilibrium solution (5.3) is asymptotically stable for the ODEs system (3.1) if the condition (3.2) holds. Substituting the chosen parameters into (3.2), we have:

$$-(24k + 1) < 4\sqrt{2},$$

which is always satisfied regardless of k. Therefore, for the chosen parameter set, we should achieve asymptotic stability of the ODEs system for any $k > 0$. We perform two different numerical tests. At the top of Figure **??**, for $k = 0.05$ and initial conditions $(u_0, v_0) = (0.2, 0.06)$, it is shown that the solutions converge towards the equilibrium $(u^*, v^*) = \left(\frac{1}{8}, \frac{7}{160}\sqrt{2}\right)$. Analogously, at the bottom of Figure **??**, for $k = 0.1$ and initial conditions $(u_0, v_0) = (0.2, 0.09)$, we can see that the solution of the ODEs system asymptotically converges towards the steady state $(u^*, v^*) = \left(\frac{1}{8}, \frac{9}{160}\sqrt{2}\right)$. The same solutions are plotted in Figure **??** in the u–v phase plane to better show the asymptotic evolution towards the steady state.

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Let us, now, consider the reaction-diffusion system (1.1) in a one–dimensional spatial domain. The initial conditions are chosen as the following sinusoidal disturbances:

$$\begin{cases} u(x,0) = u_0 \times (1 + \sin(50x)), \\ v(x,0) = v_0 + (1 + \cos(50x)). \end{cases} \quad (5.5)$$

Being $\varphi(u)$ a decreasing function, in order to achieve the global asymptotic stability of the solutions, the condition (3.48) must hold. If the following function:

$$f(u) = \frac{a-u}{\varphi_k(u)} = \frac{a-u}{\frac{\sqrt{u}}{k+u}}$$

is also decreasing, then (3.48) holds. Using the system parameters as above, it is easy to check that, if:

$$k \leq \min \left\{ \delta = \frac{1}{10}, 7 - \sqrt{48}a = 7 - \sqrt{3} \right\}, \quad (5.6)$$

then the function $f(u)$ is decreasing. We again perform two numerical tests choosing respectively $k = 0.05$ and $k = 0.1$, as both these values satisfies (5.6). The corresponding numerical simulations of the one dimensional reaction-diffusion system are respectively given in Figures **??** and **??** showing that the solutions converge towards the spatially homogeneous steady state.

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6. Conclusions

In this paper, we have studied the dynamics of a dimensionless generalized reaction–diffusion system based on the Degn–Harrison model. We have established the existence of a unique bounded solution. Moreover, we have investigated the asymptotic behaviour of the steady state solution, deriving sufficient conditions for its local and global asymptotic stability. We have also analyzed the elliptic boundary value problem, obtaining some a priori estimates of the nonconstant steady state solutions. Finally, we have obtained nonexistence conditions for nonconstant positive solutions depending on the size of the reactor, which should be large enough, and the diffusion coefficient.

The mechanism of generating the steady-state mode has not been addressed here. The existence of the non-constant positive solution will be the object of future works.

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