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Derivatives not first return integrable on a fractal set

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Dedicated to Prof. Hans Weber on the occasion of his 70^{th} *birthday*

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Abstract We extend to s-dimensional fractal sets the notion of first return integral (Definition 5) and we prove that there are s-derivatives not s-first return integrable.

Keywords s-dimensional Hausdorff measure, s-set, s-derivative, Henstock-Kurzweil integral, first return integral

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1 Introduction

Recently, D. Bongiorno and G. Corrao gave in [3] a quite general formulation of the Fundamental Theorem of Calculus on a fractal subset E of the real line, having finite and positive s-dimensional Hausdorff measure $\mathcal{H}^s$, with $0 < s < 1$. To give such a formulation (in particular, to get the integrability of the s-derivatives), the authors used an extension of the Henstok-Kurzweil integral to fractal sets.

It is well known that, in the Henstock-Kurzweil integration process, the choice of partitions is not completely free as it is in the Riemann integral. This was the reason that led U. B. Darji and M. J. Evans, borrowing the notion of first-return from dynamics, to introduce in [4] the *first-return integral*, that is an integration process that contains the Lebesgue integral, based on partitions completely free.

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Unfortunately this process is inadequate to the Riemann-improper integral, hence to the Henstock-Kurzweil integral (see [1] and [5]). A Riemann-type characterization of the Riemann-improper integral is given in [2].

However, a problem that would deserve interest is to know if the derivatives are first return integrable, hence if the first return integral can be used to give a general formulation of the Fundamental Theorem of Calculus on the real line.

In this paper, we solve negatively this problem on a fractal subset E of the real line. To this end, we extend to s-dimensional fractal sets the notion of first return integral (Definition 5) and we prove that there are s-derivatives not s-first return integrable (Theorem 1).

2 Preliminaries

Throughout this paper we denote by $\mathbf{N}$ the set of all natural numbers and by $\mathbf{R}$ the set of all real numbers.

Let $0 < s < 1$, we recall that the *s-dimensional exterior Hausdorff measure* of a subset A of the real line is defined as:

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam}(A_i))^s : A \subset \bigcup_{i=1}^{\infty} A_i,\ \operatorname{diam}(A_i) \leq \delta \right\}.$$

Moreover, we recall that $\mathcal{H}^s(\cdot)$ is a Borel regular measure and that the unique number s for which $\mathcal{H}^t(A) = 0$ if $t > s$ and $\mathcal{H}^t(A) = \infty$ if $t < s$ is called the Hausdorff dimension of A (see Mattila [8]). Whenever A is $\mathcal{H}^s$-measurable with $0 < \mathcal{H}^s(A) < \infty$, it is said that A is an s-set. So $\mathcal{H}^s$ is a Radon measure on each s-set.

In this paper we denote by E a closed s-set of $\mathbf{R}$.

Definition 1 We say that a subset $\widetilde{A}$ of E is an *E-interval* whenever there exists an interval $A \subset \mathbf{R}$ such that $\widetilde{A} = A \cap E$.

Definition 2 (Jiang-Su [6] and Parvate-Gangal [7]) Let $f\colon E \to \mathbf{R}$. We say that f is *s-Riemann integrable on E* if there exists a number I such that, for each $\varepsilon > 0$, there is a constant $\delta > 0$ with

$$\left| \sum_{i=1}^{p} f(x_i)\, \mathcal{H}^s(\widetilde{A}_i) - I \right| < \varepsilon, \tag{1}$$

for each partition $\{\widetilde{A}_i\}_{i=1}^p$ of E-intervals with $\mathcal{H}^s(\widetilde{A}_i) < \delta$, and for each $x_i \in \widetilde{A}_i$, for $i = 1, 2, ..., p$.

The number I is called the *s-Riemann integral of f on E* and we write

$$I = (R)\int_E f(t)\, d\mathcal{H}^s(t).$$

The collection of all s-Riemann integrable functions on E will be denoted by s-$R(E)$.

Definition 3 (De Guzman-Martin-Reyes [9]) Let $F : E \to \mathbf{R}$ and let $x_0 \in E$. The s-*derivatives* of F at the point x_0, on the left and on the right, are defined, respectively, as follows:

$$F_s'^{-}(x_0) = \lim_{E \ni x \to x_0^-} \frac{F(x_0) - F(x)}{\mathcal{H}^s([x, x_0] \cap E)}, \text{ if } \mathcal{H}^s([x, x_0] \cap E) > 0\,, \text{ for all } x < x_0,$$

$$F_s'^{+}(x_0) = \lim_{E \ni x \to x_0^+} \frac{F(x) - F(x_0)}{\mathcal{H}^s([x_0, x] \cap E)}, \text{ if } \mathcal{H}^s([x_0, x] \cap E) > 0\,, \text{ for all } x > x_0,$$

when these limits exist.

We say that the s-*derivative* of F at x_0 exists if $F_s'^{-}(x_0) = F_s'^{+}(x_0)$ or if the s-derivative of F on the left (resp. right) at x_0 exists and for some $\varepsilon > 0$ we have $\mathcal{H}^s([x_0, x_0+\varepsilon] \cap E) = 0$ (resp. $\mathcal{H}^s([x_0-\varepsilon, x_0] \cap E) = 0$). The s-derivative of F at x_0, when it exists, will be denoted by $F_s'(x_0)$.

3 The s-first return integral

Definition 4 We call *trajectory* on E any sequence $\Gamma \subset E$ of distinct points of E, dense in E. Given a trajectory Γ on E and an E-interval $\widetilde{J}$ we denote by $r(\Gamma, \widetilde{J})$ the first element of Γ that belongs to $\widetilde{J}$.

Definition 5 Let $f\colon E \to \mathbf{R}$ and let Γ be a trajectory on E. We say that f is *s-first return integrable on* E with respect to Γ if there exists a number I such that, for each $\epsilon > 0$, there is a constant $\delta > 0$ with

$$\left| \sum_{i=1}^{p} f(r(\Gamma, \widetilde{A}_i))\, \mathcal{H}^s(\widetilde{A}_i) - I \right| < \varepsilon, \tag{2}$$

for each partition $\{\widetilde{A}_i\}_{i=1}^{p}$ of E with $\mathcal{H}^s(\widetilde{A}_i) < \delta$.

The number I is called the *s-first return integral of* f on E with respect to a trajectory Γ and we write

$$I \;=\; (R^*)_\Gamma \int_E f(t)\, d\mathcal{H}^s(t).$$

The collection of all functions that are s-first return integrable with respect to a trajectory Γ on E will be denoted by s-$R^*(E)_\Gamma$.

It is clear that, for each each $f \in s$-$R(E)$ and for each trajectory Γ on E, it is $f \in s$-$R^*(E)_\Gamma$ with

$$(R^*)_\Gamma \int_E f(t)\, d\mathcal{H}^s(t) = (R)\int_E f(t)\, d\mathcal{H}^s(t).$$

4 Main result

Theorem 1 *There exists an s-derivative $f\colon E \to \mathbf{R}$ such that $f \notin s\text{-}R^*(E)_\Gamma$, for each trajectory Γ on E.*

Proof Let E be the ternary Cantor set. Remark that E is an s-set, with $s = \log_3 2$, and that

$$\mathcal{H}^s\left[\widetilde{\frac{2}{3^n}, \frac{1}{3^{n-1}}}\right] = \frac{1}{2^n} = \mathcal{H}^s\left[\widetilde{0, \frac{2}{3^n}}\right], \tag{3}$$

and

$$\mathcal{H}^s\left[\widetilde{\frac{2}{3^n}, \frac{7}{3^{n+1}}}\right] = \frac{1}{4^n} = \mathcal{H}^s\left[\widetilde{\frac{8}{3^{n+1}}, \frac{1}{3^{n-1}}}\right], \tag{4}$$

for $n = 1, 2, \cdots$.
Define

$$F(x) = \begin{cases} \frac{(-2)^n}{n}\,\mathcal{H}^s\big[\widetilde{\tfrac{2}{3^n}, x}\big], & x \in \big[\widetilde{\tfrac{2}{3^n}, \tfrac{7}{3^{n+1}}}\big]; \\ \\ \frac{(-2)^n}{n}\,\mathcal{H}^s\big[\widetilde{\tfrac{8}{3^{n+1}}, x}\big], & x \in \big[\widetilde{\tfrac{8}{3^{n+1}}, \tfrac{1}{3^{n-1}}}\big]; \\ \\ 0, & x = 0. \end{cases}$$

Then, for $x \in \big[\widetilde{\tfrac{2}{3^n}, \tfrac{1}{3^{n-1}}}\big]$ we have $F_s'(x) = (-2)^n/n$.
Moreover, for $x \in \big[\widetilde{\tfrac{2}{3^n}, \tfrac{7}{3^{n+1}}}\big]$, it is

$$\left|\frac{F(x) - F(0)}{\mathcal{H}^s(\widetilde{[0,x]})}\right| = \frac{2^n}{n}\,\frac{\mathcal{H}^s\big[\widetilde{\tfrac{2}{3^n}, x}\big]}{\mathcal{H}^s(\widetilde{[0,x]})} \le \frac{2^n}{n}\frac{1}{4^n}2^n = \frac{1}{n},$$

and, for $x \in \big[\widetilde{\tfrac{8}{3^{n+1}}, \tfrac{1}{3^{n-1}}}\big]$, it is

$$\left|\frac{F(x) - F(0)}{\mathcal{H}^s(\widetilde{[0,x]})}\right| = \frac{2^n}{n}\,\frac{\mathcal{H}^s\big[\widetilde{\tfrac{8}{3^{n+1}}, x}\big]}{\mathcal{H}^s(\widetilde{[0,x]})} \le \frac{2^n}{n}\frac{1}{4^n}2^n = \frac{1}{n}.$$

Thus

$$F_s'(0) = \lim_{x\to 0}\frac{F(x) - F(0)}{\mathcal{H}^s(\widetilde{[0,x]})} = 0.$$

In conclusion, the function F is s-differentiable on E with

$$F_s'(x) = \begin{cases} \frac{(-2)^n}{n}, & x \in \big[\widetilde{\tfrac{2}{3^n}, \tfrac{1}{3^{n-1}}}\big]; \\ 0, & x = 0. \end{cases} \tag{5}$$

Now let us consider the function $f(x) = F_s'(x)$, $x \in E$. By definition, f is an s-derivative on E.
In order to show that $f \notin s\text{-}R^*(E)_\Gamma$, for a given trajectory $\Gamma \equiv \{t_n\}$, it is

enough to find, for each $M > 0$ and each $\delta > 0$, a finite system of pairwise disjoints E-intervals $\widetilde{A}_i$, $i = 1, 2, \cdots, p$, such that $\bigcup_{i=1}^{p} \widetilde{A}_i = E$, $\mathcal{H}^s(\widetilde{A}_i) < \delta$ and

$$\sum_{i=1}^{p} f(r(\Gamma, \widetilde{A}_i))\, \mathcal{H}^s(\widetilde{A}_i) > M. \tag{6}$$

To this end, given two disjoint E-intervals $\tilde{J}_1, \tilde{J}_2$, we put

$$r(\Gamma, \tilde{J}_1) \prec r(\Gamma, \tilde{J}_2),$$

whenever $r(\Gamma, \tilde{J}_1) = t_n$, $r(\Gamma, \tilde{J}_2) = t_m$, and $n < m$.
We also define

$$\mathbf{N}_1 = \{n \in \mathbf{N} : r(\Gamma, \widetilde{[2/3^{2n}, 1/3^{2n-1}]}) \prec r(\Gamma, \widetilde{[2/3^{2n+1}, 1/3^{2n}]})\}$$

and

$$\mathbf{N}_2 = \mathbf{N} \setminus \mathbf{N}_1.$$

Remark that

$$\begin{aligned} r(\Gamma, \widetilde{[2/3^{2n+1}, 1/3^{2n-1}]}) &= r(\Gamma, \widetilde{[2/3^{2n}, 1/3^{2n-1}]}), \ \text{ if } n \in \mathbf{N}_1; \\ r(\Gamma, \widetilde{[2/3^{2n+1}, 1/3^{2n-1}]}) &= r(\Gamma, \widetilde{[2/3^{2n+1}, 1/3^{2n}]}), \ \text{ if } n \in \mathbf{N}_2. \end{aligned}$$

By the divergence of the series $\sum_n 1/n$ it follows that, at least one of the series $\sum_{n \in \mathbf{N}_1} 1/n$, $\sum_{n \in \mathbf{N}_2} 1/n$ is divergent; then, without loss of generality, we can assume that $\sum_{n \in \mathbf{N}_1} 1/n = +\infty$.
Now, given an arbitrary constant $\delta > 0$, let us take $k \in \mathbf{N}$ such that

$$\mathcal{H}^s(\widetilde{[2/3^{2n+1}, 1/3^{2n-1}]}) < \delta, \ \text{ for each } n \geq k. \tag{7}$$

Then, by (3) and (5), we have

$$\begin{aligned} &\sum_{k \leq n \in \mathbf{N}_1} f(r(\Gamma, \widetilde{[2/3^{2n+1}, 1/3^{2n-1}]}))\, \mathcal{H}^s(\widetilde{[2/3^{2n+1}, 1/3^{2n-1}]}) \\ &= \sum_{k \leq n \in \mathbf{N}_1} \frac{2^{2n}}{2n} \left(\frac{1}{2^{2n+1}} + \frac{1}{2^{2n}} \right) = \frac{3}{2} \sum_{k \leq n \in \mathbf{N}_1} \frac{1}{2n} = +\infty. \end{aligned}$$

So, given $M > 0$, there exists $N \in \mathbf{N}$ such that $p_1 = N - k + 1$ is an even number and

$$\begin{aligned} \sum_{k \leq n \leq N; n \in \mathbf{N}_1} & f(r(\Gamma, \widetilde{[2/3^{2n+1}, 1/3^{2n-1}]}))\, \mathcal{H}^s(\widetilde{[2/3^{2n+1}, 1/3^{2n-1}]}) \\ & > M + 3 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}. \end{aligned} \tag{8}$$

The E-intervals $\{\widetilde{[2/3^{2n+1}, 1/3^{2n-1}]}\}_{n=k}^{N}$ are pairwise disjoint, cover the portion of E contained in $[2/3^{2N+1}, 1/3^{2k-1}]$ and each of them has $\mathcal{H}^s$-measure less than δ. They constitute a first group of p_1 requested E-intervals $\{\widetilde{A}_i\}_{i=1}^{p_1}$:

$$\widetilde{A}_1 = \widetilde{[2/3^{2k+1}, 1/3^{2k-1}]},\ \widetilde{A}_2 = \widetilde{[2/3^{2k+3}, 1/3^{2k+1}]},\\ \cdots, \widetilde{A}_{p_1} = \widetilde{[2/3^{2N+1}, 1/3^{2N-1}]}.$$

Then, by (8), we have

$$\sum_{i=1}^{p_1} f(r(\Gamma, \widetilde{A}_i))\, \mathcal{H}^s(\widetilde{A}_i) > M + 3 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}. \tag{9}$$

Now we define a second group of requested pairwise disjoint E-intervals, $\{\widetilde{A}_i\}_{i=p_1+1}^{p_2}$, that cover the portions of E contained in $[0, 1/3^{2N+1}]$.
There are two possible cases:

$$r(\Gamma, \widetilde{[0, 1/3^{2N+1}]}) = 0, \quad \text{or} \qquad r(\Gamma, \widetilde{[0, 1/3^{2N+1}]}) \neq 0.$$

In the first case we define

$$\widetilde{A}_{p_1+1} = \widetilde{[0, 1/3^{2N+1}]},$$

and we have

$$f(r(\Gamma, \widetilde{A}_{p_1+1}))\, \mathcal{H}^s(\widetilde{A}_{p_1+1}) = 0. \tag{10}$$

In the second case there exists a unique $n^* > 2N + 1$ such that

$$r(\Gamma, \widetilde{[0, 1/3^{2N+1}]}) \in \widetilde{[2/3^{n^*}, 1/3^{n^*-1}]}.$$

If $n^* = 2N + 2$ we also define $\widetilde{A}_{p_1+1} = \widetilde{[0, 1/3^{2N+1}]}$, otherwise we define

$$\widetilde{A}_{p_1+1} = \widetilde{[0, 1/3^{n^*-1}]},\ \widetilde{A}_{p_1+2} = \widetilde{[2/3^{n^*-1}, 1/3^{n^*-2}]},\\ \cdots,\ \widetilde{A}_{p_2} = \widetilde{[2/3^{2N+2}, 1/3^{2N+1}]}.$$

Hence we have

$$f(r(\Gamma, \widetilde{A}_{p_1+1}))\, \mathcal{H}^s(\widetilde{A}_{p_1+1}) = \begin{cases} 1/(N+1), & \text{if } n^* = 2N+2; \\ 2 \cdot (-1)^{n^*}/n^*, & \text{otherwise.} \end{cases}$$

Consequently,

$$|f(r(\Gamma, \widetilde{A}_{p_1+1}))\, \mathcal{H}^s(\widetilde{A}_{p_1+1})| < 1. \tag{11}$$

Thus, since p_1 is even, by (9) and (11), if follows

$$\left| \sum_{i=p_1+1}^{p_2} f(r(\Gamma, \widetilde{A}_i)\, \mathcal{H}^s(\widetilde{A}_i) \right| < 1 + \sum_{n=p_1+2}^{p_2} \frac{(-1)^n}{n}$$

$$< 2 + \sum_{n=p_1+2}^{\infty} \frac{(-1)^n}{n}. \tag{12}$$

The third group of requested pairwise disjoint E-intervals that cover the portions of E contained in $[2/3^{2k-1}, 1]$ can be defined taking a generic system of pairwise disjoint E-intervals, $\{\widetilde{A}_i\}_{i=p_2+1}^{p}$, such that $\mathcal{H}^s(\widetilde{A}_i) < \delta$, for each i, and such that $[2/3^n, \widetilde{1/3^{n-1}}] = \bigcup_{i\in I_n} \widetilde{A}_i$, where $I_n \subset \{p_2+1, \cdots, p\}$ and $1 \leq n \leq 2k-1$.
By definition of f it follows $f(r(\Gamma, \widetilde{A}_i)) = f(r(\Gamma, [2/3^n, \widetilde{1/3^{n-1}}])) = (-2)^n/n$, for $1 \leq n \leq 2k-1$ and $i \in I_n$. Thus

$$\begin{aligned}\sum_{i=p_2+1}^{p} f(r(\Gamma, \widetilde{A}_i)\, \mathcal{H}^s(\widetilde{A}_i) &= \sum_{n=1}^{2k-1} \frac{(-2)^n}{n}\, \mathcal{H}^s([2/3^n, \widetilde{1/3^{n-1}}]) \\ &= \sum_{n=1}^{2k-1} \frac{(-1)^n}{n}\, . \end{aligned} \tag{13}$$

In conclusion, we have defined the required system $\{\widetilde{A}_i\}_{i=1}^{p}$ of pairwise disjoint E-intervals such that $\mathcal{H}^s(\widetilde{A}_i) < \delta$, for each i, $\bigcup_{i=1}^{p} \widetilde{A}_i = E$, and, by (9), (12), and (13), such that

$$\begin{aligned}&\left|\sum_{i=1}^{p} f(r(\Gamma, \widetilde{A}_i))\, \mathcal{H}^s(\widetilde{A}_i)\right| \\ &= \left|\sum_{i=1}^{p_1} + \sum_{i=p_1+1}^{p_2} + \sum_{i=p_2+1}^{p}\right| \\ &> M + 3 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} - \left|\sum_{i=p_1+1}^{p_2}\right| - \left|\sum_{i=p_2+1}^{p}\right| \\ &> M + 3 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} - 3 - \sum_{n=p_1+2}^{\infty} \frac{(-1)^n}{n} - \sum_{n=2}^{2k-1} \frac{(-1)^n}{n} \\ &> M.\end{aligned}$$

This completes the proof.

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