Generalized Riesz systems and orthonormal sequences in Krein spaces

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Abstract. We analyze special classes of bi-orthogonal sets of vectors in Hilbert and in Krein spaces, and their relations with generalized Riesz systems. In this way, the notion of the first/second type sequences is introduced and studied. We also discuss their relevance in some concrete quantum mechanical system driven by manifestly non self-adjoint Hamiltonians.

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I. Introduction

The employing of non self-adjoint operators for the description of experimentally observable data goes back to the early days of quantum mechanics. In the past twenty years, the steady interest in this subject grew considerably after it has been discovered [10, 12] that the spectrum of the manifestly non self-adjoint Hamiltonian

$$H = -\frac{d^2}{dx^2} + x^2(ix)^\epsilon, \qquad 0 < \epsilon < 2 \tag{1}$$

is real. It was conjectured [10] that the reality of eigenvalues of H is a consequence of its $\mathcal{PT}$-symmetry: $\mathcal{PT}H = H\mathcal{PT}$, where the space parity operator $\mathcal{P}$ and the complex conjugation operator $\mathcal{T}$ are defined as follows: $(\mathcal{P}f)(x) = f(-x)$ and $(\mathcal{T}f)(x) = \overline{f(x)}$. This gave rise to a consistent development of theory of $\mathcal{PT}$-symmetric Hamiltonians in Quantum Physics, see [4, 5, 11] and references therein.

Usually, $\mathcal{PT}$-symmetric Hamiltonians can be interpreted as self-adjoint ones for a suitable choice of *indefinite* inner product. For instance, the operator H in (1) is self-adjoint with respect to the indefinite inner product $[f,g] = \int_{-\infty}^{\infty} f(-x)\overline{g(x)}dx$ in the Krein space $(\mathcal{L}^2(\mathbb{R}), [\cdot,\cdot])$ (see Subsection III.1 for the definition of Krein spaces). The eigenstates of H lose the property of being Riesz basis in the original Hilbert space $\mathcal{L}^2(\mathbb{R})$ but they still form a complete set in $\mathcal{L}^2(\mathbb{R})$ [18, 21]. Moreover, they form a sequence which is orthonormal with respect to the indefinite inner product $[\cdot,\cdot]$. Such kind of phenomenon is typical for $\mathcal{PT}$-symmetric Hamiltonians and it gives rise to a natural problem: *What can we say about properties of vectors which form a complete set in a Hilbert space and, additionally, are orthonormal with respect to the indefinite inner product?*

The main objective of the paper is to investigate this problem with the use of theory of generalized Riesz systems (GRS) and $\mathcal{G}$-quasi bases. These two concepts were originally introduced in [15] and [2], respectively and then analyzed in a series of papers, see, e.g. [4, Chapter 3], [6, 7], [14]. The motivation for introducing GRS and $\mathcal{G}$-quasi bases was the need to put on a mathematically rigorous ground several physical models where the eigenstates of some non self-adjoint operator were usually claimed to be bases, while they were not.

The paper is structured as follows. Section II contains facts related to GRS and $\mathcal{G}$-quasi bases. We slightly change the original definition of GRS given in [15] putting in evidence the role of a self-adjoint operator Q, since it is more convenient for our purpose: *a sequence $\{\phi_n\}$ in a Hilbert space $\mathcal{H}$ is called a generalized Riesz system (GRS) if there exists a self-adjoint operator Q in $\mathcal{H}$ and an orthonormal basis (ONB) $\{e_n\}$ such that $e_n \in D(e^{Q/2}) \cap D(e^{-Q/2})$ and $\phi_n = e^{Q/2}e_n$.*

A GRS $\{\phi_n\}$ is a Riesz basis if and only if Q is a bounded operator.

For each GRS $\{\phi_n\}$, the dual GRS is determined by the formula $\{\psi_n = e^{-Q/2}e_n\}$. The dual GRS $\{\phi_n\}$ and $\{\psi_n\}$ are biorthogonal.

In Subsection II.3, the phenomenon of nonuniqueness of a self-adjoint operator Q in the formulas

$$\phi_n = e^{Q/2}e_n, \qquad \psi_n = e^{-Q/2}e_n \tag{2}$$

is investigated in the case of regular biorthogonal sequences $\{\phi_n\}$ and $\{\psi_n\}$. We firstly prove that the operator Q and ONB $\{e_n\}$ are determined uniquely for dual bases. In general, this property does not hold for regular biorthogonal sequences. To describe all possible self-adjoint operators Q in (2) we consider a positive densely defined operator G_0 which acts as $G_0\phi_n = \psi_n$ and then is extended on $D(G_0) = span\{\phi_n\}$ by the linearity. The proved statement is: *for given $\{\phi_n\}$ and $\{\psi_n\}$, the set of admissible operators Q in (2) is in one-to-one correspondence with the set of extremal extensions G of G_0,* *precisely,* $Q = -\ln G$*.* This fact allows one to characterize the important case where Q is determined uniquely.

Section III contains the main results. We begin with the simple fact that each complete sequence $\{\phi_n\}$ which is orthonormal in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ (briefly, J-orthonormal) is a GRS and therefore, it can be expressed by (2) for some choice of Q. If Q is uniquely determined, then the anti-commutation relation $JQ = -QJ$ holds. For this reason, the anti-commutation relation is reasonable to keep in the general case (if Q is not determined uniquely, then we cannot state, in general, that $JQ = -QJ$): we say that a complete J-orthonormal sequence $\{\phi_n\}$ is of *the first type* if there exists Q in (2) such that $JQ = -QJ$. Otherwise, $\{\phi_n\}$ is of *the second type*.

We proved that the formula (2) defines first type sequences if and only if Q anticommutes with J and elements of ONB $\{e_n\}$ are eigenvectors of J.

The first type sequences have a lot of useful properties. One of benefits is the fact that a first type sequence generates a $\mathcal{C}$-symmetry operator $\mathcal{C} = e^Q J$ where Q is the same operator as in (2). The latter allows one to construct the Hilbert space $(\mathcal{H}_{-Q}, \langle\cdot,\cdot\rangle_{-Q})$ involving $\{\phi_n\}$ as ONB, directly as the completion of $D(\mathcal{C})$ with respect to "$\mathcal{CPT}$-norm":

$$\langle\cdot,\cdot\rangle_{-Q} = [\mathcal{C}\cdot,\cdot] = \langle Je^Q J\cdot,\cdot\rangle = \langle e^{-Q}\cdot,\cdot\rangle. \qquad (3)$$

For a second type sequence, the inner product $\langle\cdot,\cdot\rangle_{-Q}$ generated by an operator Q from (2) cannot be expressed via $[\cdot,\cdot]$ and one should apply more efforts for the determination of $\langle\cdot,\cdot\rangle_{-Q}$, see Subsection III.2.

Assume that a complete J-orthonormal sequence $\{\phi_n\}$ consists of eigenfunctions of a Hamiltonian H corresponding to real eigenvalues $\{\lambda_n\}$. The sequence $\{\phi_n\}$ generates $\mathcal{C}$-symmetry operators (Proposition III.9). If $\{\phi_n\}$ is of the first type, then there exists at least one $\mathcal{C}$ such that the operator H restricted on $\mathrm{span}\{\phi_n\}$ turns out to be essential self-adjoint in the Hilbert space $\mathcal{H}_{-Q}$ with the inner product (3). The spectrum of the closure of H in $\mathcal{H}_{-Q}$ coincides with the closure of $\{\lambda_n\}$. Hence, we construct an isospectral realization of H in $\mathcal{H}_{-Q}$.

For a second type sequence, each operator $\mathcal{C}$ generated by $\{\phi_n\}$ gives rise to the Hilbert space $(\mathcal{H}_{-Q}, \langle\cdot,\cdot\rangle_{-Q})$ with *non-densely defined symmetric operator* H. Its extensions to self-adjoint operators in $\mathcal{H}_{-Q}$ lead to the appearance of new spectral points. Therefore, self-adjoint realizations constructed with the use of $\mathcal{CPT}$-norm cannot be isospectral. The isospectrality of self-adjoint realizations of H in $\mathcal{H}_{-Q}$ can be achieved via the construction of the Friedrichs extension $G = e^{-Q}$ of the symmetric operator $G_0\phi_n = [\phi_n, \phi_n]J\phi_n$ defined on $\mathrm{span}\{\phi_n\}$, that is quite complicated problem.

Section IV contains examples of J-orthonormal sequences. We show that the eigenstates of the shifted harmonic oscillator constitute a first type sequence. This example leads to the following conjecture: *eigenstates of a $\mathcal{PT}$-symmetric Hamiltonian H with* unbroken *$\mathcal{PT}$-symmetry [11, p. 41] form a first type sequence.*

In what follows, $\mathcal{H}$ means a complex Hilbert space with inner product linear in the first argument. Sometimes, it is useful to specify the inner product $\langle\cdot,\cdot\rangle$ associated with $\mathcal{H}$. In that case the notation $(\mathcal{H},\langle\cdot,\cdot\rangle)$ has already been used, and will be used in the following. All operators in $\mathcal{H}$ are supposed to be linear, the identity operator is denoted by I. The symbols $D(A)$ and $R(A)$ denote the domain and the range of a linear operator A. An operator A is called positive [nonnegative] if $(Af,f)>0$ $[(Af,f)\geq 0]$ for non-zero $f\in D(A)$.

II. Generalized Riesz systems in Hilbert spaces

Let $\{\phi_n\}$ be a Riesz basis in $\mathcal{H}$. Then there exists a bounded and boundedly invertible operator R such that $\phi_n = Re'_n$, where $\{e'_n\}$ is an orthonormal basis (ONB) of $\mathcal{H}$. The operator RR^* is positive and self-adjoint in $\mathcal{H}$ and it admits the presentation $RR^*=e^Q$, where Q is a bounded self-adjoint operator. The polar decomposition of R has the form $R=\sqrt{RR^*}U=e^{Q/2}U$, where U is a unitary operator in $\mathcal{H}$. The unitarity of U means that $\{e_n = Ue'_n\}$ is an ONB of $\mathcal{H}$ and we can rewrite the definition of Riesz bases as follows: *a sequence $\{\phi_n\}$ is called a Riesz basis if there exists a bounded self-adjoint operator Q in $\mathcal{H}$ and an ONB $\{e_n\}$ such that $\phi_n = e^{Q/2}e_n$.* This simple observation leads to:

Definition II.1. A sequence $\{\phi_n\}$ is called a generalized Riesz system (GRS) if there exists a self-adjoint operator Q in $\mathcal{H}$ and an ONB $\{e_n\}$ such that $e_n\in D(e^{Q/2})\cap D(e^{-Q/2})$ and $\phi_n = e^{Q/2}e_n$.

Let $\{\phi_n\}$ be a GRS. In view of Definition II.1, the sequence $\{\psi_n = e^{-Q/2}e_n\}$ is well defined and it is a biorthogonal sequence for $\{\phi_n = e^{Q/2}e_n\}$. Obviously, $\{\psi_n\}$ is a GRS which we will call *a dual GRS*. Dual GRS are Riesz bases if and only if Q is a bounded operator.

Example 1:– A first simple example of GRS can be extracted from [6]: if we take $Q=-\frac{x^2}{2}$, x being the position operator, it is clear that $D(e^{Q/2})=\mathcal{L}^2(\mathbb{R})$, while

$$D(e^{-Q/2}) = D(e^{x^2/4}) = \left\{f(x)\in\mathcal{L}^2(\mathbb{R}):\ e^{x^2/4}f(x)\in\mathcal{L}^2(\mathbb{R})\right\}.$$

This set is dense in $\mathcal{L}^2(\mathbb{R})$, since contains each eigenfunction of the quantum harmonic oscillator

$$e_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2}, \quad H_n(x) = e^{x^2/2}\left(x - \frac{d}{dx}\right)^n e^{-x^2/2}. \qquad (4)$$

The Hermite functions $\{e_n(x)\}$ form an orthonormal basis of $\mathcal{L}^2(\mathbb{R})$.

Following [6], we have $\phi_n(x) = e^{Q/2}e_n(x) = \frac{1}{\sqrt{2^n\, n! \sqrt{\pi}}} H_n(x)\, e^{-\frac{3x^2}{4}}$, $\psi_n(x) = e^{-Q/2}e_n(x) = \frac{1}{\sqrt{2^n\, n! \sqrt{\pi}}} H_n(x)\, e^{-\frac{x^2}{4}}$ and both $\{\phi_n\}$ and $\{\psi_n\}$ are GRS, one the dual of the other. Of course, due to the unboundedness of Q, they are not Riesz bases.

In general, the inner product

$$\langle f, g\rangle_{-Q} := \langle e^{-Q} f, g\rangle = \langle e^{-Q/2} f, e^{-Q/2} g\rangle, \qquad f, g \in D(e^{-Q}) \qquad (5)$$

is not equivalent to $\langle \cdot, \cdot \rangle$ and the linear space $D(e^{-Q})$ endowed with $\langle \cdot, \cdot\rangle_{-Q}$ is only a pre-Hilbert space. Denote by $\mathcal{H}_{-Q}$ the completion of $D(e^{-Q})$ with respect to $\langle \cdot, \cdot\rangle_{-Q}$. Analogously, the Hilbert space $(\mathcal{H}_Q, \langle \cdot, \cdot\rangle_Q)$ is defined as the completion of $D(e^Q)$ with respect to the inner product

$$\langle f, g\rangle_Q := \langle e^Q f, g\rangle = \langle e^{Q/2} f, e^{Q/2} g\rangle, \qquad f, g \in D(e^Q). \qquad (6)$$

Generally, the Hilbert spaces $\mathcal{H}_{-Q}$ and $\mathcal{H}_Q$ differ from $\mathcal{H}$ (as the sets of elements). We can just say, in view of (5) and (6) that $D(e^{-Q/2}) \subset \mathcal{H}_{-Q}$ and $D(e^{Q/2}) \subset \mathcal{H}_Q$.

Lemma II.2. *Let $\{e_n\}$, $e_n \in D(e^{Q/2}) \cap D(e^{-Q/2})$, be a complete set in $\mathcal{H}$. Then $\{\phi_n = e^{Q/2}e_n\}$ and $\{\psi_n = e^{-Q/2}e_n\}$ are complete sets in the Hilbert spaces $\mathcal{H}_{-Q}$ and $\mathcal{H}_Q$, respectively.*

Proof. In view of (5),

$$\|f\|^2_{-Q} = \langle e^{-Q} f, f\rangle = \|e^{-Q/2} f\|^2 \quad \text{for all} \quad f \in D(e^{-Q}). \qquad (7)$$

Therefore, $\{f_n\}$ ($f_n \in D(e^{-Q})$) is a Cauchy sequence in $\mathcal{H}_{-Q}$ if and only if $\{e^{-Q/2}f_n\}$ is a Cauchy sequence in $\mathcal{H}$.

By construction, the vectors ϕ_n belong to $D(e^{-Q/2})$. Hence, $\phi_n \in \mathcal{H}_{-Q}$. Assume that $f \in \mathcal{H}_{-Q}$ is orthogonal to $\{\phi_n\}$ and consider a sequence $\{f_m\}$ ($f_m \in D(e^{-Q})$) such that $f_m \to f \in \mathcal{H}_{-Q}$ (wrt. $\|\cdot\|_{-Q}$). Then, $\{e^{-Q/2}f_m\}$ is a Cauchy sequence in $\mathcal{H}$ and $e^{-Q/2}f_m$ converges to some $g \in \mathcal{H}$ (wrt. $\|\cdot\|$). This means that

$$0 = \langle f, \phi_n\rangle_{-Q} = \lim_{m\to\infty} \langle f_m, e^{Q/2}e_n\rangle_{-Q} = \lim_{m\to\infty} \langle e^{-Q/2} f_m, e_n\rangle = \langle g, e_n\rangle$$

and, as a result, $g = 0$ since $\{e_n\}$ is a complete set in $\mathcal{H}$. This means that $f = 0$ and the set $\{\phi_n\}$ is complete in $\mathcal{H}_{-Q}$. Completeness of $\{\psi_n\}$ in $\mathcal{H}_Q$ is established in a similar manner.

□

Proposition II.3. *Every dual GRS $\{\phi_n = e^{Q/2}e_n\}$ and $\{\psi_n = e^{-Q/2}e_n\}$ are ONB of the Hilbert spaces $(\mathcal{H}_{-Q}, \langle\cdot,\cdot\rangle_{-Q})$ and $(\mathcal{H}_Q, \langle\cdot,\cdot\rangle_Q)$, respectively.*

Proof. Due to (5), the sequence $\{\phi_n = e^{Q/2}e_n\}$ is orthonormal in $\mathcal{H}_{-Q}$. Its completeness follows from Lemma II.2. The case $\{\psi_n\}$ is considered similarly with the use of (6). □

Dual GRS could be used to define manifestly non self-adjoint Hamiltonians

$$H_{\phi,\psi}f = \sum_{n=1}^{\infty}\lambda_n\langle f,\psi_n\rangle\phi_n, \qquad H_{\psi,\phi}g = \sum_{n=1}^{\infty}\lambda_n\langle g,\phi_n\rangle\psi_n \tag{8}$$

with known complex eigenvalues $\{\lambda_n\}$ and eigenvectors $\{\phi_n\}$ and $\{\psi_n\}$, respectively. We refer to [6, 7] for the connection between $H_{\phi,\psi}$ and the adjoint of $H_{\psi,\phi}$ and for the analysis of ladder operators associated to similar bi-orthogonal sets, and how these ladder operators can be used to factorize the Hamiltonians above.

II.1. Dual GRS and $\mathcal{G}$-quasi bases

Dual GRS $\{\phi_n\}$ and $\{\psi_n\}$ can be considered as examples of more general object: $\mathcal{G}$-quasi bases. These are biorthogonal sets originally introduced in [2], and then analyzed in a series of papers (see [4] for a relatively recent review).

Definition II.4. Let $\mathcal{G}$ be a dense linear manifold in $\mathcal{H}$. Biorthogonal sequences $\{\phi_n\}$ and $\{\psi_n\}$ are called $\mathcal{G}$-quasi bases, if for all $f, g \in \mathcal{G}$, the following holds:

$$\langle f,g\rangle = \sum_{n=1}^{\infty}\langle f,\phi_n\rangle\langle\psi_n,g\rangle = \sum_{n=1}^{\infty}\langle f,\psi_n\rangle\langle\phi_n,g\rangle. \tag{9}$$

Proposition II.5. *Dual GRS $\{\phi_n\}$ and $\{\psi_n\}$ are $\mathcal{G}$-quasi bases with $\mathcal{G} = D(e^{Q/2}) \cap D(e^{-Q/2})$.*

Proof. If $\{\phi_n\}$ and $\{\psi_n\}$ are dual GRS, then there exists a self-adjoint operator Q and ONB $\{e_n\}$ such that (2) hold. Hence, for all $f, g \in \mathcal{G} = D(e^{Q/2}) \cap D(e^{-Q/2})$,

$$e^{Q/2}f = \sum\langle e^{Q/2}f, e_n\rangle e_n = \sum\langle f,\phi_n\rangle e_n$$

and

$$e^{-Q/2}g = \sum\langle e^{-Q/2}g, e_n\rangle e_n = \sum\langle g,\psi_n\rangle e_n.$$

The last relations yield

$$\langle f,g\rangle = \langle e^{Q/2}f, e^{-Q/2}g\rangle = \sum\langle f,\phi_n\rangle\langle\psi_n,g\rangle.$$

Similarly, $\langle f,g\rangle = \langle e^{-Q/2}f, e^{Q/2}g\rangle = \sum\langle f,\psi_n\rangle\langle\phi_n,g\rangle$. To complete the proof, it suffices notice that $\mathcal{G}$ is dense in $\mathcal{H}$ since each vector of ONB $\{e_n\}$ belongs to $D(e^{Q/2}) \cap D(e^{-Q/2})$. □

Remark II.6. Proposition II.5 implies that Example 1 above of GRS provides also an example of $\mathcal{G}$-quasi bases, with $\mathcal{G} = D(e^{x^2/4})$, in agreement with what was found in [6].

II.2. Regular biorthogonal sequences and dual GRS

We say that biorthogonal sequences $\{\phi_n\}$ and $\{\psi_n\}$ are *regular* if $\{\phi_n\}$ and $\{\psi_n\}$ are complete sets in $\mathcal{H}$. In other words, a biorthogonal sequence $\{\psi_n\}$ is defined uniquely by $\{\phi_n\}$ and vice-versa.

Theorem II.7. *Regular biorthogonal sequences* $\{\phi_n\}$ *and* $\{\psi_n\}$ *are dual GRS.*

Proof. Let $\{\phi_n\}$ and $\{\psi_n\}$ be regular biorthogonal sequences. Then an operator G_0 defined initially as

$$G_0\phi_n = \psi_n, \qquad n \in \mathbb{N} \tag{10}$$

and extended on $D(G_0) = span\{\phi_n\}$ by the linearity is densely defined and positive. The later follows from the fact that

$$\langle G_0 f, f\rangle = \sum_{n=1}^{k}\sum_{m=1}^{k} c_n\overline{c}_m\langle\psi_n, \phi_m\rangle = \sum_{n=1}^{k}|c_n|^2$$

for all $f = \sum_{n=1}^{k} c_n\phi_n \in D(G_0)$.

Let G be the Friedrichs extension of G_0. The operator G is positive. Indeed, assuming that $Gg = 0$ for $g \in D(G)$ we obtain $0 = \langle Gg, \phi_n\rangle = \langle g, G\phi_n\rangle = \langle g, \psi_n\rangle$. Therefore, $g = 0$ since $\{\psi_n\}$ is a complete set in $\mathcal{H}$. The positivity of G allows one to state that $G = e^{-Q}$, where Q is a self-adjoint operator in $\mathcal{H}$. Denote $e_n = e^{-Q/2}\phi_n$. Due to (10), $e_n = e^{Q/2}\psi_n$. Therefore, $e_n \in D(e^{Q/2}) \cap D(e^{-Q/2})$ and

$$\phi_n = e^{Q/2}e_n, \qquad \psi_n = e^{-Q/2}e_n. \tag{11}$$

The sequence $\{e_n\}$ is orthonormal in $\mathcal{H}$ since

$$\langle e_n, e_m\rangle = \langle e^{-Q/2}\phi_n, e^{Q/2}\psi_m\rangle = \langle\phi_n, \psi_m\rangle = \delta_{nm}.$$

Let us assume that $\gamma \in \mathcal{H}$ is orthogonal to $\{e_n\}$. Then there exists a sequence $\{f_m\}$ ($f_m \in D(e^{-Q})$) such that $e^{-Q/2}f_m \to \gamma$ in $\mathcal{H}$ (because $e^{-Q/2}D(e^{-Q})$ is a dense set in $\mathcal{H}$). In this case, due to (7), $\{f_m\}$ is a Cauchy sequence in $\mathcal{H}_{-Q}$ and therefore, f_m tends to some $f \in \mathcal{H}_{-Q}$. This means that

$$0 = \langle\gamma, e_n\rangle = \lim_{m\to\infty}\langle e^{-Q/2}f_m, e_n\rangle = \lim_{m\to\infty}\langle f_m, \phi_n\rangle_{-Q} = \langle f, \phi_n\rangle_{-Q}. \tag{12}$$

We note that the set $D(G_0) = span\{\phi_n\}$ is dense in the Hilbert space $(\mathcal{H}_{-Q}, \langle\cdot,\cdot\rangle_{-Q})$ (since $G = e^{-Q}$ is the Friedrichs extension of G_0 [1]). In view of (12), $f = 0$ that means

$\lim_{m\to\infty}\|f_m\|_{-Q} = \lim_{m\to\infty}\|e^{-Q/2}f_m\| = 0$ and therefore, $\gamma = 0$. This means that the orthonormal sequence $\{e_n\}$ is complete in $\mathcal{H}$. Hence $\{e_n\}$ is an ONB. $\square$

Remark II.8. Another proof of Theorem II.7 can be found in [14, Theorem 2.1].

II.3. The uniqueness of Q and $\{e_n\}$ for regular biorthogonal sequences

Let $\{\phi_n\}$ be a basis in $\mathcal{H}$. Then $\{\phi_n\}$ is a regular sequence because its biorthogonal sequence $\{\psi_n\}$ has to be a basis. By Theorem II.7, $\{\phi_n\}$ is a GRS, i.e., there exists a self-adjoint operator Q and an ONB $\{e_n\}$ such that $\phi_n = e^{Q/2}e_n$.

Proposition II.9. *The operator Q and ONB $\{e_n\}$ in Definition II.1 are determined* uniquely *for every basis $\{\phi_n\}$.*

Proof. Let γ be orthogonal to $R(G_0+I)$. Then, in view of (10), $\langle\gamma,\phi_n\rangle = -\langle\gamma,\psi_n\rangle$ and the basis property of $\{\phi_n\}$ and $\{\psi_n\}$ leads to the relation:

$$\gamma = \sum\langle\gamma,\psi_n\rangle\phi_n = \sum\langle\gamma,\phi_n\rangle\psi_n = -\sum\langle\gamma,\psi_n\rangle\psi_n. \qquad (13)$$

By virtue of (13), the sequence $\gamma_m = \sum_{n=1}^{m}\langle\gamma,\psi_n\rangle\phi_n$ tends to γ, while $G_0\gamma_m = \sum_{n=1}^{m}\langle\gamma,\psi_n\rangle\psi_n$ tends to $-\gamma$. Therefore,

$$-\|\gamma\|^2 = \lim_{m\to\infty}\langle G_0\gamma_m,\gamma_m\rangle = \lim_{m\to\infty}\sum_{n=1}^{m}|\langle\gamma,\psi_n\rangle|^2 = \sum_{n=1}^{\infty}|\langle\gamma,\psi_n\rangle|^2$$

that is possible when $\gamma = 0$. Hence, $R(G_0+I)$ is a dense set in $\mathcal{H}$ and, as a result, G_0 is an essentially self-adjoint operator in $\mathcal{H}$. Its closure $\overline{G}_0$ gives a unique positive self-adjoint extension G which determines a unique self-adjoint operator $Q = -\ln G$ (i.e. $G = e^{-Q}$). Moreover, because of the equality $e_n = e^{-Q/2}\phi_n$, the ONB $\{e_n\}$ is also determined uniquely. $\square$

In view of Proposition II.9 a natural question arise: *is the operator Q determined uniquely for a given GRS $\{\phi_n\}$?*

The choice of the Friedrichs extension $G = e^{-Q}$ of G_0 in the proof of Theorem II.7 was inspired by the fact that the sequence $\{\phi_n\}$ must be complete in the Hilbert space $\mathcal{H}_{-Q}$ (that, in view of (12) and Lemma II.2, is equivalent to the completeness of orthonormal system $\{e_n\}$ in $\mathcal{H}$). Generally, there are many self-adjoint extensions G of G_0 which preserve this property and each of them can be used instead of the Friedrichs extension.

We recall [1] that a nonnegative self-adjoint extension G of G_0 is called extremal if

$$\inf_{\phi\in D(G_0)}\langle G(f-\phi),(f-\phi)\rangle = 0 \quad \text{for all} \quad f\in D(G).$$

The Friedrichs extension and the Krein-von Neumann extension of G_0 are examples of extremal extensions.

In the case of regular bi-orthogonal sequences $\{\phi_n\}$ and $\{\psi_n\}$, the symmetric operator G_0 is positive and each nonnegative self-adjoint extension G of G_0 must also be *positive*. Indeed, if $\langle Gf, f\rangle = 0$ for some $f \in D(G)$, then $Gf = 0$ and

$$0 = \langle Gf, \phi_n\rangle = \langle f, G\phi_n\rangle = \langle f, G_0\phi_n\rangle = \langle f, \psi_n\rangle$$

that implies $f = 0$. Therefore, each nonnegative self-adjoint extension G of G_0 is positive and it has the form $G = e^{-Q}$. This means that $\langle G(f-\phi), (f-\phi)\rangle$ coincides with $\|f-\phi\|^2_{-Q}$ due to (7). For this reason, the definition of extremal extensions can be rewritten as follows: let G_0 be determined by (10), where $\{\phi_n\}$ and $\{\psi_n\}$ are regular biorthogonal sequences. A self-adjoint extension $G = e^{-Q}$ of G_0 is called *extremal* if

$$\inf_{\phi \in D(G_0)} \|f-\phi\|^2_{-Q} = 0 \quad \text{for all} \quad f \in D(e^{-Q}).$$

Therefore, the extremality of a self-adjoint extension e^{-Q} of G_0 is equivalent to the completeness of the sequence $\{\phi_n\}$ in $\mathcal{H}_{-Q}$. This means that for each extremal self-adjoint extension e^{-Q} one can repeat the proof of Theorem II.7 and establish the relations (11), where $\{e_n\}$ is an ONB. Summing up, we prove the equivalence of statements (i) and (ii) in:

Proposition II.10. *Let $\{\phi_n\}$ and $\{\psi_n\}$ be a regular biorthogonal sequences. The following are equivalent:*

(i) the self-adjoint operator Q and the ONB $\{e_n\}$ are determined uniquely in (11);

(ii) the symmetric operator G_0 in (10) has a unique extremal extension $G = e^{-Q}$.

(iii)
$$\inf_{\phi\in D(G_0)} \frac{\langle G_0\phi, \phi\rangle}{|\langle \phi, g\rangle|^2} = 0, \quad \text{for all nonzero} \quad g \in \ker(I + G_0^*), \tag{14}$$
where G_0^* means the adjoint operator of G_0 with respect to $\langle\cdot,\cdot\rangle$.

Proof. It suffices to establish the equivalence (ii) and (iii). Indeed, the set of extremal extensions involves the Friedrichs G_F and the Krein-von Neumann G_K extensions of G_0 and it contains only one element when $G_F = G_K$ [1]. The last equality is equivalent to (14) due to [17, Theorem 9]. □

III. Orthonormal sequences in Krein space

III.1. Elements of the Krein spaces theory

Here all necessary results of Krein spaces theory are presented in a form convenient for our exposition. The chapters [4, Chap. 6] and [11, Chap. 8] are recommended as complementary reading on the subject.

An operator J is called *fundamental symmetry* in a Hilbert space $\mathcal{H}$ if J is a bounded self-adjoint operator in $\mathcal{H}$ and $J^2 = I$.

Let J be a non-trivial fundamental symmetry, i.e., $J \neq \pm I$. The Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ equipped with the indefinite inner product $[\cdot,\cdot] := \langle J\cdot,\cdot\rangle$ is called a Krein space $(\mathcal{H}, [\cdot,\cdot])$.

The principal difference between the initial inner product $\langle\cdot,\cdot\rangle$ and the indefinite inner product $[\cdot,\cdot]$ is that there exist nonzero elements $f \in \mathcal{H}$ such that $[f,f] < 0$. An element $f \neq 0$ is called *positive* or *negative* if $[f,f] > 0$ or $[f,f] < 0$, respectively. A closed subspace $\mathfrak{L}$ of the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is called *positive* or *negative* if all nonzero elements $f \in \mathfrak{L}$ are, respectively, positive or negative. A positive (negative) subspace $\mathfrak{L}$ is called *uniformly positive* (*uniformly negative*) if there exists $\alpha > 0$ such that

$$[f,f] \geq \alpha\langle f,f\rangle \qquad (-[f,f] \geq \alpha\langle f,f\rangle) \quad \forall f \in \mathfrak{L}.$$

In each of these classes we can define maximal subspaces. For instance, a positive subspace $\mathfrak{L}$ is called *maximal positive* if $\mathfrak{L}$ is not a subspace of another positive subspace in $\mathcal{H}$. The maximality of a (negative, uniformly positive, uniformly negative) closed subspace is defined similarly.

Let a subspace $\mathfrak{L}$ be a maximal positive (negative). Then its orthogonal complement with respect to the indefinite inner product $[\cdot,\cdot]$

$$\mathfrak{L}^{[\perp]} = \{f \in \mathcal{H} \ : \ [f,g] = 0, \ \forall g \in \mathfrak{L}\}$$

is a maximal negative (positive) subspace and the J-orthogonal sum

$$\mathfrak{L}[\dot{+}]\mathfrak{L}^{[\perp]} \tag{15}$$

is dense in the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ (the symbol $[\dot{+}]$ in (15) indicates that the subspaces $\mathfrak{L}$ and $\mathfrak{L}^{[\perp]}$ are orthogonal with respect to $[\cdot,\cdot]$, i.e. J-orthogonal).

The J-orthogonal sum (15) coincides with $\mathcal{H}$, i.e.,

$$\mathcal{H} = \mathfrak{L}[\dot{+}]\mathfrak{L}^{[\perp]} \tag{16}$$

if and only if $\mathfrak{L}$ is a maximal uniformly positive (uniformly negative) subspace (in this case, $\mathfrak{L}^{[\perp]}$ is maximal uniformly negative (uniformly positive)).

Remark III.1. The decomposition (16) is called *the fundamental decomposition* of $\mathcal{H}$ and it is often used for (an equivalent) definition of Krein spaces. Precisely, let $\mathcal{H}$ be a complex linear space with a Hermitian sesquilinear form $[\cdot,\cdot]$ (i.e. a mapping

$[\cdot,\cdot] : \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ such that $[\alpha_1 f_1+\alpha_2 f_2, g]=\alpha_1[f_1,g]+\alpha_2[f_2,g]$ and $[f,g]=\overline{[g,f]}$ for all $f_1,f_2,f,g\in\mathcal{H}$, $\alpha_1,\alpha_2\in\mathbb{C}$). Then $(\mathcal{H},[\cdot,\cdot])$ is called a Krein space if $\mathcal{H}$ admits a decomposition (16) such that the linear manifolds $(\mathfrak{L},[\cdot,\cdot])$ and $(\mathfrak{L}^{[\perp]},-[\cdot,\cdot])$ are Hilbert spaces (here we suppose for definiteness that $\mathfrak{L}$ is positive).

Each fundamental decomposition (16) is uniquely determined by a bounded operator $\mathcal{C}$ which coincides with the identity operator on the positive subspace $\mathfrak{L}_+:=\mathfrak{L}$ and with the minus identity operator on the negative subspace $\mathfrak{L}_-:=\mathfrak{L}^{[\perp]}$. By the construction, $\mathfrak{L}_\pm=(I\pm\mathcal{C})\mathcal{H}$ and $\mathcal{C}^2=I$. Moreover, the operator $J\mathcal{C}$ is positive self-adjoint since

$$\langle J\mathcal{C}f,f\rangle=[\mathcal{C}f,f]=[f_+,f_+]-[f_-,f_-]>0 \quad \text{for non-zero} \quad f=f_++f_-,\ f_\pm\in\mathfrak{L}_\pm.$$

Hence, $J\mathcal{C}=e^{-Q}$, where Q is a bounded self-adjoint operator. The relations $\mathcal{C}^2=I, J\mathcal{C}=e^{-Q}>0$ and [19, Theorem 2.1] imply that

$$JQ=-QJ. \tag{17}$$

Similar reasonings applied to the J-orthogonal sum (15) gives rise to the collection of unbounded operators $\mathcal{C}=Je^{-Q}=e^{Q}J$, where unbounded Q anticommutes with J. The subspaces in (15) are recovered as $\mathfrak{L}_\pm=(I\pm\mathcal{C})(\mathfrak{L}_+[\dot{+}]\mathfrak{L}_-)$.

Summing up: *the fundamental decompositions (16) of a Krein space are in one-to-one correspondence with the set of bounded operators $\mathcal{C}=Je^{-Q}=e^{Q}J$.*

The J-orthogonal sums (15) of maximal positive/maximal negative subspaces are in one-to-one correspondence with the set of unbounded operators $\mathcal{C}=Je^{-Q}=e^{Q}J$. In both cases, Q anticommutes with J.

The operator $\mathcal{C}$ is called *a $\mathcal{C}$-symmetry operator* and this notion is widely used in $\mathcal{PT}$-symmetric approach in Quantum Mechanics [11].

Remark III.2. If Q in (17) is unbounded, then we understood (17) as the identity $JQf=-QJf$, where $f\in D(Q)$ and J leaves $D(Q)$ invariant. From now on, we will adopt this simplifying notation.

A $\mathcal{C}$-symmetry operator allows one to define a new inner product via the indefinite inner product $[\cdot,\cdot]$:

$$\langle f,g\rangle_{-Q}:=[\mathcal{C}f,g]=\langle J^2e^{-Q}f,g\rangle=\langle e^{-Q}f,g\rangle, \qquad f,g\in D(\mathcal{C})=D(e^{-Q}). \tag{18}$$

The corresponding norm $\|\cdot\|_{-Q}$ is equivalent to the original norm of $\mathcal{H}$ when $\mathcal{C}$ is bounded. If $\mathcal{C}$ is unbounded, then the completion of $D(\mathcal{C})$ with respect to $\|\cdot\|_{-Q}$ leads to the Hilbert space $(\mathcal{H}_{-Q},\langle\cdot,\cdot\rangle_{-Q})$ defined in Section II.

Remark III.3. It is maybe worth mentioning that the unboundedness of the $\mathcal{C}$ operator, and of the related metric, is a serious issue in $\mathcal{PT}$ Quantum Mechanics. For example, [4], it may happen that the basis property of the eigenvectors of a $\mathcal{PT}$-symmetric Hamiltonian, obtained by considering a suitable deformation of a self-adjoint operator, is lost. This is the case, for instance, of the Swanson model and of the shifted harmonic oscillator, [3]. We meet similar difficulties also when working with Krein spaces, as it will appear clear in the remaining part of the paper.

III.2. J-orthonormal sequences of the first and of the second type

A sequence $\{\phi_n\}$ is called orthonormal in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ (briefly, J-orthonormal) if $|[\phi_n, \phi_m]| = \delta_{nm}$. For each J-orthonormal sequence $\{\phi_n\}$ there exists a biorthogonal one

$$\psi_n = [\phi_n, \phi_n] J\phi_n. \tag{19}$$

Obviously, $\{\psi_n\}$ is J-orthonormal and $[\phi_n, \phi_n] = [\psi_n, \psi_n]$. In view of (19), the positive symmetric operator G_0 in (10) acts as

$$G_0\phi_n = [\phi_n, \phi_n] J\phi_n, \qquad n \in \mathbb{N}. \tag{20}$$

In what follows we assume that $\{\phi_n\}$ is complete in the Hilbert space $\mathcal{H}$. Then $\{\psi_n\}$ in (19) is complete too. Therefore, $\{\phi_n\}$ and $\{\psi_n\}$ are regular biorthogonal sequences and, by Theorem II.7, they are dual GRS. Thus, *each complete J-orthonormal sequence is a GRS.* The corresponding operator $Q = -\ln\ G$ in (11) can be determined by every extremal extension $G = e^{-Q}$ of G_0. Such kind of freedom allows us to select an appropriative operator Q which fits well with the J-orthonormality of $\{\phi_n\}$.

Theorem III.4. *Let $\{\phi_n\}$ be a complete J-orthonormal sequence. If a self-adjoint operator Q in (11) is determined uniquely, then the relation (17) holds.*

Proof. Separating the sequence $\{\phi_n\}$ by the signs of $[\phi_n, \phi_n]$:

$$\phi_n = \begin{cases} \phi_n^+ & \text{if} \quad [\phi_n, \phi_n] = 1, \\ \phi_n^- & \text{if} \quad [\phi_n, \phi_n] = -1 \end{cases} \tag{21}$$

we obtain two sequences of positive $\{\phi_n^+\}$ and negative $\{\phi_n^-\}$ elements. Denote by $\mathfrak{L}_+^0$ and $\mathfrak{L}_-^0$ the closure (in the Hilbert space $\mathcal{H}$) of the linear spans generated by the sets $\{\phi_n^+\}$ and $\{\phi_n^-\}$, respectively. By construction, $\mathfrak{L}_\pm^0$ are positive/negative subspaces and their J-orthogonal sum $\mathfrak{L}_+^0[\dot{+}]\mathfrak{L}_-^0$ coincides with the domain of the closure $\overline{G}_0$ of G_0 determined by (20) on *span*$\{\phi_n\}$ [19, Lemma 4.1][1].

[1]) in [19], the notation G_0 is used for $\overline{G}_0$

By Proposition II.10, the uniqueness of Q means that the symmetric operator $\overline{G}_0$ has a unique extremal extension $G = e^{-Q}$. This is possible when G coincides with the Friedrichs extension of G_0 as well as with the Krein-von Neumann extension of G_0. This fact, by virtue of [19, Theorem 4.3], means that $Je^{-Q}f = e^{Q}Jf$ for $f \in D(e^{-Q})$. The last relation and [19, Theorem 2.1] justify (17). □

In view of Theorem III.4, it seems natural to consider the anti-commutation relation (17) in the case where Q is not determined uniquely. Taking into account that (17) is equivalent to the relation

$$JGf = G^{-1}Jf, \qquad f \in D(G), \tag{22}$$

where $G = e^{-Q}$ is an extremal extension of G_0, we reduce the choice of Q which satisfies (17) to the choice of an extremal extension G satisfying (22).

If extremal extensions G of G_0 are not determined uniquely, then not each $Q = -\ln G$ will anticommute necessarily with J. In particular, the operator Q that corresponds to the Friedrichs extension $G = e^{-Q}$ of G_0 does not satisfy (17) [16].

Definition III.5. A complete J-orthonormal sequence $\{\phi_n\}$ is of the first type if there exists a self-adjoint operator Q such that the formulas (11) hold with the additional property $JQ = -QJ$. Otherwise, $\{\phi_n\}$ is of the second type.

In view of Theorem III.4, each complete J-orthonormal sequence $\{\phi_n\}$ with the unique operator Q in (11) is the first type. In particular, every J-orthonormal basis is a first type sequence. The example of a second type sequence can be found in [16, Subsection 6.2]. In what follows, considering a first type sequence, we assume that Q anti-commutes with J.

Proposition III.6. *Let a complete sequence $\{\phi_n\}$ be a GRS. The following are equivalent:*

(i) $\{\phi_n\}$ is the first type;

(ii) the operator Q in (11) can be chosen in such a way that $JQ = -QJ$ and the vectors e_n are eigenvectors of J (i.e., $Je_n = e_n$ or $Je_n = -e_n$).

Proof. $(i) \to (ii)$. By virtue of (11) and (19),

$$J\phi_n = Je^{Q/2}e_n = e^{-Q/2}Je_n = [\phi_n, \phi_n]\psi_n = [\phi_n, \phi_n]e^{-Q/2}e_n.$$

Comparing the third and the fifth terms in the equality above we get $Je_n = [\phi_n, \phi_n]e_n$ that implies (ii).

(ii) $\to$ *(i)*. Since $\{\phi_n\}$ is complete in $\mathcal{H}$ by assumption, it suffices to verify the J-orthonormality of $\{\phi_n\}$: $[\phi_n, \phi_m] = \langle Je^{Q/2}e_n, e^{Q/2}e_m\rangle = \langle e^{-Q/2}Je_n, e^{Q/2}e_m\rangle = [e_n, e_n]\delta_{nm}$. □

Remark III.7. The studies of the first type sequences began in [16], where they were called "quasi bases". Proposition III.6 is a part of [16, Theorem 6.3]. We present here a simpler proof.

For the first type sequence, the inner product in $(\mathcal{H}_{-Q}, \langle\cdot,\cdot\rangle_{-Q})$ is *directly determined by the known indefinite inner product* $[\cdot,\cdot]$, see (24) below. Let us briefly explain this important fact (see [16] for details).

Since Q anticommutes with J, the J-orthogonal sum $\mathfrak{L}_+^0[\dot{+}]\mathfrak{L}_-^0$ of the subspaces $\mathfrak{L}_\pm^0$ defined in the proof of Theorem III.4 can be extended to the J-orthogonal sum

$$D(G) = D(e^{-Q}) = \mathfrak{L}_+[\dot{+}]\mathfrak{L}_-, \qquad \mathfrak{L}_\pm^0 \subset \mathfrak{L}_\pm,$$

where $\mathfrak{L}_\pm$ are maximal positive/negative subspaces in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ and they are uniquely determined by the choice of Q: $\mathfrak{L}_\pm = (I \pm Je^{-Q})D(e^{-Q})$. The last relation and (5) imply that for $f = (I + Je^{-Q})u$ and $g = (I + Je^{-Q})v$ from $\mathfrak{L}_+$:

$$[f, g] = \langle Jf, g\rangle = \langle J(I + Je^{-Q})u, (I + Je^{-Q})v\rangle = 2([u, v] + \langle e^{-Q}u, v\rangle) =$$
$$\langle e^{-Q}(I + Je^{-Q})u, (I + Je^{-Q})v\rangle = \langle e^{-Q}f, g\rangle = \langle f, g\rangle_{-Q}.$$

Therefore, the indefinite inner product $[\cdot,\cdot]$ coincides with $\langle\cdot,\cdot\rangle_{-Q}$ on $\mathfrak{L}_+$. Similar calculations show that $[\cdot,\cdot]$ coincides with $-\langle\cdot,\cdot\rangle_{-Q}$ on $\mathfrak{L}_-$. Moreover, the subspaces $\mathfrak{L}_\pm$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{-Q}$ since

$$\langle f, \gamma\rangle_{-Q} = \langle e^{-Q}(I + Je^{-Q})u, (I - Je^{-Q})w\rangle = 0,$$

where $\gamma = (I - Je^{-Q})w$ and $u, w \in D(e^{-Q})$. This leads to the conclusion that

$$\mathcal{H}_{-Q} = \widehat{\mathfrak{L}}_+[\oplus_{-Q}]\widehat{\mathfrak{L}}_-, \tag{23}$$

where $\widehat{\mathfrak{L}}_\pm$ are the completion of the pre-Hilbert spaces $(\mathfrak{L}_\pm, \pm[\cdot,\cdot])$ and $[\oplus_{-Q}]$ indicates the orthogonality with respect to $\langle\cdot,\cdot\rangle_{-Q}$ and with respect to $[\cdot,\cdot]$. Keeping the same notation for the extension of $[\cdot,\cdot]$ onto $\mathcal{H}_{-Q}$ we obtain the new Krein space $(\mathcal{H}_{-Q}, [\cdot,\cdot])$ with the fundamental decomposition (23). For every $f, g \in \mathcal{H}_{-Q}$ $(f_\pm, g_\pm \in \widehat{\mathfrak{L}}_\pm)$,

$$\langle f, g\rangle_{-Q} = [f_+, g_+] - [f_-, g_-]. \tag{24}$$

For the second type sequences, there are no operators Q in (11) which anticommute with J. The space $\mathcal{H}_{-Q}$ cannot be presented as in (23). This implies that $\langle\cdot,\cdot\rangle_{-Q}$ cannot be directly expressed via $[\cdot,\cdot]$ and one should apply much more efforts for calculation of $\langle\cdot,\cdot\rangle_{-Q}$.

Let $\{\phi_n\}$ be the first type. Then the linear manifold $\mathcal{G} = D(e^{Q/2}) \cap D(e^{-Q/2})$ in Proposition II.5 is invariant with respect to J and the formula (9) can be rewritten as

$$[f,g] = \sum_{n=1}^{\infty} \delta_n [f,\phi_n][\phi_n,g] = \sum_{n=1}^{\infty} \delta_n [f,\psi_n][\psi_n,g], \qquad f,g \in \mathcal{G},$$

where $\delta_n = [\phi_n,\phi_n] = [\psi_n,\psi_n]$. Moreover, for all $f \in \mathcal{H}_{-Q}$,

$$f = \sum_{n=1}^{\infty} \delta_n [f,\phi_n]\phi_n, \tag{25}$$

where the series converges in $(\mathcal{H}_{-Q}, \langle\cdot,\cdot\rangle_{-Q})$. Indeed, $f = \sum_{n=1}^{\infty} \langle f,\phi_n\rangle_{-Q}\phi_n$ since $\{\phi_n\}$ is ONB of $\mathcal{H}_{-Q}$ (Proposition II.3). By virtue of (21), (23) and (24),

$$\langle f,\phi_n\rangle_{-Q} = \begin{cases} [f,\phi_n^+] & (\text{if } \phi_n = \phi_n^+) \\ -[f,\phi_n^-] & (\text{if } \phi_n = \phi_n^-) \end{cases} = \delta_n [f,\phi_n]$$

that implies (25).

Assume that $\sum_{n=1}^{\infty} c_n\phi_n$ converges to an element f in $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. In this case, due to (19), $c_n = \langle f,\psi_n\rangle = \delta_n[f,\phi_n]$. In general, we cannot state that this series converges unconditionally in $\mathcal{H}$. Denote

$$\mathcal{D}_{un} = \{f \in \mathcal{H} : \text{ the series } \sum_{n=1}^{\infty} \delta_n[f,\phi_n]\phi_n \text{ converges unconditionally to } f \text{ in } \mathcal{H}\}.$$

Proposition III.8. *Let $\{\phi_n\}$ be a first type sequence. Then $\mathcal{D}_{un} \subseteq \mathfrak{L}_+^0 \dot{+} \mathfrak{L}_-^0$, where $\mathfrak{L}_\pm^0$ are defined in the proof of Theorem III.4.*

Proof. Let $f \in \mathcal{D}_{un}$. Then, simultaneously with (25), the series

$$\sum_{n}^{\infty} [f,\phi_n^+]\phi_n^+ \qquad \text{and} \qquad -\sum_{n}^{\infty} [f,\phi_n^-]\phi_n^-$$

(the vectors $\phi_n^\pm$ are determined by (21)) converge to elements $f_\pm$ in the Hilbert space $\mathcal{H}$ (see, e.g., [13, Theorem 3.10]). By the construction $f_\pm \in \mathfrak{L}_\pm^0$. Therefore, $f = f_+ + f_-$ belongs to $\mathfrak{L}_+^0 \dot{+} \mathfrak{L}_-^0$. □

III.3. J-orthonormal sequences and operators of $\mathcal{C}$-symmetry

We say that an J-orthonormal sequence $\{\phi_n\}$ *generates a $\mathcal{C}$-symmetry operator* $\mathcal{C} = Je^{-Q} = e^{Q}J$ (the operator Q anti-commutes with J) if

$$\mathcal{C}\phi_n^+ = \phi_n^+, \qquad \mathcal{C}\phi_n^- = -\phi_n^-,$$

where $\phi_n^{\pm}$ are defined in (21).

With each operator $\mathcal{C}$ one can associate a Hilbert space $(\mathcal{H}_{-Q}, \langle\cdot,\cdot\rangle_{-Q})$ (see Subsection III.1). In view of (18),

$$\langle\phi_n, \phi_m\rangle_{-Q} = [\mathcal{C}\phi_n, \phi_m] = \begin{cases} [\phi_n^+, \phi_m] & (\text{if } \phi_n = \phi_n^+) \\ -[\phi_n^-, \phi_m] & (\text{if } \phi_n = \phi_n^-) \end{cases} = \delta_{nm}.$$

Therefore, $\{\phi_n\}$ is an orthonormal system in $\mathcal{H}_{-Q}$.

Proposition III.9 and Corollary III.10 follow from [16, Sections 5, 6].

Proposition III.9. *Each complete J-orthonormal sequence $\{\phi_n\}$ generates at least one operator of $\mathcal{C}$-symmetry. The sequence $\{\phi_n\}$ is the first type if and only if it generates a $\mathcal{C}$-symmetry operator $\mathcal{C} = e^Q J$ such that $\{\phi_n\}$ is an ONB of $\mathcal{H}_{-Q}$. This operator $\mathcal{C}$ is determined uniquely or there are infinitely many such operators.*

Obviously, if $\{\phi_n\}$ is the first type, then its bi-orthogonal sequence $\{\psi_n\}$ is also the first type.

Corollary III.10. *If $\{\phi_n\}$ is the first type and $\{\lambda_n\}$ are real numbers, then there exists a $\mathcal{C}$-symmetry operator $\mathcal{C} = e^Q J$ such that the operators $H_{\phi,\psi}$ and $H_{\psi,\phi}$ defined in (8) on the domains $D(H_{\phi,\psi}) = span\{\phi_n\}$ and $D(H_{\psi,\phi}) = span\{\psi_n\}$, respectively are essentially self-adjoint in the Hilbert spaces $\mathcal{H}_{-Q}$ and $\mathcal{H}_Q$. The spectra of $H_{\phi,\psi}$ and $H_{\psi,\phi}$ coincides with the closure of $\{\lambda_n\}$.*

Remark III.11. Corollary III.10 can be easy extended for the general case of GRS $\{\phi_n = e^{Q/2}e_n\}$. Indeed, in view of Proposition II.3, $\{\phi_n\}$ and $\{\psi_n\}$ are ONB of the Hilbert spaces $\mathcal{H}_{-Q}$ and $\mathcal{H}_Q$, respectively. Therefore, the operators $H_{\phi,\psi}$ and $H_{\psi,\phi}$ defined on span$\{\phi_n\}$ and span$\{\psi_n\}$ have to be essentially self-adjoint in $\mathcal{H}_{-Q}$ and $\mathcal{H}_Q$. This approach can be used for J-orthonormal sequences of the second type. The principal difference is: for the first type sequence, the new scalar product in $\mathcal{H}_{-Q}$ is directly determined by the known indefinite inner product $[\cdot,\cdot]$, see (24). For the second type sequence, the inner product $\langle\cdot,\cdot\rangle_{-Q}$ cannot be expressed via $[\cdot,\cdot]$ and it becomes more complicated to determine $\langle\cdot,\cdot\rangle_{-Q}$.

IV. Examples

IV.1. Eigenfunctions of the shifted harmonic oscillator

In the space $\mathcal{L}^2(\mathbb{R})$ we define a fundamental symmetry J as the space parity operator $\mathcal{P}f(x) = f(-x)$. The indefinite inner product is

$$[f, g] = \langle \mathcal{P}f, g\rangle = \int_{-\infty}^{\infty} f(-x)\overline{g(x)}dx.$$

The Hermite functions $e_n(x)$ in (4) form an ONB of $\mathcal{L}^2(\mathbb{R})$ and $\mathcal{P}e_n = (-1)^n e_n$.

Define

$$\phi_n(x) = e_n(x+ia), \quad \psi_n(x) = e_n(x-ia), \quad a \in \mathbb{R}\backslash\{0\}, \quad n = 0, 1, 2, \ldots \tag{26}$$

using the fact that $e_n(x)$ are entire functions. The functions $\{\phi_n\}$ and $\{\psi_n\}$ are eigenvectors of the shifted harmonic oscillators

$$H = -\frac{d^2}{dx^2} + x^2 + 2iax \quad \text{and} \quad H^* = -\frac{d^2}{dx^2} + x^2 - 2iax,$$

respectively. It follows from [20, Lemma 2.5] that $\{\phi_n\}$ and $\{\psi_n\}$ are regular biorthogonal sequences and hence, they are dual GRS (Theorem II.7).

To find a self-adjoint operator Q in (11), we calculate the Fourier transform $F\phi_n = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}\phi_n(x)dx$. In view of (26), $F\phi_n = e^{-a\xi}Fe_n$. Therefore, $\phi_n = F^{-1}e^{-a\xi}Fe_n$. The last relation can be rewritten as

$$\phi_n = e^{Q/2}e_n, \qquad e^{Q/2} = F^{-1}e^{-a\xi}F, \tag{27}$$

where $Q = 2ai\frac{d}{dx}$ is an unbounded self-adjoint operator in $\mathcal{L}^2(\mathbb{R})$ that anticommutes with $\mathcal{P}$. By virtue of Proposition III.6, $\{\phi_n\}$ and $\{\psi_n\}$ are $\mathcal{P}$-orhonormal sequences of the first type. The operator of $\mathcal{C}$-symmetry generated $\{\phi_n\}$ coincides with $\mathcal{C} = e^{ai\frac{d}{dx}}\mathcal{P}$ and it is determined uniquely.

IV.2. Eigenfunctions of perturbed anharmonic oscillator

Let $\{e_n\}$ be eigenfunctions of the anharmonic oscillator

$$H_\beta = -\frac{d^2}{dx^2} + |x|^\beta, \qquad \beta\rangle 2$$

Without loss of generality we assume that $\|e_n\| = 1$. The eigenfunctions $\{e_n\}$ form an orthonormal basis in $\mathcal{L}^2(\mathbb{R})$ and they are eigenvectors of $\mathcal{P}$.

Consider the sequences

$$\phi_n(x) = e^{p(x)}e_n(x), \qquad \psi_n(x) = e^{-p(x)}e_n(x)$$

where a real-valued odd function $p \in C^2(\mathbb{R})$ satisfies [20, Assumption II]. The functions $\{\phi_n\}$ and $\{\psi_n\}$ are eigenvectors of the perturbed anharmonic oscillators

$$H = H_\beta + 2p'(x)\frac{d}{dx} + p''(x) - (p'(x))^2 \quad \text{and} \quad H^* = H_\beta - 2p'(x)\frac{d}{dx} - p''(x) - (p'(x))^2,$$

respectively. The functions $\{\phi_n\}$ and $\{\psi_n\}$ are determined by (11) with the operator of multiplication $Q = 2p(x)$ in $\mathcal{L}^2(\mathbb{R})$ which anticommutes with $\mathcal{P}$. Proposition III.6 implies that $\{\phi_n\}$ and $\{\psi_n\}$ are $\mathcal{P}$-orhonormal and they are sequences of the first type.

IV.3. Back to the harmonic oscillator

Let us go back to Example 1 in Section II. The vectors $\phi_n(x)$ and $\psi_n(x)$ are respectively eigenstates of H and H^*, where

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} - x\frac{d}{dx} + \frac{1}{2}\left(\frac{3x^2}{2} - 1\right)\right):$$

$H\phi_n = E_n\phi_n$ and $H^*\psi_n = E_n\psi_n$, where $E_n = n + \frac{1}{2}$. Simple computations show that $H^* = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x\frac{d}{dx} + \frac{1}{2}\left(\frac{3x^2}{2} + 1\right)\right)$, see [6]. The sets $\{\phi_n\}$ and $\{\psi_n\}$ are both complete in $\mathcal{L}^2(\mathbb{R})$. Hence they are regular and dual GRS, as a consequence of Theorem II.7.

The interesting aspect of this example is that the operator $Q = -\frac{x^2}{2}$ does not anticommute with the space parity operator $\mathcal{P}$. In fact, we have $Q\mathcal{P} = \mathcal{P}Q$. Then Proposition III.6 suggests that the set $\{\phi_n\}$ cannot be $\mathcal{P}$-orthonormal. In fact, it is possible to check that this is what happens. To do that, we first notice that, because of the parity properties of the Hermite polynomials, we have

$$[\phi_n, \phi_m] = \frac{(-1)^n}{\sqrt{2^{n+m}\, n!\, m!\, \sqrt{2}}} \int_{\mathbb{R}} H_n(x)\, H_m(x) e^{-\frac{3x^2}{2}}\, dx.$$

This integral is zero if $n+m$ is odd. But, if $n+m$ is even, the result is non zero. In fact, after some computations, we get

$$|[\phi_n, \phi_m]| = \sqrt{\frac{2^{n+m+1}}{3^{n+m+1}\, \pi\, n!\, m!}}\, \Gamma\left(\frac{n+m+1}{2}\right) {}_2F_1\left(-m, n; \frac{1-m-n}{2}; \frac{3}{2}\right),$$

where Γ and ${}_2F_1$ are respectively the Gamma and the Hypergeometric functions. The result is not zero, if $n+m$ is even. Hence the set $\{\phi_n\}$ is not $\mathcal{P}$-orthonormal, as expected.

V. Conclusions

In this paper, motivated by many (already existing or) possible applications to $\mathcal{PT}$-quantum mechanics, we have derived some properties of vectors which are complete in a given Hilbert space, and orthonormal with respect to an indefinite inner product. In this analysis we have heavily used generalized Riesz systems and $\mathcal{G}$-quasi bases, and we have introduced two different types of J-orthonormal sequences, those of the first and those of the second type, depending on the validity of a certain anti-commutation relation between J and Q. Examples of both kind have been proposed, all arising from harmonic or anharmonic oscillators.

Among our future projects we plan to study physical operators constructed, following [7, 14], by the bi-orthogonal sets considered along this paper and to check under which conditions a given Hamiltonian can be factorized. When this is possible, we will also consider the properties of its SUSY partner.

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