# Heat solitons and thermal transfer of information along thin wires

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**Abstract**

The aim of this paper is to consider soliton propagation of heat signals along a cylinder whose heat exchange with the environment is a non-linear function of the difference of temperatures of the cylinder and the environment and whose heat transfer along the system is described by the Maxwell-Cattaneo equation. To find the soliton solutions we use the auxiliary equation method. Our motivation is to obtain and compare the speed of propagation, the maximum rate of information transfer, and the energy necessary for the transfer of one bit of information for different solitons, by assuming that a localized soliton may carry a bit of information. It is shown that a given total power (energy/time) may be used either to send a few bits in a fast way, or many bits in a slower way. This may be controlled by choosing the initial condition imposed at one end of the wire.

**Keywords:** Heat waves. Thermal solitons. Maxwell-Cattaneo law. Radiative transfer; auxiliary equation method.

**AMS Subject Classifications:** 35C08 (Soliton solutions), 35C07 (Traveling wave solutions), 74J30 (Nonlinear waves), 35Q51 (Soliton-like equations), 35Q79 (PDEs in connection with classical thermodynamics and heat transfer).

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## 1 Introduction

The increasing possibilities of control of heat transfer has stimulated the research of thermal transfer and manipulation of information [1, 2, 3]. Assuming that one localized soliton may carry a bit of information, one may search for different kinds of solitons in heat transfer, to compare the propagation speed, the maximum rate of information transfer, and the energy required for the transmission of one bit for each kind of soliton or, given a kind of soliton characterized by a given width, how to choose this width (dependent on the initial conditions imposed on the system) in order to optimize the speed of transmission, the transmission rate, or the energy per bit.

Single-soliton solutions (those having the form of localized travelling pulses, as "sech"-type) would be especially interesting for transmitting heat energy without external loss nor internal dispersion, because both effects (nonlinear effects and dispersion) compensate each other. This allows to transport heat packets to long distance. These solutions could have interest in the so-called phononics, for computations based on heat pulses.

Heat rectifiers (diodes) and amplifiers (transistors) which are usually considered in steady state situations, should be also considered in the context of soliton heat signals to take advantage of these new possibilities in the context of soliton heat signals. We are aware that other kinds of solitons would be possible (such as “tanh”-type, kink and anti-kink solitons), but here we are interested to the sinlge-soliton solution, also called bright solitons (“sech”-type).

The propagation of second sound, or heat waves, in continuous media has been one of the sources of inspiration for generalized heat transport equations in the last fifty years [4, 5, 6, 7, 8, 9]. This has stimulated the analysis not only of generalized heat transport equations, as Maxwell-Cattaneo, Guyer-Krumhansl, double lagging or thermomass, but also of the foundations of non-equilibrium thermodynamics, in order to make such generalized transport equations compatible with enlarged formulations of the second law [10, 11, 12, 13, 14, 15, 16, 17, 18].

Much of the attention has been focused on small-amplitude waves described by linear equations. In this paper, instead, we consider nonlinear equations and we seek the conditions for having single-soliton solution by means of the Maxwell-Cattaneo law. Our aim is to find some interesting cases of high-amplitude thermal solitons as exact solutions [19, 20, 21], a topic which has not received much interest up to now, despite the wide interest arised by soliton propagation in many other fields [22, 23, 24, 25, 26, 27, 28, 29, 30, 31] and which may provide strategies for the efficient transmission of thermal signals along thin wires, without dispersion nor losses, a topic of interest in the emerging field of phononics [32], [33], [34].

In fact, solitons have been obtained in the context of heat transfer combining Fourier’s law with some nonlinear heat producing process, as for instance, exothermic chemical reactions, and phase transitions with latent heat [35]. Here, instead of using the Fouriers law, we consider the relaxational Maxwell-Cattaneo equation, and instead of chemical reactions or phase transitions, a nonlinear heat exchange between the system and the enviroment is assumed. This problem was considered in [36, 37]; we deal with it here with a different mathematical method (the so-called auxiliary equation method, explained in Section 3) and with a different physical motivation (namely, the propagation of a bit of information by means of thermal signals). Furthermore, we are aware of only two other papers [38, 39] having dealt with solitons in the context of heat transfer with memory, but in different context than the nonlinearity of the lateral heat transfer, namely, by introducing a nonlinearity in of the heat transfer equation along the system, or in a heat production term.

Among the different methods, in this paper we apply the auxiliary equation method [40], [41], [42] to find exact solutions of the nonlinear equations in terms of hyperbolic functions (in particular “sech”-type), because it allows us to find the exact degree of the nonlinearity in the mathematical model for appearance of the single-soliton solution. Further solutions can be found in terms of elliptic or hyperelliptic functions [43, 44, 45], which is the aim of a future paper. We are not meaning that all kinds of possible solitons may be obtained through this method, but it is very effective in exploring concrete specific solutions.

In our analysis, we consider heat propagation in thin wires, in such a way that the heat exchange between the wires and the environment is important. In particular, we consider that such heat exchange has nonlinear contributions in the difference of temperatures between the cylinder and the environment. In this way we are going beyond previous linear approximations to heat waves in nanowires [46, 47]. We have in mind nonlinear effects of radiative transfer with the surrounding vacuum (considered in Section 4).

The paper is organized as follows. In Section 2 we present the mathematical model, in Section 3 we explain the mathematical method used for searching single-soliton solution of the proposed equations; in Section 4 we present some nonlinear wave solutions in radiative heat exchange of a cylinder, in Section 5 we study the consequences of the soliton solution on the information transfer properties of thermal bits along the wire, and Section 6 is for conclusions.

## 2 The mathematical model

We consider heat propagation along a heat-conducting wire of radius $r$, composed of a material of mass density $\rho$ and specific heat per unit mass $c$. The energy balance equation in a cylindrical wire with lateral heat flux is:

$$\rho c \frac{\partial T}{\partial t} = -\frac{\mathrm{d}}{\mathrm{d}z} q - \frac{2}{r} q_t \tag{2.1}$$

with $q_t$ the transverse heat exchange between the cylinder and the environment through the surface of the cylinder.

We assume that the axial heat flux $q$ is described by the so-called Maxwell-Cattaneo equation [38, 39]

$$\tau \frac{\partial q}{\partial t} = -q - \lambda \frac{\mathrm{d}}{\mathrm{d}z} T, \tag{2.2}$$

with $\tau$ the relaxation time of the heat flux and $\lambda$ the thermal conductivity of the material.

By differentiating the first equation (2.1) with respect to the time and using equation (2.2), the following equation for the temperature is obtained

$$\tau \rho c \frac{\partial^2 T}{\partial t^2} = \lambda \frac{\mathrm{d}^2}{\mathrm{d}z^2} T - \rho c \frac{\partial T}{\partial t} - \frac{2}{r}\left(q_t + \tau \frac{\partial q_t}{\partial t}\right). \tag{2.3}$$

For a small relaxation time $\tau$, equation (2.2) becomes the Fourier's law, which combined with (2.1) yields [35]:

$$\rho c \frac{\partial T}{\partial t} = \lambda \frac{\mathrm{d}^2}{\mathrm{d}z^2} T - \frac{2}{r} q_t. \tag{2.4}$$

which is rather different than equation (2.3).

If we assume that the last term in equation (2.3) is a function of the temperature, namely

$$\frac{2}{r}\left(q_t + \tau \frac{\partial q_t}{\partial t}\right) \equiv g(T), \tag{2.5}$$

then (2.3) becomes:

$$\tau \rho c \frac{\partial^2 T}{\partial t^2} = \lambda \frac{\mathrm{d}^2}{\mathrm{d}z^2} T - \rho c \frac{\partial T}{\partial t} - g(T). \tag{2.6}$$

Note that expression (2.6) is still valid for a change of the reference temperature, i.e. $T \to T - T_0$, with $T_0$ the homogeneous temperature outside the cylinder, which will be taken as the reference temperature. When $g(T) = 0$, equation (2.6) yields a telegraphist equation for temperature evolution, which for high frequencies and short wavelengths leads to temperature waves propagating with speed $(\lambda/\rho c \tau)^{1/2}$.

Equation (2.6) in dimensionless form is

$$\frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 u}{\partial z_1^2} - \frac{\partial u}{\partial t_1} - \tilde{g}(u), \tag{2.7}$$

where we have used

$$t_1 = t/\tau, \quad z_1 = z\sqrt{\frac{\rho c}{\lambda \tau}}, \quad u = \frac{(T - T_0)}{T_0}, \quad \tilde{g}(u) = \frac{\tau}{\rho c T_0} g(T - T_0) \tag{2.8}$$

There is another equation which can be inferred from (2.6) under the hypotheses that $\lambda \to \infty$, $\tau \to \infty$ but $\lambda/\tau$ finite, which in dimensionless form is

$$\frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 u}{\partial z_1^2} - \tilde{g}(u). \tag{2.9}$$

In this paper we look for single-soliton solutions of equations (2.7) and (2.9) by means of the auxiliary equation method [40, 41, 42], considering some nonlinear expressions for the lateral heat flux. For these purposes, we use the similarity variable $\xi = kz_1 - \omega t_1$ and we introduce a parameter $a$ in order to include both equations (2.7) and (2.9)

$$(\omega^2 - k^2)\frac{\partial^2 u}{\partial \xi^2} - \omega a \frac{\partial u}{\partial \xi} + \tilde{g}(u) = 0. \tag{2.10}$$

Indeed, for $a = 1$ we recover equation (2.7) in the moving frame of reference and for $a = 0$ we recover equation (2.9). The relation between the dimensionless $k$ and $\omega$ and the corresponding dimensional quantities $k_d$ and $\omega_d$ are $k_d = (\rho c/(\lambda\tau))^{1/2}$ and $\omega_d = \omega/\tau$.

## 3 Auxiliary method for travelling waves

In this section we recall the main steps of the Auxiliary method [40], [41], [42], which allows to find some exact travelling wave solutions of the $1 + 1$ nonlinear equation ($t$ time and $z$ the 1D spatial coordinate along the axis of the wire):

$$E(z, t, u, u_z, u_t, ...) = 0. \tag{3.1}$$

The first step is to transform equation (3.1) in an ordinary nonlinear equation, $E(\xi, u, u_\xi, u_{\xi\xi}, ...) = 0$, by means of the transformation $\xi = kz - \omega t$, which is typical for searching for travelling wave solutions.

The second step is to choose for $u(\xi)$ a polynomial form

$$u(\xi) = \sum_{i=0}^{n} u_i y(\xi)^i, \tag{3.2}$$

where $u_i$ are constants to be determined and the functions $y(\xi)$ are solutions of the auxiliary equation, which in this paper is

$$y(\xi)'^2 = y(\xi)^2(1 - y(\xi)^2) \tag{3.3}$$

which has the solution $y(\xi) = \text{sech}(\xi)$, having the form of a propagating pulse.

The third step is to determine the coefficients $u_i$ in the expression (3.2). This is achieved after introduction of (3.2) into (2.10) taking into account of (3.3). The value of $n$ (the maximum value of the exponents of $y(\xi)$ in (3.2)) is determined by balancing the higher-order linear term with the higher nonlinear term of the equation (3.1).

Soliton is a nonlinear travelling wave, $U(\xi = kx - \omega t)$, solution of a non-linear evolution equation (partial differential equation), which at every moment of time is localized in a bounded domain of space, such that the size of the domain remains bounded in time.

# 4 Some travelling wave solutions for nonlinear radiative exchange

Here we consider lateral heat exchange of a cylinder in a dilute gas or in vacuum, in such a way that radiative exchange is dominating. In this case we assume that, according to Stefan-Boltzmann law, one has for $g_t$ as defined in (2.5)

$$g(T) = \frac{2}{r}\sigma_{SB}(T^4 - T_0^4) \tag{4.1}$$

with $\sigma_{SB}$ the Stefan-Boltzmann constant. Note that in writing (4.1) we are in fact assuming a Stefan-Boltzmann law with a relaxation term, in the same way as Fourier's law has been generalized in (2.2) by incorporating a relaxation term. Other situations could be considered assuming a different relaxation time for the relaxational generalization of the Fourier's law and Stefan-Boltzmann law, but here we will concentrate on the case of equal relaxation times.

Note that Stefan-Boltzmann law is valid for systems with characteristic size bigger than the average thermal wavelength, as given by the Wien's law, and which for $T_0 = 300$ K is of the order of $10^4$ nm. For smaller systems, near-field descriptions must be used [48, 49]. Here, we use Stefan-Boltzmann (valid for wires with diameter bigger than $10^4$ nm) and we will comment the approach to thinner wires in the final session.

In (4.1) we write $T = T_0 + \Delta T$, and expanding the fourth power we get for the rate of heat exchange a polynomial of fourth order in $\Delta T$. Thus, equation (2.6) becomes

$$\tau\rho c\frac{\partial^2 \Delta T}{\partial t^2} = \lambda\nabla^2\Delta T - \rho c\frac{\partial \Delta T}{\partial t} - \frac{2}{r}\sigma_{SB}\left(4T_0^3\Delta T + 6T_0^2(\Delta T)^2 + 4T_0(\Delta T)^3 + (\Delta T)^4\right) \tag{4.2}$$

Equation (4.2) can be truncated to lower degrees in $\Delta T$ if $|\Delta T| < T_0$ for any $(z, t)$. We will see that the appearance of "sech"-type soliton solutions requires the truncation of the polynomial to second degree in $\Delta T$.

In dimensionless form equation (4.2) becomes

$$\frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 u}{\partial z_1^2} - a\frac{\partial u}{\partial t_1} - b\left(4u + 6u^2 + 4u^3 + u^4\right) \tag{4.3}$$

with $b = \frac{2}{r}\frac{\tau\sigma_{SB}T_0^3}{\rho c}$.

In order to explore travelling wave solutions we must work in the moving frame of reference $\xi = kz - \omega t = k(z - vt)$, with the speed $v$ given by $v = \omega/k$. Then, equation (4.3) becomes:

$$(\omega^2 - k^2)\frac{\partial^2 u}{\partial \xi^2} - a\omega\frac{\partial u}{\partial \xi} + b\left(4u + 6u^2 + 4u^3 + u^4\right) = 0, \tag{4.4}$$

which is a particular case of (2.10) with $\tilde{g}(u) = b\left(4u + 6u^2 + 4u^3 + u^4\right)$.

Now we apply the Auxiliary equation method to obtain some travelling waves of equation (4.4). For our aims, we are looking for linear combinations of the functions $\text{sech}^i(\xi)$, which is solution of the auxiliary equation $y(\xi)'^2 = y(\xi)^2(1 - y(\xi)^2)$ with the solution $y(\xi) = \text{sech}(\xi)$. Thus, we first consider the full expansion of (4.1) (which is valid for any amplitude of the wave solution) and in case of negative answer we consider a truncated expansion of it, which will be valid for $|\Delta T| < T_0$ for any $(z, t)$.

Following the procedure explained in Section 3, we balance the highest nonlinear term $u^4$ with the highest linear term $\dfrac{\partial^2 u}{\partial \xi^2}$ keeping in mind (3.3), from which we find the value $n = 2/3$, which is not an integer. Thus, we consider the third degree $u^3$, namely a weaker nonlinearity in the equation. In this case we find $n = 1$ but the procedure does not lead to any single-soliton solution. The last attempt is by a still weaker nonlinearity, given by $u^2$, which leads to $n = 2$ and to some single-soliton solutions (see below) with the truncation $\tilde{g}(u) = b\left(4u + 6u^2\right)$.

Thus, higher nonlinearities, namely $u^3$ and $u^4$ do not lead to linear combination of $\text{sech}^i(\xi)$, but it does not necessarily mean that other kind of solitonic solutions do not exist in such cases. The existence of localized solitons for the truncated expansion means that the ratio of the temperature difference between the perturbed system and the environment $\Delta T$ should be less than the reference temperature $T_0$, in order that higher-order terms are negligible. Thus, we consider the truncated equation:

$$(\omega^2 - k^2)\frac{\partial^2 u}{\partial \xi^2} - a\omega\frac{\partial u}{\partial \xi} + b\left(4u + 6u^2\right) = 0, \tag{4.5}$$

The polynomial $u(\xi) = u_0 + u_1 y(\xi) + u_2 y(\xi)^2$ is substituted in (4.5) with the auxiliary equation $y(\xi)'^2 = y(\xi)^2(1 - y(\xi)^2)$. Setting all the coefficients of the polynomial in $y(\xi)$ and $y'(\xi)$ equal to zero, we find solutions the single-soliton solutions for both cases $a = 0$ and $a = 1$. Note that only the case $a = 0$ leads to travelling solitons (equation (4.6) and (4.9)) while the case $a = 1$ leads to the stationary solitons (equation (4.12) and (4.14)).

**Case** $a = 0$

By setting $a = 0$ in equation (4.5) we find the soliton solution corresponding to $u_2 = 1$, $u_1 = 0$ and $u_0 = -2/3$:

$$u(z_1, t_1) = \text{sech}^2(k(z_1 - vt_1)) - \frac{2}{3} \tag{4.6}$$

with $\omega^2 = k^2 + b$ and $v = \omega/k$.

Solution (4.6) in dimensional form is

$$\frac{\Delta T(z, t)}{T_0} = \text{sech}^2\left(k_d z - \omega_d t\right) - \frac{2}{3} \tag{4.7}$$

with $\omega_d^2 = \dfrac{2\sigma_{SB} T_0^3 + r\lambda k_d^2}{r\tau\rho c}$ and

$$v = \omega_d/k_d = \sqrt{\frac{\lambda}{\tau\rho c}}\sqrt{1 + \frac{2\sigma_{SB} T_0^3}{r\lambda k_d^2}}. \tag{4.8}$$

There is also the soliton solution corresponding to $u_2 = -1$, $u_1 = 0$ and $u_0 = 0$

$$u(z_1, t_1) = -\text{sech}^2(k(z_1 - vt)) \tag{4.9}$$

with $\omega^2 = k^2 - b$ and $v = \omega/k$.

The soliton (4.9) and the corresponding relations for $\omega$ and $v$ can be written in the dimensional form

$$\frac{\Delta T(z,t)}{T_0} = -\text{sech}^2\left(k_d z - \omega_d t\right) \tag{4.10}$$

with $\omega_d^2 = \dfrac{r\lambda k_d^2 - 2\sigma_{SB}T_0^3}{r\tau\rho c}$ and

$$v = \omega_d/k_d = \sqrt{\frac{\lambda}{\tau\rho c}}\,\sqrt{1 - \frac{2\sigma_{SB}T_0^3}{r\lambda k_d^2}}. \tag{4.11}$$

Note that in (4.7) the speed is higher than that of high-frequency linear waves $\sqrt{\frac{\lambda}{\tau\rho c}}$, whereas in (4.10) it is lower. This is of conceptual interest because one of the motivations to incorporate relaxation term in the transport equation (2.2) is to avoid infinite speed of propagation for thermal pulses. This is indeed achieved, because thermal pulses are related to the limit $\omega_d \to \infty$, in which case the limit value of $v$ in (4.7) and in (4.9) is finite and the same than in linear theory.

According to the restrictions for the truncated equation, only the soliton solution (4.7) should be admitted because $\Delta T < T_0$ for any $\xi$, whereas for solution (4.10) $|\Delta T| = T_0$ at $k_d z = \omega_d t$.

### Case $a = 1$

Setting $a = 1$ in equation (4.5) we find the stationary (non-propagating) and localized solution, corresponding to $u_2 = 1$, $u_1 = 0$ and $u_0 = -\dfrac{2}{3}$,

$$u(z_1, t_1) = -\frac{1}{3}(2 - 3\text{sech}^2(kz_1)) \tag{4.12}$$

with $\omega = 0$ and $k^2 = -b$, in dimensional form it becomes

$$\frac{\Delta T(z,t)}{T_0} = -\frac{1}{3}(2 - 3\text{sech}^2(k_d z)) \tag{4.13}$$

with $\omega = 0$ and $\lambda k_d^2 = -\frac{2}{r}\sigma_{SB}T_0^3$. The latter expression implies that $k_d$ is imaginary and that "sech" becomes "sec". In the latter case, since $sec(h\pi/2) = \infty$ ($h$ being a integer number) then the solution may have a physical meaning in a restricted interval of $\xi$.

Another stationary and localized solution, corresponding to $u_2 = -1$, $u_1 = u_0 = 0$, is

$$u(z_1, t_1) = -\text{sech}^2(kz_1) \tag{4.14}$$

with $\omega = 0$ and $k^2 = b$. In dimensional form (4.14) becomes

$$\frac{\Delta T(z,t)}{T_0} = -\text{sech}^2(k_d z) \tag{4.15}$$

with $\omega_d = 0$ and $\lambda k_d^2 = \frac{2}{r}\sigma_{SB}T_0^3$. The perturbation does not propagate for this particular form of the perturbation profile and this particular value of $k_d$. These stationary solutions correspond to an inhomogeneous nonequilibrium steady state.

The truncation considered for the labeled equation requires that $|\Delta T| < T_0$ for any $z$, which is satisfied only by the solution (4.13).

Note that $\omega = 0$ is required to finding the single-soliton solution, which sets zero the second term of equation (4.5), namely we are again in the case $a = 0$. From a practical perspective, maybe those non-propagating solitons could be useful as thermal memories, but this is an open topic for future discussion.

We are aware that these single-soliton solutions can be found directly instead of using the Auxiliary equation method. But, we have preferred to use it for practical use to search single soliton solutions for higher degree too.

# 5 Application to thermal transmission of bits

We have obtained two special cases of heat solitons along a heat conducting cylinder when the lateral heat transfer between the cylinder and the environment is not given by the linear Newton's law of heat transfer (namely, heat flow proportional to $T - T_0$), but by a nonlinear expression: the travelling single soliton solution and the stationary one.

The peculiar behaviour of solitons is to propagate in a medium without losing energy. Their appearance is a consequence of combined nonlinear and dispersive effects in the medium. Because of these peculiarities, we wonder for a possible propagation of thermal solitons along thin wires, for the transmission of bits of information, as in optical transmission. Among the soliton solutions found in the previous sections the most interesting solution is (4.7), corresponding to $a = 0$, namely to high thermal conductivity and high relaxation time. In this section we obtain four physical quantities related to the bit transmission: 1) speed; 2) temporal width; 3) maximum rate of information transfer; 4) energy per bit.

From a practical point of view, a strategy for sending localized heat pulses as a basis for sending information (a pulse corresponding to state 1 of a bit, and absence of pulse to state 0 of a bit, as it occurs in fiber optics) should be based on (4.7).

## 5.1 Transmission speed, rate of information transfer, energy per bit

Typical parameters used in optical trasmission are the velocity of the soliton, which in our case is given by (4.8), and the temporal width of a soliton, which is usually referred to as the Full Width at Half Maximum (FWHM), namely the "width of the soliton" shown in Fig. 1 (see the black arrow), which corresponds to the width of the shape of the soliton at half of its heigth. In the case of soliton (4.7), we evaluate the width of the function $U(z,t) = \mathrm{sech}^2(k_d z - \omega_d t) = \mathrm{sech}^2\left(\omega_d\left(\frac{z}{v} - t\right)\right)$, with the maximum value $U(0,0) = 1$ kept at the distance $z = \dfrac{\omega_d}{k_d}t$ or at the time $t = \dfrac{k_d}{\omega_d}z$. The half of the maximum value is $1/2$, and the equation $\mathrm{sech}^2(\omega_d t) = \dfrac{1}{2}$ can be solved in terms of $t$ in order to find the temporal width of the soliton $\tau_A$ (FWHM)

$$\tau_A = \frac{2}{\omega_d}\mathrm{sech}^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{2}{\omega_d}\ln(1+\sqrt{2}) = 2\ln(1+\sqrt{2})\sqrt{\frac{r\tau\rho c}{2\sigma_{SB}T_0^3 + r\lambda k_d^2}}. \tag{5.1}$$

where we have used the value $\omega_d^2 = \dfrac{2\sigma_{SB}T_0^3 + r\lambda k_d^2}{r\tau\rho c}$ below (4.7).

The Rate of Information Transfer (RIT) (i.e. bits/second) is

$$\text{RIT} = \frac{1}{\tau_A} = \frac{1}{2\ln(1+\sqrt{2})}\sqrt{\frac{2\sigma_{SB}T_0^3 + r\lambda k_d^2}{r\tau\rho c}}. \tag{5.2}$$

The initial energy per bit (energy carried by a soliton at $t = 0$) will be

$$\frac{Energy}{bit} = \int_{-\frac{1}{k_d}\ln(1+\sqrt{2})}^{\frac{1}{k_d}\ln(1+\sqrt{2})} \rho c \left(\Delta T\right)_{|t=0} \pi r^2 dz \tag{5.3}$$

where $2\dfrac{1}{k_d}\ln(1+\sqrt{2})$ is the spatial width of the soliton at $t = 0$, which can be evaluated as $\tau_A$ in (5.1) setting $t = 0$ instead of setting $z = 0$. The amount $\pi r^2 \dfrac{2}{k_d}\ln(1+\sqrt{2})$ is the characteristic volume occupied by a soliton. Recall that $\rho c$ is the specific heat per unit volume. Then, in view of (5.3) we have

$$\begin{aligned}\frac{Energy}{bit} &= \int_{-\frac{1}{k_d}\ln(1+\sqrt{2})}^{\frac{1}{k_d}\ln(1+\sqrt{2})} \rho c T_0 \left(\operatorname{sech}^2(k_d z) - \frac{2}{3}\right)\pi r^2 dt = \\ &= \frac{\rho c T_0 \pi r^2}{k_d}\left[\sqrt{2} - \frac{4}{3}\text{Ln}(1+\sqrt{2})\right] = 0.239\frac{\rho c T_0 \pi r^2}{k_d}\end{aligned} \tag{5.4}$$

These quantities (speed, full width at half maximum, rate of information, and energy per bit) depend on $k_d$. In particular for higher $k_d$ the soliton has slower velocity, approaching the asymptotic value $\sqrt{\lambda/(\tau\rho c)}$ for high $k_d$ (narrow solitons). In contrast, the rate of information transfer grows because the temporal width of the soliton decreases. This means that higher $k_d$ is preferable in the case of high amount of bits to send. Furthermore, the energy per bit is lower for higher $k_d$. Instead, smaller $k_d$ is preferable in the case of a few solitons, because of the higher speed of the soliton.

Furthermore, it seen that small $r$ is preferible; indeed, to increase the speed (4.8) of soliton (4.7), $r$ must be increased. But according to (5.3) an increase in $r$ implies an increase in the energy per bit; but the increase in the energy per bit goes as $r^2$, whereas the increase of the speed goes only as the square root of $r$, at best, according to (4.8).

The same arguments may be applied to the soliton (4.10), but this soliton does not satisfy the condition that $|\Delta T| < T_0$ for any $(z, t)$. Anyway, it may be illustrative to consider how the propagation properties of this soliton would differ from the ones of the previous soliton, in order to grasp a qualitative understanding of non-linear propagation properties.

The temporal width of soliton (4.10) is given by:

$$\tau_A = \frac{2}{\omega_d}\operatorname{sech}^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{2}{\omega_d}\ln(1+\sqrt{2}) = 2\ln(1+\sqrt{2})\sqrt{\frac{r\tau\rho c}{r\lambda k_d^2 - 2\sigma_{SB}T_0^3}}. \tag{5.5}$$

where we have used the value $\omega_d^2 = \dfrac{r\lambda k_d^2 - 2\sigma_{SB}T_0^3}{r\tau\rho c}$ below (4.10) and the Rate of Information Transfer (RIT) (i.e. bits/second) is

$$\text{RIT} = \frac{1}{\tau_A} = \frac{1}{2\ln(1+\sqrt{2})}\sqrt{\frac{r\lambda k_d^2 - 2\sigma_{SB}T_0^3}{r\tau\rho c}}. \tag{5.6}$$

Instead, the energy per bit for the soliton (4.10) is:

$$\frac{Energy}{bit} = \int_{-\frac{1}{k_d}\ln(1+\sqrt{2})}^{\frac{1}{k_d}\ln(1+\sqrt{2})} \rho c \left(|\Delta T|\right)_{|t=0} \pi r^2 dz = \int_{-\frac{1}{k_d}\ln(1+\sqrt{2})}^{\frac{1}{k_d}\ln(1+\sqrt{2})} \rho c T_0 \left(\mathrm{sech}^2\left(k_d z\right)\right) \pi r^2 dz = \rho c T_0 \pi r^2 \frac{\sqrt{2}}{k_d} \tag{5.7}$$

These values of $\tau_A$, RIT and energy per bit are different to those of the first soliton (4.7), namely (5.3), (5.2) and (5.1). Thus, for both solitons a given total power (energy/time) may be used either to send a few bits in a fast way, or many bits in a slower way.

Furthermore, the shape of the two solitons (4.7) and (4.10) does not depend on the material and their shape does not change in time but it is kept constant. Instead, the dispersion relation and hence the speed of the solitons depend on the material through the relaxation time $\tau$, the specific heat $c$, the thermal conductivity $\lambda$ and the radius $r$ of the cylinder.

## 5.2 Influence of initial conditions

The propagation of the solitons along the wire depends on the initial conditions of the thermal pulse. The best choice would be sending exactly soliton (4.7) at $t = 0$, as sketched for the sake of illustration in Figure 2. From a practical point of view the corresponding temperature profile could be achieved by surrounding the initial region of the wire with an helicoidal set of lasers radially directed towards the center of the wire. At the initial moment of transmission $t = 0$, the region from $z = 0$ to $z = 1/k_d$ could be submitted to a set of simultaneous laser pulses giving to each point the necessary energy to get the required temperature profile (each laser at a given $z$ position should supply the required amount of energy in order to get the temperature needed at the corresponding point $z$, as given by (4.7) with $t = 0$).

In the case this were too difficult to achieve in the practice, we remind that a similar problem appears in fiber optics where optical laser usually launches Gaussian-shape pulses in the fiber. Studies in fiber optics show that the propagation of a Gaussian pulse is affected by group velocity dispersion and the frequency chirping due to nonlinearity of the medium [22]. The former (the group velocity dispersion) broadens the pulse (because each Fourier component of the pulse travels at different velocity) whereas the latter (changing the domain of the Fourier component of the pulse) reduces the width of the pulse. When both effects balance each other the soliton appears. Since nonlinearity is stronger for higher amplitude of the solitons, the propagation of soliton occurs if the initial pulse (even though Gaussian or similar) has an amplitude higher than the theoretical one. Thus, both the amplitude and the width of the Gaussian pulse must be controlled, the latter one being $1/k_d$.

In our case, it is expected that a similar situation should occur in the propagation of nonlinear thermal waves along thin wire, although nonlinearity and dispersion here work in a different way than in the fiber optics. The conditions on the parameters for the propagation of thermal soliton should be found from experiments, but we can assert that, as it occurs in fiber optics, if the amplitude of the soliton is smaller than the theoretical one then pulse broadens and is dispersed; otherwise, pulse degenerates to the soliton and disperses the exceeding energy. Thus, the initial condition for transmission of thermal soliton would be that the peak of the pulse should be $\frac{1}{3}T_0$ higher than $T_0$ and the ground of the pulse should be $-\frac{2}{3}T_0$ smaller than $T_0$.

### 5.3 Numerical estimation of the values

Let's consider a concrete example to make these results clearer. Let's choose a Si thin wire with two different values for $k_d$ in order to focus the different behaviour for different $k_d$. We choose the following parameters: diameter of the wire $d = 15000$ nm, temperature $T_0 = 300$ K, and $\sigma_{SB} = 5.67 \times 10^8 \mathrm{W/(m^2K^4)}$, $\rho = 2330 \mathrm{kg/m^3}$, $\lambda = 148 \mathrm{W/(m\ K)}$, $c = 700 \mathrm{J/(kg\,K)}$ and $\tau = 50\mathrm{ps} = 5 \cdot 10^{-11}\mathrm{s}$. Thus, thermal soliton (4.7) becomes

$$\Delta T(z,t) = T_0 \left( \mathrm{sech}^2\left(k_d z - \omega_d t\right) - \frac{2}{3} \right) = 300 \left( \mathrm{sech}^2\left(k_d z - \omega_d t\right) - \frac{2}{3} \right) \tag{5.8}$$

with $\omega_d^2 = \dfrac{2\sigma_{SB}T_0^3 + r\lambda k_d^2}{r\tau\rho c} = \dfrac{2 \cdot 5.67 \times 10^8\,\mathrm{W\ m^2\,K^{-4}} \cdot 300^3\,\mathrm{K^3} + 15000/2 \cdot 148\,\mathrm{W(m\ \ K)^{-1}} \cdot k_d^2}{15000/2 \cdot 5 \cdot 10^{-11}\,\mathrm{s} \cdot 2330\,\mathrm{kg\,m^{-3}} \cdot 700\,\mathrm{J(kg\,K)^{-1}}} =$ $5.00601 \cdot 10^{16} + 1.81484 \cdot 10^6 k_d^2$ or $\omega_d = \sqrt{5.00601 \cdot 10^{16} + 1.81484 \cdot 10^6 k_d^2}\,\mathrm{s^{-1}}$ and

$$v = \omega_d / k_d = \sqrt{\frac{5.00601 \cdot 10^{16}}{k_d^2} + 1.81484 \cdot 10^6} \tag{5.9}$$

The other quantities are the temporal width of the soliton (FWHM):

$$\tau_A = \frac{2\ln(1+\sqrt{2})}{\sqrt{5.00601 \cdot 10^{16} + 1.81484 \cdot 10^6 k_d^2}}. \tag{5.10}$$

the Rate of Information Transfer (RIT) (i.e. bits/second)

$$\mathrm{RIT} = \frac{1}{\tau_A} = \frac{\sqrt{5.00601 \cdot 10^{16} + 1.81484 \cdot 10^6 k_d^2}}{2\ln(1+\sqrt{2})} \tag{5.11}$$

and the energy per bit (energy carried by a soliton)

$$\frac{Energy}{bit} = 2330 \mathrm{kg/m^3} \cdot 700 \mathrm{J/(kg\,K)} \cdot 300\mathrm{K} \cdot \pi (15000/2)^2 \frac{\sqrt{2}}{k_d} = \frac{1.22282 \cdot 10^{17}}{k_d} \tag{5.12}$$

Now, we choose two different values of $k_d$ in order to understand better how these amounts change by them. For instace, we can choose $k_{d1} = 10^7\,\mathrm{m^{-1}}$ and $k_{d2} = 10^8\,\mathrm{m^{-1}}$ for which we find:

$$\omega_{d1} = 1.34734 \cdot 10^{10}\,\mathrm{s^{-1}} \qquad \omega_{d2} = 1.34716 \cdot 10^{11}\,\mathrm{s^{-1}} \tag{5.13}$$

$$v_1 = 1347.34\mathrm{m/s} \qquad v_2 = 1347.16\mathrm{m/s} \tag{5.14}$$

$$\tau_{A1} = 1.30831 \cdot 10^{-10} \qquad \tau_{A2} = 1.30849 \cdot 10^{-11} \tag{5.15}$$

$$\mathrm{RIT}_1 = 7.64344 \cdot 10^9 \qquad \mathrm{RIT}_2 = 7.64239 \cdot 10^{10} \tag{5.16}$$

$$\left(\frac{Energy}{bit}\right)_1 = 1.22282 \cdot 10^{10} \qquad \left(\frac{Energy}{bit}\right)_2 = 1.22282 \cdot 10^9 \tag{5.17}$$

From these results we can say that higher wavenumbers (namely, narrower solitons) means higher frequency and rate of information transfer (RIT) but smaller velocity and temporal width of the soliton and lower energy per bit transmitted.

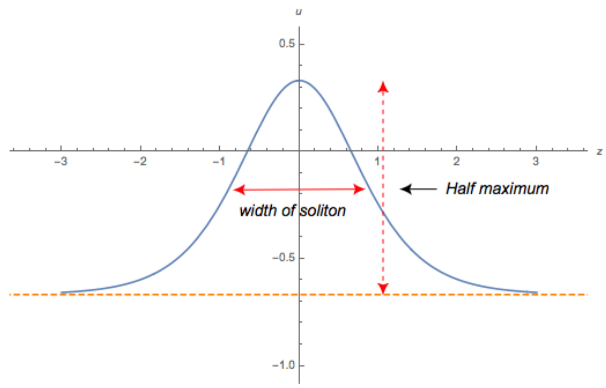


Figure 1: The Full Width at Half Maximum (FWHM) for the temporal width of the soliton, as usually defined in optics communication. For this definition, one needs to consider the half-height of the soliton, which corresponds to black arrow in the figure. At this height the soliton has a precise width, which is the called "width of the soliton" or FWHM.

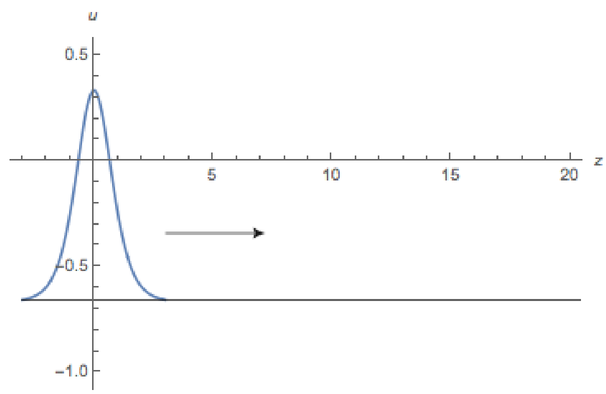


Figure 2: Sketch of the propagation of thermal soliton along the wire. In the figure the soliton shape is in the origin of the axis ($z = 0$) for $t = 0$. Since the shape of the soliton does not change along the wire during the propagation, it is expected to be the same everywhere along the wire. The intitial profile could be imposed on the wire by means of a suitable set of radial laser pulses.

# 6 Conclusions

We have studied two kinds of of thermal solitons appearing in hyperbolic transport of heat along a wire in the presence of nonlinear radiative exchange. A similar problem was considered in [36, 37] in 1984–1987. The new aspects in our approach are: a) the mathematical model used and the exact soliton solutions are found by means of the so-called auxiliary equation method; b) the concrete physical motivation, which in our case is to obtain the physical quantities related to the transport of a bit of information by means of thermal signals (a topic which has become of current interest because of the growth of attention on phononics since 2005).

The assumption that the radiative exchange is described by the Stefan-Boltzmann law is restricted to systems bigger than the average wavelength of thermal radiation as described by Wien's law. For smaller systems, more complicated models of near-field radiation must be taken into account [48, 49]. Such models imply a higher value of the radiative power. For systems at 300 K, the average wavelength of thermal radiation is of the order of $10^{-5}$ m ($10^{4}$

nm). Thus, for wires with a diameter smaller than $10^4$ nm, the Stefan-Boltzmann law is not accurate.

This is not a big drawback for our analysis, because in it we have considered a radiation power described as $A(T - T_0) + B(T - T_0)^2$, with $A$ and $B$ numerical coefficients, and with emphasis on the role of the quadratic term on the possibility of thermal solitons. In the near-field radiation, coefficients $A$ and $B$ are different (higher) than in the black-body radiation, but the mathematical aspect of our model remains the same from a qualitative point of view.

In this paper we have found 4 soliton solutions: two travelling solitons (4.7) and (4.10) and two stationary (non-propagating) solitons (4.13) and (4.15). Among them, two of them are not admitted because do not satisfy the condition that $\Delta T/T_0 < 1$ required in the truncation of the nonlinear term in the labeled equation. In Section 5 we have considered the travelling soliton (4.7) and its application in a transmission of thermal bit, giving the temporal width FWHM $\tau_A$, the rate of transfer information and the energy per bit. We have estimated the corresponding numerical values to the propagation of thermal soliton in silicon thin wire. The stationary solution is not suitable for bit transmission but it may be useful as thermal storage of a bit of infomation, similarly to the stationary solitons predicted in proteins by the Davydov model [26].

The exploration of some other kinds of solitons in heat transport may have an interest in connection with information processing. Of course, heat solitons, up to now, are only relatively speculative proposal for information transfer, but they are worth of analysis from the information perspective, which brings a new stimulus for their study.

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