Path Integral approach via Laplace's method of integration for nonstationary response of nonlinear systems

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Abstract

In this paper the nonstationary response of a class of nonlinear systems subject to broad-band stochastic excitations is examined. A version of the Path Integral (PI) approach is developed for determining the evolution of the response Probability Density Function (PDF). Specifically, the PI approach, utilized for evaluating the response PDF in short time steps based on the Chapman-Kolmogorov equation, is here employed in conjunction with the Laplace's method of integration. In this manner, an approximate analytical solution of the integral involved in this equation is obtained, thus circumventing the repetitive integrations generally required in the conventional numerical implementation of the procedure. Further, the method is extended to nonlinear oscillators, approximately modeling the amplitude of the system response as a one-dimensional Markovian process. Various nonlinear systems are considered in the numerical applications, including Duffing and Van der Pol oscillators. Appropriate comparisons with Monte Carlo simulation data are presented, demonstrating the efficiency and accuracy of the proposed approach.

1 Introduction

In the last decades several research efforts have been focused on the development of novel and efficient methods for the analysis of randomly excited mechanical and structural systems. A number of different approaches have been developed, which in many instances relate to the case of linear and nonlinear systems excited by Gaussian white noise. A broad class of structural systems, however, is subject to excitations such as ocean waves, winds, and seismic motions, whose realistic modeling involves the representation by nonstationary stochastic processes. Undoubtedly, approaches based on Monte Carlo simulations are among the most versatile ones to address these cases [1], especially when multi-degree-of-freedom (MDOF) systems are considered. Indicatively, other methods for the nonstationary response determination may resort to statistical linearization schemes [2], complex spectral moments for cross-correlations and cross power spectral densities representation [3-4], Galerkin approach coupled with stochastic averaging method [5-8], harmonic wavelets based statistical linearization [9], and Wiener Path Integral solution [10, 11].

In this context, the so-called Path Integral (PI) approach represents an alternative method which proved particularly accurate for determining the response probability density function (PDF) and statistics of nonlinear low-dimensional systems subject to stationary or nonstationary random excitations. In essence, the PI method could be considered as a discretized version of the Chapman-Kolmogorov (C-K) equation, valid for Markov processes [12, 13], which is applied within a step-by-step procedure to propagate the PDF in short time steps. In this regard, the response PDF at a certain time instant can be computed simply evaluating an integral whose kernel involves the PDF in a previous time instant, and the Conditional PDF (CPDF) of the system. Further, if short time steps are used, then Gaussian distribution of the CPDF can be assumed (short time Gaussian approximation), even for nonlinear systems. Since its origin [14-16], where the numerical implementation of the method has been firstly addressed, the PI approach has been extensively applied to a number of different problems [17-21], also involving nonstationary response of systems excited by a time-modulated normal white noise [22]. Further, the method has been also extended to non-normal excitation cases, such as Poisson white noise [23], combined normal and Poisson white noise [24], Parametric Poissonian excitation [25], and Lèvy white noise [26]. Recently the PI approach has been used for determining the response PDFs of nonlinear systems under stochastic excitations characterized by non-separable evolutionary power spectrum, relying on a combination of statistical linearization and of stochastic averaging [27]. Further, other current contributions have been focused on different schemes for an efficient implementation of the approach for MDOF systems [28-30]. Notably, although several progresses have been made in this field, beneficial features of the PI approach, such as accuracy even for low probability levels and applicability to practically any form of nonlinearity, warrant further investigations in alternative solution schemes based on this method.

In this regard, in this paper a version of the PI approach is developed based on the so-called Laplace's method of integration [31, 32], a mathematical tool generally used for an approximate evaluation of integrals whose kernels comprise exponential functions. Specifically, the aforementioned method, recently used for reliability analyses of systems subject to normal and Poisson white noise [33, 34], is here exploited to perform fast analytical approximate evaluation of the repeated integrations involved in the discretized numerical implementation of the C-K equation. In this manner, the evolution of the response PDF of one-dimensional systems can be easily obtained, even for nonstationary excitations. Further, the approach is also extended to nonlinear oscillators under broad-band nonstationary stochastic excitation with separable evolutionary power spectrum, relying on the Markovian approximation of the response amplitude process. Several numerical examples are considered, and juxtaposition of the proposed approach-based results with pertinent MCS data demonstrates the accuracy of the method.

2 Preliminary remarks on Laplace's method of integration

Consider an integral of the form

$$I = \int_a^b e^{-\lambda g(y)} h(y)\, dy \tag{1}$$

Assuming that the parameter $\lambda > 0$ is large, $g(y)$ and $h(y)$ are both smooth real-valued functions, and $g(y)$ has a local minimum at y^* in the interval $[a,b]$, it can be argued that the main contribution to the integral I in Eq. (1) is essentially entirely originating from the neighborhood around y^*. In this regard, let the functions $g(y)$ and $h(y)$ be expanded in Taylor series around y^* up to the second order, as

$$g(y) \cong g(y^*) + g^I(y^*)(y-y^*) + g^{II}(y^*)\frac{(y-y^*)^2}{2} \tag{2}$$

and

$$h(y) \cong h(y^*) + h^I(y^*)(y-y^*) + h^{II}(y^*)\frac{(y-y^*)^2}{2} \tag{3}$$

where the apexes in roman letters in Eq. (2) and (3) stand for the order of the derivatives with respect to the integration variable in Eq. (1).

Since y^* is a local minimum, then $g^I(y^*) = 0$. Thus, substituting in Eq. (1) yields [31]

$$\begin{aligned} I = \int_a^b e^{-\lambda g(y)} h(y)\, dy \cong e^{-\lambda g(y^*)} h(y^*) \int_{-\infty}^{\infty} e^{-\lambda g^{II}(y^*)\frac{(y-y^*)^2}{2}} dy + e^{-\lambda g(y^*)} h^I(y^*) \int_{-\infty}^{\infty} (y-y^*) e^{-\lambda g^{II}(y^*)\frac{(y-y^*)^2}{2}} dy \\ + e^{-\lambda g(y^*)} h^{II}(y^*) \int_{-\infty}^{\infty} \frac{(y-y^*)^2}{2} e^{-\lambda g^{II}(y^*)\frac{(y-y^*)^2}{2}} dy \end{aligned} \tag{4}$$

Recalling that the integrals at the right-hand side of Eq. (4) are Gaussian integrals, Eq. (4) can be approximately evaluated as

$$I \cong e^{-\lambda g(y^*)} \sqrt{\frac{2\pi}{\lambda g^{II}(y^*)}} \left\{ h(y^*) + \frac{1}{2\lambda g^{II}(y^*)} h^{II}(y^*) \right\} \tag{5}$$

Alternatively, incorporating $h(y)$ in $g(y)$ as

$$\tilde{g}(y) = g(y) - \frac{1}{\lambda}\ln\left[h(y)\right] \tag{6}$$

the following second order approximation can be obtained

$$I = \int_a^b e^{-\lambda g(y)} h(y)\,dy = \int_a^b e^{-\lambda \tilde{g}(y)} dy \cong e^{-\lambda g\left(\tilde{y}^*\right)} \sqrt{\frac{2\pi}{\lambda \tilde{g}^{II}\left(\tilde{y}^*\right)}} \left\{1 + \frac{1}{2\lambda \tilde{g}^{II}\left(\tilde{y}^*\right)}\right\} \tag{7}$$

where $\tilde{y}^*$ now maximizes $\tilde{g}(y)$.

The aforementioned formulation is generally referred to as Laplace's method of integration [31, 32], and it is valid also for infinite and semi-infinite intervals of integration in Eq. (1). Note that Eq. (5) is a rather well-known results, for instance exploited for the approximate evaluation of the Gamma function through the so-called Stirling's formula.

Clearly, accuracy of the approximation can be improved by further expansion in Eq. (2) and (3). For instance, if a fourth order expansion of $g(y)$ is considered, then Eq. (5) becomes [31]

$$I \cong e^{-\lambda g\left(y^*\right)} \sqrt{\frac{2\pi}{\lambda g^{II}\left(y^*\right)}} \left\{ h\left(y^*\right) + \frac{1}{\lambda}\left[\frac{h^{II}\left(y^*\right)}{2g^{II}\left(y^*\right)} - \frac{h\left(y^*\right)}{8}\frac{g^{IV}\left(y^*\right)}{g^{II}\left(y^*\right)^2} - \frac{h^{I}\left(y^*\right)}{2}\frac{g^{III}\left(y^*\right)}{g^{II}\left(y^*\right)^2} + \frac{5h\left(y^*\right)}{24}\frac{g^{III}\left(y^*\right)^2}{g^{II}\left(y^*\right)^3} \right]\right\} \tag{8}$$

3 Path Integral based on Laplace's method

Consider a nonlinear system whose motion is governed by the differential equation

$$\begin{cases} \dot{X}(t) = f(X,t) + V(t) \\ X(0) = X_0 \end{cases} \tag{9}$$

where a dot over a variable denotes differentiation with respect to time *t*, $f(X,t)$ is a nonlinear function of the response process $X(t)$, and X_0 is the initial condition which may be either deterministic or a random variable with assigned PDF. Further, $V(t)$ denotes a zero-mean modulated white noise, that is

$$V(t) = \xi(t) W(t) \tag{10}$$

where $\xi(t)$ is a deterministic modulating function and $W(t)$ is a zero mean stationary Gaussian white noise process with constant (two-sided) power spectral density equal to S_0, so that the time dependent spectral density of $V(t)$ is $S_V(t) = \xi^2(t) S_0$.

Based on Eq. (9), $X(t)$ is a one-dimensional Markov process, which satisfies the Chapman-Kolmogorov (CK) equation

$$p_X(x,t+\tau)=\int_{-\infty}^{\infty} p_X(x,t+\tau\,|\,\bar{x},t)\,p_X(\bar{x},t)\,d\bar{x} \tag{11}$$

where $p_X(x,t+\tau\,|\,\bar{x},t)$ is the so-called Conditional PDF (CPDF) of the response process $X(t)$. Note that hereinafter stochastic processes will be denoted with capital letters, and pertinent domains with the corresponding lower-case letters.

Equation (11) represents the basis of the PI approach. In this regard, based on Eq. (11), to evaluate the PDF $p_X(x,t+\tau)$ of the response process $X(t)$ at the time instant $(t+\tau)$, it suffices to determine the CPDF in $(t+\tau)$ for an assigned (deterministic) initial condition $\bar{x}$ in (t), once the PDF of $X(t)$ in a previous time instant (t) is already known. Notably, this equation is valid for any value of τ, and the aforementioned CPDF can be obtained evaluating the unconditional PDF $p_{\bar{X}}(x,\tau)$ in τ of the following system [23, 24]

$$\begin{cases} \dot{\bar{X}}(\rho)=f(\bar{X},\rho)+V(t+\rho) \\ \bar{X}(0)=\bar{x} \end{cases};\ 0\le\rho\le\tau \tag{12}$$

taking into account that

$$p_X(x,t+\tau\,|\,\bar{x},t)=p_{\bar{X}}(x,\tau) \tag{13}$$

However, if τ is sufficiently small, it can be approximately assumed that the CPDF follows a Gaussian distribution of the form

$$p_X(x,t+\tau\,|\,\bar{x},t)=\frac{1}{\sqrt{4\pi^2 S_0\,\xi^2(t)\,\tau}}\exp\left[-\frac{\left(x-\bar{x}-f(\bar{x},t)\,\tau\right)^2}{4\pi S_0\,\xi^2(t)\,\tau}\right] \tag{14}$$

which constitutes the so-called short time Gaussian approximation [19].

Further, considering the time interval $\left[0,t_f\right]$, where t_f is the final time instant, discretized so that the generic time instant is $t_k=k\Delta t$ with $k=0,\ldots,N$ and $\Delta t=t_f/N$ is a short time step, then Eq. (11) can be rewritten as

$$p_X(x,t_k+\Delta t)=\int_{-\infty}^{\infty} p_X(x,t_k+\Delta t\,|\,\bar{x},t_k)\,p_X(\bar{x},t_k)\,d\bar{x} \tag{15}$$

assuming $\tau=\Delta t$. It can be readily seen from this equation that a step-by-step application of Eq. (15) yields the entire evolution of the response PDF. Clearly, as shown in Eq. (15), the PI method requires a repetitive numerical integration in the $\bar{x}$ domain for each time step, which often constitutes the highest contribution to the computational cost of the method itself, especially for higher dimensional systems. Fast approximate evaluation of the integral could however be obtained using the previously introduced Laplace's method of integration. Specifically, since short time step Δt are commonly used (often chosen smaller than $10^{-2} s$), and generally both the functions $p_X\left(x,t_k+\Delta t \mid \bar{x},t_k\right)$ and $p_X\left(\bar{x},t_k\right)$ are smooth, an approximate solution of the discretized version of the CK equation Eq. (15) can be obtained based on Eq. (7). In this regard, taking into account Eq. (14), Eq. (15) can be rewritten as

$$p_X\left(x,t_k+\Delta t\right)=\frac{1}{\sqrt{4\pi^2 S_0\,\xi^2\left(t_k\right)\Delta t}}\int_{-\infty}^{\infty} p_X\left(\bar{x},t_k\right)e^{-\tilde{\lambda} g_k\left(x,\bar{x}\right)}\,d\,\bar{x} \tag{16}$$

which is now given in a form similar to Eq. (1), with

$$g_k\left(x,\bar{x}\right)=\frac{\left[x-\left(\bar{x}+f\left(\bar{x},t_k\right)\Delta t\right)\right]^2}{2\pi S_0\xi^2\left(t_k\right)} \tag{17}$$

and $\tilde{\lambda}=1/\left(2\Delta t\right)$. Note that, since Δt is small, then $\tilde{\lambda}\gg 0$.

Let $\bar{x}_k^*$ be the point such that

$$\left.\frac{\partial g_k\left(x,\bar{x}\right)}{\partial \bar{x}}\right|_{\bar{x}=\bar{x}_k^*}=g_k^I\left(x,\bar{x}_k^*\right)=0 \tag{18.a}$$

and

$$\left.\frac{\partial^2 g_k\left(x,\bar{x}\right)}{\partial \bar{x}^2}\right|_{\bar{x}=\bar{x}_k^*}=g_k^{II}\left(x,\bar{x}_k^*\right)>0 \tag{18.b}$$

Then, taking into account Eqs. (5) and (16), the approximate solution of the CK equation Eq. (15) can be given as

$$p_X\left(x,t_k+\Delta t\right)=\frac{e^{-\tilde{\lambda} g_k\left(x,\bar{x}_k^*\right)}}{\sqrt{\pi S_0\,\xi^2\left(t_k\right)}}\frac{1}{\sqrt{g_k^{II}\left(x,\bar{x}_k^*\right)}}\left[p_X\left(\bar{x}_k^*,t_k\right)+\frac{1}{2\tilde{\lambda} g_k^{II}\left(x,\bar{x}_k^*\right)}p_X^{II}\left(\bar{x}_k^*,t_k\right)\right] \tag{19}$$

where $p_X\left(\bar{x}_k^*,t_k\right)$ and $p_X^{II}\left(\bar{x}_k^*,t_k\right)$ are, respectively, the value of the response PDF and the value of the second order derivative (with respect to $\bar{x}$) of the response PDF evaluated in $\bar{x}_k^*$ at the time instant t_k.

The significance of Eq. (19) relates to the fact that the response PDF of the nonlinear system in Eq. (9) can be readily computed via a step-by-step application of this approximate analytical expression.

Note that, in this manner the potentially computationally demanding repetitive numerical integration involved in Eq. (15) are avoided. Further, it is noted that $g_k^{II}\left(x,\bar{x}_k^*\right)$ and $g_k^{II}\left(x,\bar{x}_k^*\right)$, defined in Eq. (18), are deterministic functions which do not depend on the PDF of the response process. Thus, they can be evaluated once beforehand, since they do not change during the step-by-step application of Eq. (19), hence further reducing the computational effort.

Finally, it is worth mentioning that Eq. (19) is based on a second order Taylor series expansion, which could generally lead to sufficiently accurate results. However, if a more accurate approximation is required (for instance in case of nonlinear systems with high degree on nonlinearity), a fourth order expansion as in Eq. (8) can be adopted, leading to

$$p_X\left(x,t_k+\Delta t\right)=\frac{e^{-\tilde{\lambda}g_k\left(x,\bar{x}_k^*\right)}}{\sqrt{\pi S_0\,\xi^2\left(t_k\right)}}\frac{1}{\sqrt{g_k^{II}\left(x,\bar{x}_k^*\right)}}\left\{p_X\left(\bar{x}_k^*,t_k\right)+\frac{1}{\tilde{\lambda}}\left[\frac{p_X^{II}\left(\bar{x}_k^*,t_k\right)}{2g_k^{II}\left(x,\bar{x}_k^*\right)}-\frac{p_X\left(\bar{x}_k^*,t_k\right)}{8}\frac{g_k^{IV}\left(x,\bar{x}_k^*\right)}{g_k^{II}\left(x,\bar{x}_k^*\right)^2}\right.\right.$$
$$\left.\left.-\frac{p_X^{I}\left(\bar{x}_k^*,t_k\right)}{2}\frac{g_k^{III}\left(x,\bar{x}_k^*\right)}{g_k^{II}\left(x,\bar{x}_k^*\right)^2}+\frac{5\,p_X\left(\bar{x}_k^*,t_k\right)}{24}\frac{g_k^{III}\left(x,\bar{x}_k^*\right)^2}{g_k^{II}\left(x,\bar{x}_k^*\right)^3}\right]\right\} \tag{20}$$

3.1 Analysis for the response amplitude process

Equation (9) and the following approximate PI approach in Eqs. (19) and (20) refer to a generic first-order nonlinear system. Notably, analogous procedure can be derived focusing on the response amplitude process of a nonlinear single-degree-of-freedom system. To this aim, let the equation of motion of the system be given as

$$\ddot{X}+\beta_0\dot{X}+z\left(t,X,\dot{X}\right)=\tilde{V}\left(t\right) \tag{21}$$

where $z\left(t,X,\dot{X}\right)$ is an arbitrarily chosen restoring force, β_0 is a linear damping coefficient, and $\tilde{V}\left(t\right)$ is a Gaussian zero-mean nonstationary stochastic process possessing a power spectrum $S_{\tilde{V}}\left(\omega,t\right)$. In the ensuing analysis, it is assumed that $\tilde{V}\left(t\right)$ is a separable random process, thus it can be recast in the form

$$\tilde{V}\left(t\right)=\xi\left(t\right)\tilde{W}\left(t\right) \tag{22}$$

and the corresponding evolutionary power spectrum is

$$S_{\tilde{V}}\left(\omega,t\right)=\left|\xi\left(t\right)\right|^2 S_{\tilde{W}}\left(\omega\right) \tag{23}$$

where $\tilde{W}\left(t\right)$ is a stationary random process whose power spectrum is $S_{\tilde{W}}\left(\omega\right)$.

If the system is lightly damped, the response process $X\left(t\right)$ exhibits a pseudo-harmonic behavior, which can be described by

$$X(t) = A(t)\cos\left[\omega(A) + \theta(t)\right] \tag{24}$$

and

$$\dot{X}(t) = -A(t)\omega(A)\sin\left[\omega(A) + \theta(t)\right] \tag{25}$$

where the amplitude $A(t)$ and phase $\theta(t)$ processes are slowly varying functions of time, and $\omega(A)$ denotes an amplitude-dependent angular frequency of the system. Further, based on Eqs. (24) and (25), the amplitude can be expressed as

$$A(t) = \left[X^2(t) + \frac{\dot{X}^2(t)}{\omega^2(A)}\right]^{\frac{1}{2}} \tag{26}$$

In this manner, following [27, 35], a linearized version of Eq. (21) can be cast as

$$\ddot{X} + c_e(A)\dot{X} + \omega^2(A)X = \tilde{V}(t) \tag{27}$$

where the equivalent damping $c_e(A)$, and $\omega(A)$ are, respectively, given as

$$c_e(A) = \beta_0 + \frac{S(A)}{A\omega(A)} \tag{28}$$

and

$$\omega^2(A) = \frac{C(A)}{A} \tag{29}$$

Note that in Eqs. (28) and (29) the variables

$$S(A) = -\frac{1}{\pi}\int_0^{2\pi} z\left[t, A\cos\phi, -\omega(A)A\sin\phi\right]\sin\phi \, d\phi \tag{30}$$

and

$$C(A) = \frac{1}{\pi}\int_0^{2\pi} z\left[t, A\cos\phi, -\omega(A)A\sin\phi\right]\cos\phi \, d\phi \tag{31}$$

have been introduced.

Next, differentiating Eq. (26) with respect to time, taking into account Eqs. (21-25), and proceeding with standard procedure of stochastic averaging [27], yields

$$\dot{A} = K_1(A,t) + K_2(A,t)\eta(t) \tag{32}$$

where $K_1(A,t)$ and $K_2(A,t)$ are the so-called drift and diffusion coefficients, respectively given as

$$K_1(A,t) = -\frac{1}{2}c_e(A)A(t) + \frac{\pi\xi^2(t)S_{\tilde{W}}[\omega(A)]}{2\omega^2(A)A(t)} \tag{33}$$

and

$$K_2(A,t) = \frac{\sqrt{\pi\xi^2(t)S_{\tilde{W}}[\omega(A)]}}{\omega(A)} \tag{34}$$

Further, $\eta(t)$ is a unitary intensity zero-mean delta correlated Gaussian process, i.e. $E[\eta(t)\eta(t+\tau)] = \delta(\tau)$, with $\delta(\tau)$ being the Dirac delta function and $E[\cdot]$ the mathematical expectation operator.

Note that, Eq. (32) approximately governs the evolution in time of the amplitude process $A(t)$, decoupled from the phase, and modelled as a one-dimensional Markov process. Therefore, the CK equation in the form

$$p_A(a,t_k+\Delta t) = \int_0^\infty p_A(a,t_k+\Delta t \mid \bar{a},t_k)\, p_A(\bar{a},t_k)\, d\bar{a} \tag{35}$$

is satisfied, where $p_A(a,t_k+\Delta t \mid \bar{a},t_k)$ represents the CPDF of $A(t)$. In this regard, relying once more on the short time Gaussian approximation, the CPDF can be given as [27]

$$p_A(a,t_k+\Delta t \mid \bar{a},t_k) = \frac{1}{\sqrt{2\pi K_2^2(\bar{a},t)\Delta t}} \exp\left[-\frac{\left(a-\bar{a}-K_1(\bar{a},t)\Delta t\right)^2}{2K_2^2(\bar{a},t)\Delta t}\right] \tag{36}$$

Substituting Eq. (36) into Eq. (35) and manipulating, yields

$$p_A(a,t_k+\Delta t) = \frac{1}{\sqrt{2\pi\,\Delta t}} \int_0^\infty \beta_A(\bar{a},t_k)\, e^{-\tilde{\lambda}\tilde{g}_k(a,\bar{a})}\, d\bar{a} \tag{37}$$

where

$$\tilde{g}_k(a,\bar{a}) = -\frac{\left(a-\bar{a}-K_1(\bar{a},t_k)\Delta t\right)^2}{K_2^2(\bar{a},t_k)} \tag{38}$$

and

$$\beta_A(\bar{a},t_k) = \frac{p_A(\bar{a},t_k)}{K_2(\bar{a},t_k)} \tag{39}$$

In this manner, Eq. (37) is rewritten as in Eq. (1), and the Laplace's method of integration can be exploited, since $\tilde{\lambda} = 1/(2\Delta t)$ and $\tilde{\lambda} \gg 0$ for small values of Δt. To this aim, denote as $\bar{a}_k^*$ the point such that

$$\left.\frac{\partial \tilde{g}_k\left(a,\bar{a}\right)}{\partial \bar{a}}\right|_{\bar{a}=\bar{a}_k^*} = \tilde{g}_k^I\left(a,\bar{a}_k^*\right) = 0 \tag{40.a}$$

and

$$\left.\frac{\partial^2 \tilde{g}_k\left(a,\bar{a}\right)}{\partial \bar{a}^2}\right|_{\bar{a}=\bar{a}_k^*} = \tilde{g}_k^{II}\left(a,\bar{a}_k^*\right) > 0 \tag{40.b}$$

Then, taking into account Eqs. (5) and (37), the approximate solution of the CK equation Eq. (37) can be given as

$$p_A\left(a,t_k+\Delta t\right) = \sqrt{\frac{2}{\tilde{g}_k^{II}\left(a,\bar{a}_k^*\right)}}\, e^{-\tilde{\lambda}\tilde{g}_k\left(a,\bar{a}_k^*\right)}\left[\beta_A\left(\bar{a}_k^*,t_k\right)+\frac{1}{2\tilde{\lambda}\tilde{g}_k^{II}\left(a,\bar{a}_k^*\right)}\beta_A^{II}\left(\bar{a}_k^*,t_k\right)\right] \tag{41}$$

where $\beta_A\left(\bar{a}_k^*,t_k\right)$ and $\beta_A^{II}\left(\bar{a}_k^*,t_k\right)$ are, respectively, the value of $\beta_A\left(\bar{a},t_k\right)$ in Eq. (39) and the value of its second order derivative evaluated in $\bar{a}_k^*$ at the time instant t_k. Again, if a more accurate approximate solution of the CK equation is required, a fourth-order expansion of $\tilde{g}_k\left(a,\bar{a}\right)$ can be used, and Eq. (41) reverts to

$$\begin{aligned} p_A\left(a,t_k+\Delta t\right) = \sqrt{\frac{2}{\tilde{g}_k^{II}\left(a,\bar{a}_k^*\right)}}\, e^{-\tilde{\lambda}\tilde{g}_k\left(a,\bar{a}_k^*\right)}\Bigg\{\beta_A\left(\bar{a}_k^*,t_k\right)+\frac{1}{\tilde{\lambda}}\Bigg[\frac{\beta_A^{II}\left(\bar{a}_k^*,t_k\right)}{2\tilde{g}_k^{II}\left(a,\bar{a}_k^*\right)}-\frac{\beta_A\left(\bar{a}_k^*,t_k\right)}{8}\frac{\tilde{g}_k^{IV}\left(a,\bar{a}_k^*\right)}{\tilde{g}_k^{II}\left(a,\bar{a}_k^*\right)^2} \\ -\frac{\beta_A^{I}\left(\bar{a}_k^*,t_k\right)}{2}\frac{\tilde{g}_k^{III}\left(a,\bar{a}_k^*\right)}{\tilde{g}_k^{II}\left(a,\bar{a}_k^*\right)^2}+\frac{5\beta_A\left(\bar{a}_k^*,t_k\right)}{24}\frac{\tilde{g}_k^{III}\left(a,\bar{a}_k^*\right)^2}{\tilde{g}_k^{II}\left(a,\bar{a}_k^*\right)^3}\Bigg]\Bigg\} \end{aligned} \tag{42}$$

As far as the implementation of Eqs. (41) or (42) are concerned, note that the function $\beta_A\left(\bar{a}_k^*,t_k\right)$ and its derivatives must be evaluated in each time step, since they depend on the PDF $p_A\left(\bar{a},t_k\right)$ of the response amplitude. On the other hand, $\tilde{g}_k\left(a,\bar{a}_k^*\right)$ and its derivatives are deterministic functions, since they only depend on the drift and diffusion coefficients as shown in Eq. (38). Thus, they could be computed separately once beforehand for higher computational efficiency.

4 Numerical applications

In this section, the proposed approach is applied to three different nonlinear systems. Specifically, a nonlinear first order system, as in Eq. (9), whose stationary response PDF is known, is considered to show the accuracy of the proposed approach. Further, the evolution of the amplitude PDFs of two nonlinear oscillators is examined. Analyses are conducted taking into account also nonstationary excitations. Specifically, in these numerical applications a modulating function of the exponential type

$$\xi(t) = 4\left[\exp\left(-\frac{t}{4}\right) - \exp\left(-\frac{t}{2}\right)\right] \tag{43}$$

is considered. Further, the well-known Kanai-Tajimi power spectrum [36-38]

$$S_{\tilde{W}}(\omega) = S_0 \frac{\omega_g^4 + 4\zeta_g^2\omega_g^2\omega^2}{\left(\omega_g^2 - \omega^2\right)^2 + 4\zeta_g^2\omega_g^2\omega^2}, \ -\infty < \omega < \infty \tag{44}$$

is employed for non-white excitations, where ω_g and ζ_g are the soil natural frequency and damping ratio, respectively. In these cases, comparison with pertinent Monte Carlo simulations (MCS) data is used to establish the accuracy of the procedure.

As far as the numerical implementation of the proposed approach for the response amplitude process is concerned, Eq. (42) is used since it generally leads to higher level of accuracy. This is due to the fact that, for this process, the drift coefficient in Eq. (33) exhibits a singularity at the origin [22, 27], thus potentially leading to an additional source of numerical errors.

In the following examples, the values $\omega_g = 8\pi\, rad/s$ and $\zeta_g = 0.8$ are used in Eq. (44), while the chosen time step Δt is $10^{-3}\, s$, and 15000 samples have been used for the MCS data.

4.1 Nonlinear first order system

Consider a nonlinear system, whose equation of motion is given in Eq. (9) with $f(X,t) = -bX(t) - \varepsilon X^3(t)$, with the nonlinear parameter $\varepsilon > 0$.

Assuming a white noise excitation, that is $S_V(t) = S_0$ and $\xi(t) = 1$, Eq. (17) yields

$$g_k(x,\bar{x}) = \frac{\left[x - \bar{x} + b\,\bar{x}\,\Delta t + \varepsilon\,\bar{x}^3\Delta t\right]^2}{2\pi S_0} \tag{45}$$

Next, using Eqs. (18.a) and (18.b) the point $\bar{x}_k^*$ can be analytically found as

$$\bar{x}_k^* = \mathrm{Re}\left[\frac{\left(1 - i\sqrt{3}\right)\left(-1 + b\,\Delta t\right)}{\sqrt[3]{4}\,\alpha(x)} - \frac{\left(1 + i\sqrt{3}\right)\alpha(x)}{\varepsilon\,\Delta t\, 6\sqrt[3]{2}}\right] \tag{46}$$

where

$$\alpha(x) = \left[-27x\,\varepsilon^2\Delta t^2 + \sqrt{729x^2\varepsilon^4\,\Delta t^4 + 108\,\varepsilon^3\,\Delta t^3\left(-1 + b\,\Delta t\right)^3}\right]^{\frac{1}{3}} \tag{47}$$

In this manner the proposed approach can be directly applied using Eq. (20). In this regard, Figs. 1 and 2 show the evolution of the response PDF, corresponding to the parameter values $b = 0.1$ and $S_0 = 1/2\pi$, for two different values of the nonlinear parameter ε, considering as initial condition $p_X(x,0)$ a normal distribution with standard deviation equal to 0.1.

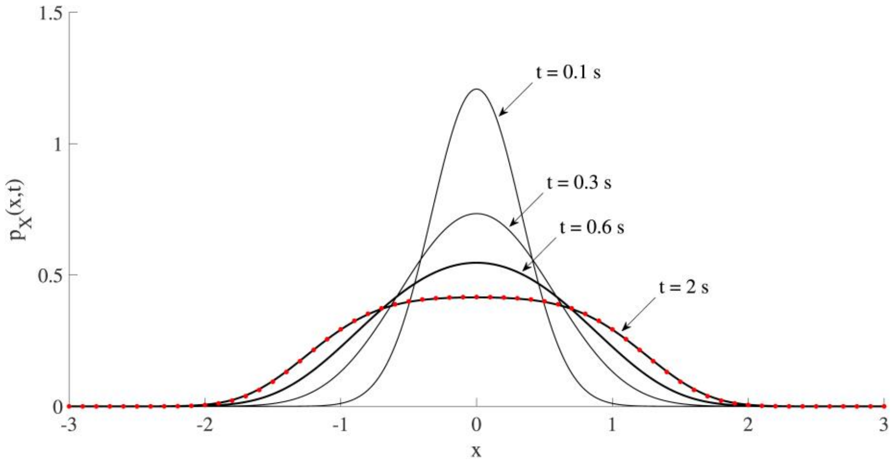

Fig. 1 Probability density function for various time instants. Proposed approach (lines) vis-à-vis stationary analytical solution (dots) for $(\varepsilon = 0.5)$.

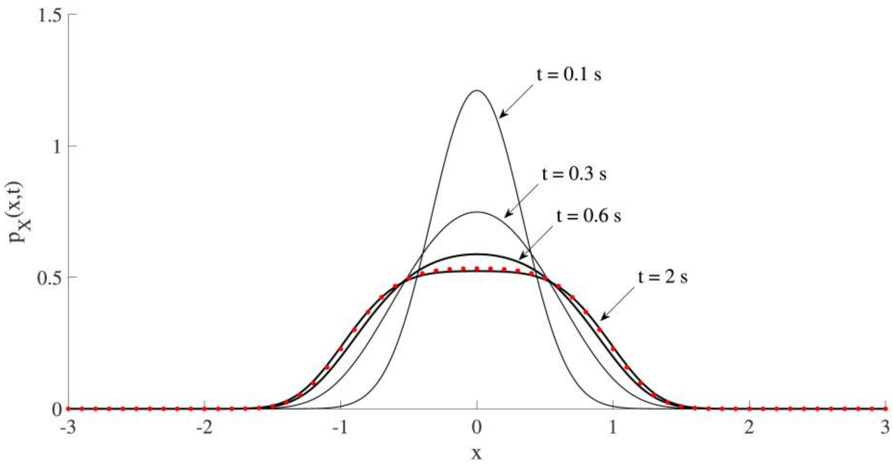

Fig. 2 Probability density function for various time instants. Proposed approach (lines) vis-à-vis stationary analytical solution (dots) for $(\varepsilon = 1.5)$.

As it can be seen, proposed approach-based stationary PDFs are in a very good agreement with the pertinent analytical stationary PDFs, given as

$$p_X(x) = K \exp\left[\frac{1}{\pi S_0}\left(b\frac{x^2}{2} + \varepsilon\frac{x^4}{4}\right)\right] \tag{48}$$

where K is a constant, obtained imposing the normalization condition to Eq. (48).

Consider next the case where the excitation is a modulated white noise as in Eq. (10), with modulating function given in Eq. (43). Taking into account Eq. (17) yields

$$g_k\left(x,\overline{x}\right)=\frac{\left[x-\overline{x}+b\,\overline{x}\,\Delta t+\varepsilon\,\overline{x}^3\Delta t\right]^2}{2\pi S_0\xi^2\left(t_k\right)} \tag{49}$$

while the point $\overline{x}_k^*$ which fulfils Eqs. (18.a) and (18.b) is again given as in Eq. (46); thus, it is independent on the modulating function. In this manner, taking into account Eq. (20), the evolution of the response PDF can be determined. In this regard, in Figs. 3 and 4 the response PDF is plotted for various time instants. Comparisons with MCS reveal a satisfactory level of accuracy.

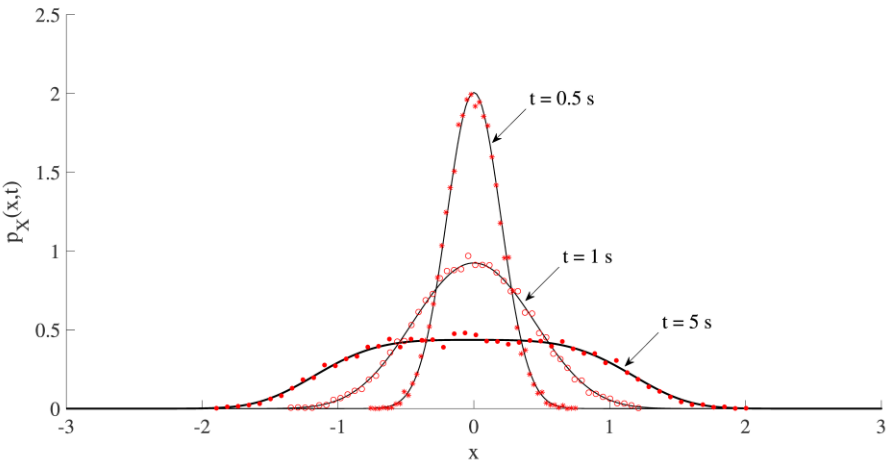

Fig. 3 Probability density function for various time instants. Proposed approach (lines) vis-à-vis MCS data (symbols) for $(\varepsilon=0.5)$.

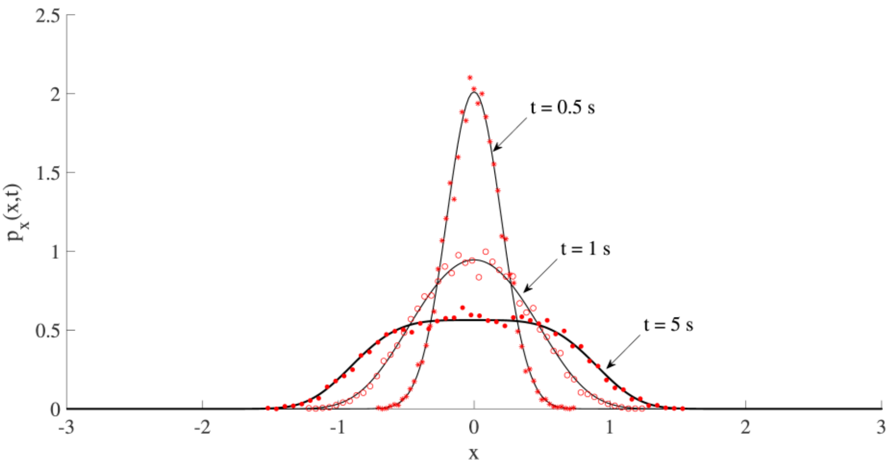

Fig. 4 Probability density function for various time instants. Proposed approach (lines) vis-à-vis MCS data (symbols) for $(\varepsilon=1.5)$.

4.2 Duffing Oscillator

Next, consider a Duffing nonlinear oscillator subject to a white noise excitation, whose equation of motion is

$$\ddot{X} + \beta_0 \dot{X} + \omega_0^2 X + \varepsilon \omega_0^2 X^3 = W(t) \tag{50}$$

where ω_0 is the natural frequency, $\varepsilon > 0$ is the nonlinear parameter of the system, and $W(t)$ is a Gaussian white noise with power spectrum $S_{\tilde{V}}(\omega, t) = S_0$.

For this system Eqs. (28) and (29) lead to [27]

$$c_e(A) = \beta_0 \tag{51}$$

and

$$\omega^2(A) = \omega_0^2 \left(1 + \frac{3}{4}\varepsilon A^2\right) \tag{52}$$

In this manner, Eq. (38) yields

$$\tilde{g}_k(a, \bar{a}) = \frac{\left[\bar{a}\left(4 + 3\bar{a}^2\varepsilon\right)\left(2a + \bar{a}\beta_0\Delta t - 2\bar{a}\right)\omega_0^2 - 4\pi S_0 \Delta t\right]^2}{16\, \bar{a}^2 \pi S_0 \left(4 + 3\bar{a}^2\varepsilon\right)\omega_0^2} \tag{53}$$

and the points $\bar{a}_k^*$ can be found, either analytically or numerically, taking into account Eqs. (40.a) and (40.b).

To show the accuracy of the proposed approach, in Figs. 5 and 6 the corresponding response amplitude PDFs are compared with the stationary response amplitude PDF possessing the following analytical expression [39, 40]

$$p_A(a) = \frac{a + \varepsilon a^3}{\sigma^2} \exp\left[-\left(\frac{a^2}{2} + \varepsilon \frac{a^4}{4}\right)\frac{1}{\sigma^2}\right] \tag{54}$$

where $\sigma^2 = \pi S_0 / \beta_0 \omega_0^2$, assuming as parameters values $\left(\omega_0^2 = 10, S_0 = 0.1, \beta_0 = 0.05\right)$, and for two different values of the nonlinear parameter, specifically $(\varepsilon = 0.5)$ and $(\varepsilon = 1)$. Further, proposed approach-based response amplitude PDFs are also plotted for different time instants, and comparison with pertinent MCS data is provided, assuming the system initially at rest. As it can be seen, reasonable agreement is achieved with both MCS data and stationary analytical solution.

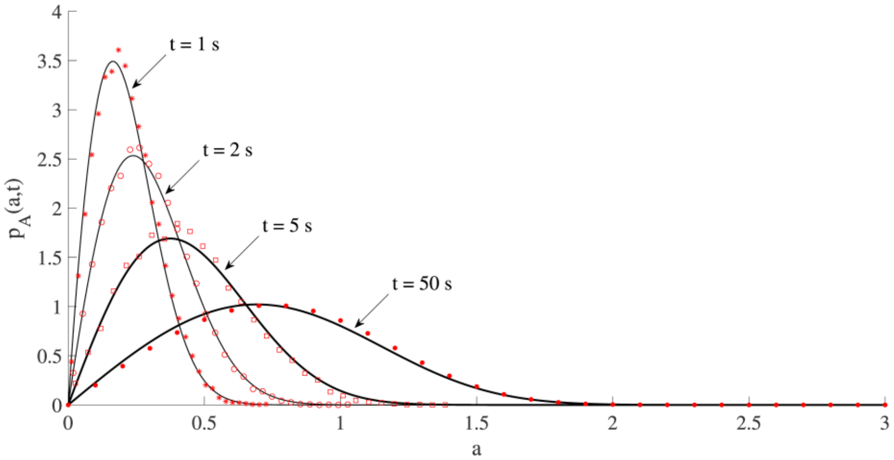

Fig. 5 Response amplitude probability density function for various time instants. Proposed approach (lines) vis-à-vis MCS data (symbols) and stationary analytical solution (dots) for $(\varepsilon = 0.5)$.

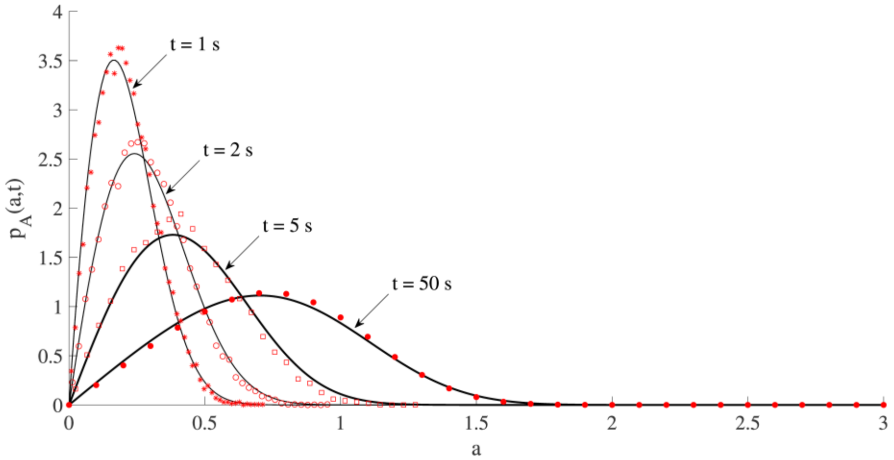

Fig. 6 Response amplitude probability density function for various time instants. Proposed approach (lines) vis-à-vis MCS data (symbols) and stationary analytical solution (dots) for $(\varepsilon = 1)$.

4.3 Van der Pol Oscillator

As a last application, consider a randomly excited Van der Pol oscillator, whose equation of motion can be given as

$$\ddot{X} + \beta_0\left(-1 + \varepsilon X^2\right)\dot{X} + \omega_0^2 X = \tilde{V}(t) \tag{55}$$

where ω_0 is the natural frequency, $\varepsilon > 0$ is a nonlinear parameter of the system, and $\tilde{V}(t)$ is a separable random process defined in Eqs. (22) and (23), with modulating function and power spectrum expressed in Eqs. (43) and (44).

Considering Eqs. (28) and (29), the equivalent damping and natural frequency of the system are [27]

$$c_e(A) = \beta_0\left(-1 + \frac{\varepsilon A^2}{4}\right) \tag{56}$$

and

$$\omega^2(A) = \omega_0^2 \tag{57}$$

In this manner, Eq. (38) yields

$$\tilde{g}_k(a,\bar{a}) = \frac{\mathrm{e}^{t_k}\omega_0^2\left[\omega_0^4 + 2\left(2\zeta_g^2 - 1\right)\omega_0^2\omega_g^2 + \omega_g^4\right]}{16\pi S_0\left(\mathrm{e}^{t_k/4} - 1\right)^2\left(4\zeta_g^2\omega_0^2\omega_g^2 + \omega_g^4\right)}\left\{-a + \bar{a} + \Delta t\left[-\frac{1}{8}\bar{a}\beta_0\left(\bar{a}^2\varepsilon - 4\right) + \frac{8\mathrm{e}^{-t_k}\left(\mathrm{e}^{t_k/4} - 1\right)^2\pi S_0\omega_g^2\left(4\zeta_g^2\omega_0^2 + \omega_g^2\right)}{\bar{a}\omega_0^2\left(\omega_0^4 + 2\left(2\zeta_g^2 - 1\right)\omega_0^2\omega_g^2 + \omega_g^4\right)}\right]\right\}^2 \tag{58}$$

and the points $\bar{a}_k^*$ can be found numerically based on Eqs. (40.a) and (40.b).

The proposed PI approach is applied to the oscillator in Eq. (55) considering $\left(\omega_0^2 = 10, S_0 = 0.1, \beta_0 = 0.05\right)$. Further, two different values of the nonlinear parameter ε are used, specifically $(\varepsilon = 0.5)$ and $(\varepsilon = 1)$. In this regard, in Figs. 7 and 8, the proposed approach-based response amplitude PDFs are compared with pertinent MCS results, for different time instants. As it can be seen, a good agreement is achieved for both values of the parameter ε, for each time instant.

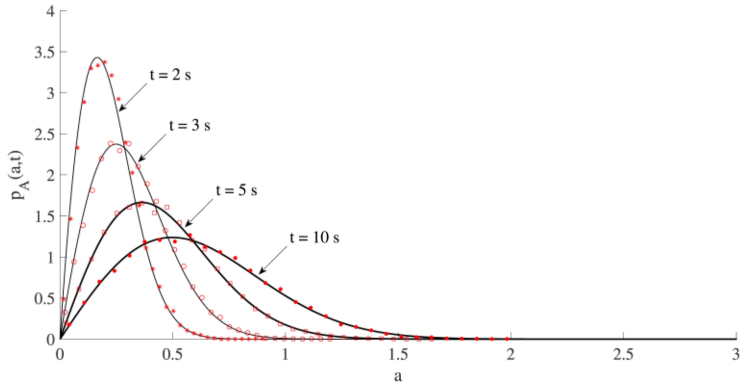

Fig. 7 Response amplitude probability density function for various time instants. Proposed approach (lines) vis-à-vis MCS data (symbols) for $(\varepsilon = 0.5)$.

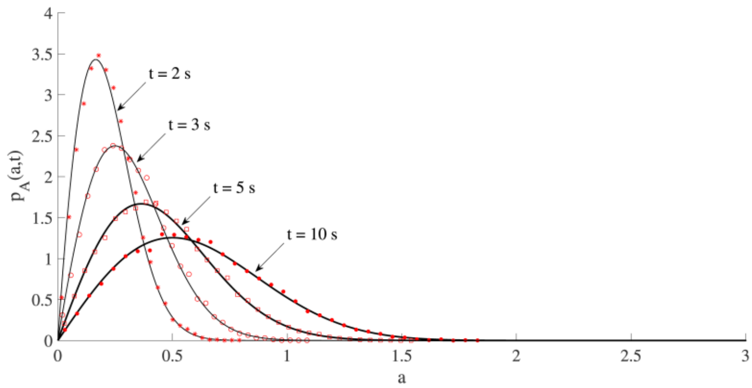

Fig. 8 Response amplitude probability density function for various time instants. Proposed approach (lines) vis-à-vis MCS data (symbols) for $(\varepsilon = 1)$.

Concluding Remarks

The nonstationary response of nonlinear systems subject to broad-band random excitations has been studied. Specifically, a version of the Path Integral approach, which is based on a discretization of the Chapman-Kolmogorov equation in short time steps, has been proposed to determine the evolution of the response probability density function (PDF) of these systems. An approximate analytical solution of this equation has been derived resorting to the Laplace's method of integration, often employed to evaluate approximately integrals whose kernels comprise exponential functions. In this manner, the repetitive integrations, required by the classical numerical implementation of the Path Integral approach for each time step, can be circumvented and the evolution of the response PDF can be sought in short time steps by a direct application of the obtained approximate analytical expression. The proposed approach has been developed for one-dimensional nonlinear systems under modulated white noise. Further, extension to nonlinear oscillators subject to separable evolutionary broad-band processes has been provided, relying on the approximate model of the system response amplitude as a one-dimensional Markov process. Applications to three different nonlinear systems have been considered. Specifically, results pertinent to a first-order nonlinear system, a Duffing oscillator, and a Van der Pol oscillator have been presented. Analyses have been carried out for several broad-band excitations and degree of nonlinearities. The perusal of the response PDFs determined by the Path Integral approach based on Laplace's method of integration vis-à-vis pertinent Monte Carlo simulations data has demonstrated the accuracy of the proposed procedure. Notably, considering the non-negligible computational cost required by the classical application of the Path Integral method, especially for higher-dimensional systems, the proposed approach could offer some advantages for an efficient analytical, albeit approximate, evaluation of the involved integrals.

References

[1] Proppe C, Pradlwarter HJ, Schueller GI (2003) Equivalent linearization and Monte Carlo simulation in stochastic dynamics. Probab Eng Mech 18:1-15.

[2] Smyth AW, Masri SF (2002) Non-stationary response of nonlinear systems using equivalent linearization with a compact analytical form of the excitation process. Probab Eng Mech 17:97-108.

[3] Alotta G, Di Paola M, Pinnola FP (2017) Cross-correlation and cross-power spectral density representation by complex spectral moments. Probab Eng Mech 94:20-27.

[4] Pinnola FP (2016) Statistical correlation of fractional oscillator response by complex spectral moments and state variable expansion. Commun. Nonlinear Sci. Numer. Simul. 39: 343-359.

[5] Spanos PD, Sofi A, Di Paola M (2007) Nonstationary response envelope probability densities of nonlinear oscillators. J Appl Mech 74:315-324.

[6] Spanos, P.D., Di Matteo, A., Cheng, Y., Pirrotta, A., Li, J.: Galerkin scheme-based determination of survival probability of oscillators with fractional derivative elements. J. Appl. Mech. 83(12), 121003 (2016)

[7] Di Matteo, A., Spanos, P.D., Pirrotta, A.: Approximate survival probability determination of hysteretic systems with fractional derivative elements. Probab. Eng. Mech. 54, 138– 146 (2018)

[8] Vanvinckenroye H, Kougioumtzoglou IA, Denoël V (2019) Reliability function determination of nonlinear oscillators under evolutionary stochastic excitation via a Galerkin projection technique. Nonlinear Dyn 95:293-308

[9] Spanos PD, Kougioumtzoglou IA (2012) Harmonic wavelets based statistical linearization for response evolutionary power spectrum determination. Probab Eng Mech 27:57-68.

[10] Kougioumtzoglou IA, Spanos PD (2012) An analytical Wiener path integral technique for non-stationary response determination of nonlinear oscillators. Probab Eng Mech 28:125-131.

[11] Kougioumtzoglou IA, Di Matteo A, Spanos PD, Pirrotta A, Di Paola M (2015) An Efficient Wiener Path Integral Technique Formulation for Stochastic Response Determination of Nonlinear MDOF Systems. J Appl Mech 82:101005

[12] Gardiner CW (1985) Handbook for stochastic methods for physics, chemistry and the natural sciences. Springer, New York

[13] Risken H (1984) The Fokker-Planck equation, methods of solution and applications. Springer, New York.

[14] Wehner MF, Wolfer WG (1983) Numerical evaluation of path integral solutions to Fokker-Planck equations. Phys Rev A 27:2663-2670.

[15] Wehner MF, Wolfer WG (1983) Numerical evaluation of path-integral solutions to Fokker-Planck equations. II. Restricted stochastic processes. Phys Rev A 28:3003-3011.

[16] Wehner MF, Wolfer WG (1987) Numerical evaluation of path integral solutions to Fokker-Planck equations. III. Time and functionally dependent coefficients. Phys Rev A 35:1795-1801.

[17] Naess A, Johnsen JM (1993) Response statistics of nonlinear, compliant offshore structures by the path integral solution method. Probab Eng Mech 8:91-106.

[18] Lin H, Yim SCS (1996) Nonlinear rocking motions. II: Overturning under random excitations. J Eng Mech 122:728-735.

[19] Barone G, Navarra G, Pirrotta A (2008) Probabilistic response of linear structures equipped with nonlinear damper devices (PIS method). Probab Eng Mech 23:125-133.

[20] Iourtchenko DV, Mo E, Naess A (2006) Response probability density functions of strongly non-linear systems by the path integration method. Int J Non-Linear Mech 41:693-705.

[21] Naess A, Moe V (2000) Efficient path integration methods for nonlinear dynamic systems. Probab Eng Mech 15:221-231.

[22] Naess A, Moe V (1996) Stationary and non-stationary random vibration of oscillators with bilinear hysteresis. Int J Non-linear Mech 31:553-562.
[23] Di Paola M, Santoro R (2008) Path integral solution for nonlinear system enforced by Poisson White Noise. Probab Eng Mech 23:164-169
[24] Pirrotta A, Santoro R (2011) Probabilistic response of nonlinear systems under combined normal and Poisson white noise via path integral method. Probab Eng Mech 26:26-32
[25] Di Matteo A, Di Paola M, Pirrotta A (2016) Path integral solution for nonlinear systems under parametric Poissonian white noise input. Probab Eng Mech 44:89-98
[26] Bucher C, Di Matteo A, Di Paola M, Pirrotta A (2016) First-passage problem for nonlinear systems under Lévy white noise through path integral method. Nonlinear Dyn 85:1445-1456
[27] Kougioumtzoglou IA, Spanos PD (2013) Response and First-Passage Statistics of Nonlinear Oscillators via a Numerical Path Integral Approach. J Eng Mech 139:1207-1217
[28] Alevras P, Yurchenko D (2016) GPU computing for accelerating the numerical Path Integration approach. Comput Struct 171:46-53
[29] Ren Z, Xu W, Qiao Y (2019) Local averaged path integration method approach for nonlinear dynamic systems. Appl Math Comp 344-345:68-77
[30] Yue X, Xu W, Xu Y, Sun JQ (2019) Non-stationary response of MDOF dynamical systems under combined Gaussian and Poisson white noises by the generalized cell mapping method. Probab Eng Mech. https://doi.org/10.1016/j.probengmech.2019.01.001
[31] Bender CM, Orszag SA (1999) Advanced Mathematical methods for scientists and Engineers I: Asymptotic methods and perturbation theory. Springer, New York.
[32] Miller PD (2006) Applied Asymptotic Analysis. American Mathematical Society, Providence
[33] Deng ML, Fu Y, Huang ZL (2014) Asymptotic Analytical Solutions of First-Passage Rate to Quasi- Nonintegrable Hamiltonian Systems. J Appl Mech 81:081012
[34] Xu Y, Li H, Wang H, Jia W, Yue X, Kurths J (2017) The Estimates of the Mean First Exit Time of a Bistable System Excited by Poisson White Noise. J Appl Mech 84:091004
[35] Roberts JB, Spanos PD (2003) Random vibration and statistical linearization. Dover Publications, New York.
[36] Kanai K (1957) Semi-empirical formula for the seismic characteristics of the ground. Univ Tokyo, Bull Earthq Res Inst.
[37] Tajimi H (1960) A statistical method for determining the maximum response of a building structure during an earthquake. Proceedings of the 2nd World Conference on Earthquake Engineering, Tokyo.
[38] Alotta G, Di Paola M, Pirrotta A (2014) Fractional Tajimi–Kanai model for simulating earthquake ground motion. Bull Earthquake Eng 12:2495-2506
[39] Lin YK (1967) Probabilistic theory of structural dynamics. McGraw Hill, New York.
[40] Iwan WD, Spanos PD (1978) Response envelope statistics for nonlinear oscillators with random excitation. J Appl Mech 45:170-174.