DE GRUYTER Adv. Geom. 2020; 20 (1):91–108

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Voisin's conjecture for zero-cycles on Calabi–Yau varieties and their mirrors

DOI 10.1515/advgeom-2019-0008. Received 19 October, 2017; revised 4 January, 2018

Abstract: We study a conjecture, due to Voisin, on 0-cycles on varieties with $p_g = 1$. Using Kimura's finite dimensional motives and recent results of Vial's on the refined (Chow–)Künneth decomposition, we provide a general criterion for Calabi–Yau manifolds of dimension at most 5 to verify Voisin's conjecture. We then check, using in most cases some cohomological computations on the mirror partners, that the criterion can be successfully applied to various examples in each dimension up to 5.

Keywords: Algebraic cycles, Chow groups, motives, finite-dimensional motives, Calabi–Yau varieties.

2010 Mathematics Subject Classification: Primary 14C15, 14C25, 14C30

Communicated by: I. Coskun

1 Introduction

For a smooth projective variety X over $\mathbb{C}$, let $A^j(X)$ denote the Chow groups of codimension j algebraic cycles on X modulo rational equivalence. Chow groups of cycles of codimension larger than 1 are still mysterious. As an example, we recall the famous Bloch Conjecture, namely:

Conjecture 1.1 (Bloch, [8]). *Let X be a smooth projective complex variety of dimension n. The following are equivalent:*

(i) $A^n(X) \cong \mathbb{Q}$;

(ii) *the Hodge numbers $h^{j,0}(X)$ are 0 for all $j > 0$.*

The implication from (i) to (ii) is actually a theorem, see [11]. The conjectural part is the implication from (ii) to (i), which has been verified for surfaces not of general type in [9], but it is wide open for surfaces of general type despite several significant cases that have been dealt with over the years; see e.g. [2; 67; 3; 71; 53].

A natural next step is to consider varieties X with geometric genus $p_g = 1$. Here, the kernel $A^n_{AJ}(X)$ of the Albanese map is huge; in a sense that can be made precise: it is "infinite-dimensional", see [50] and [70]. Yet, this huge group should have controlled behaviour on the self-product $X \times X$, according to a conjecture due to Voisin, which is motivated by the Bloch–Beilinson conjectures (see [72, Section 4.3.5.2] for a detailed discussion).

Conjecture 1.2 ([68], see [72] Conjecture 4.37 for this precise form). *Let X be a smooth projective complex variety of dimension n with $h^{j,0}(X) = 0$ for $0 < j < n$. The following are equivalent:*

(i) *For any zero-cycles $a, a' \in A^n(X)$ of degree zero, we have $a \times a' = (-1)^n a' \times a$ in $A^{2n}(X \times X)$; here $a \times a'$ is shorthand for the cycle class $p_1^*(a) \cdot p_2^*(a') \in A^{2n}(X \times X)$, where p_1, p_2 denote the projections on the first, respectively second factor.*

(ii) *the geometric genus $p_g(X)$ is at most 1.*

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Again, the implication from (i) to (ii) is actually a theorem (this can be proven à la Bloch–Srinivas [11], see Lemma 2.1 below). The conjectural part is the implication from (ii) to (i), which is still wide open for a general $K3$ surface; cf. [68], [41], [40], [43], [44] for some cases where this conjecture is verified.

In the present article we present a general criterion to check Voisin's conjecture (or a weak variant of it, cf. Theorem 4.12) for specific varieties (see Section 1 for all the relevant definitions and explanations).

Theorem (=Theorem 4.1). *Let X be a smooth projective variety of dimension $n \leq 5$ with $h^{i,0}(X) = 0$ for $0 < i < n$ and $p_g(X) = 1$. Assume moreover that*

(i) *X is rationally dominated by a variety X' of dimension n, that X' has finite-dimensional motive and that $B(X')$ is true;*
(ii) *X is $\widetilde{N}^1$-maximal;*
(iii) *$\widetilde{N}^1 H^i(X) = H^i(X)$ for $0 < i < n$;*
(iv) *X is rationally dominated by a variety X'' of dimension n and the Hodge conjecture is true for $X'' \times X''$.*

Then conjecture 1.2 is true for X, i.e. any $a, a' \in A^n_{hom}(X)$ satisfy $a \times a' = (-1)^n a' \times a$ in $A^{2n}(X \times X)$.

The proof of Theorem 4.1 relies, among other things, on results by Vial [63] on the refined Chow–Künneth decomposition, from which the hypotheses on X' are thus inherited.

The hypotheses of Theorem 4.1 may seem very stringent. Yet, there are some examples satisfying all the hypotheses. Most of these examples are given by hypersurfaces of Fermat type in a (weighted) projective space (see Section 5 for all the examples). The first and third hypotheses of our criterion hold for any Fermat hypersurface, while the fourth holds for low degree Fermat hypersurfaces, see [57]. As for the second, it seems the most delicate to verify in practice. In certain cases it is possible to check the second hypothesis by direct computation, e.g. for the Fermat sextic X in $\mathbb{P}^5$, using results by Beauville, Movasati and the classical inductive structure of Fermat hypersurfaces (Proposition 5.11). Hence, we obtain the following explicit example:

Corollary (=Proposition 5.11). *Let $X \subset \mathbb{P}^5(\mathbb{C})$ be the sextic fourfold defined as $x_0^6 + \cdots + x_5^6 = 0$. Then conjecture 1.2 is true for X, i.e. any $a, a' \in A^4_{hom}(X)$ satisfy $a \times a' = a' \times a$ in $A^8(X \times X)$.*

In other cases (for instance for the Fermat quintic 3-fold), despite the fact that the dimension of $H^n(X)$ is quite large, it is possible to control the dimension of $H^n_{tr}(X)$ by passing to the mirror partner of X, which can be explicitly described in the Fermat case. Among other examples, we obtain in this way the $\widetilde{N}^1$-maximality and therefore Voisin's conjecture in the following case:

Corollary (=Proposition 5.9). *Let $X \subset \mathbb{P}(1^4, 2)$ be the Calabi–Yau threefold defined as $x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_4^3 = 0$. Then conjecture 1.2 is true for X, i.e. any $a, a' \in A^3_{hom}(X)$ satisfy $a \times a' = -a' \times a$ in $A^6(X \times X)$.*

Conventions. In this paper, the word *variety* refers to a reduced irreducible scheme of finite type over $\mathbb{C}$. *All Chow groups will be with rational coefficients:* For a variety X, we write $A_j(X)$ for the Chow group of j-dimensional cycles on X with $\mathbb{Q}$-coefficients. If X is smooth of dimension n, the notations $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably. The notations $A^j_{hom}(X)$ and $A^j_{AJ}(X)$ are used to indicate the subgroups of homologically, respectively Abel–Jacobi trivial cycles. The (contravariant) category of Chow motives (i.e. pure motives with respect to rational equivalence as in [56], [51]) is denoted $\mathcal{M}_{rat}$.

We write $H^j(X)$ for the singular cohomology $H^j(X, \mathbb{Q})$.

2 Preliminaries

2.1 Warm-up. We begin with the following result for which we could not find a reference in the literature, although it may be well-known to experts.

Lemma 2.1. *Let X be a smooth projective complex variety of dimension n with $h^{j,0}(X) = 0$ for $0 < j < n$. Consider the following two conditions:*

(i) *For any zero-cycles a, $a' \in A^n(X)$ of degree zero, we have $a \times a' = (-1)^n a' \times a$ in $A^{2n}(X \times X)$; here $a \times a'$ is shorthand for the cycle class $(p_1)^*(a) \cdot (p_2)^*(a') \in A^{2n}(X \times X)$, where p_1, p_2 denote the projections on the first, respectively second factor.*

(ii) *the geometric genus $p_g(X)$ is at most* 1.

Then (i) *implies* (ii).

Proof. This is a "decomposition of the diagonal" argument à la Bloch–Srinivas: we define a correspondence

$$\pi := \Delta_X - x \times X - X \times x \in A^n(X \times X),$$

where Δ_X denotes the diagonal and $x \in X$. Then we consider the correspondence

$$p := (\Delta_X - (-1)^n \Gamma_\iota) \circ (\pi \times \pi) \in A^{2n}((X \times X) \times (X \times X)),$$

where ι is the involution on $X \times X$ switching the two factors. Hypothesis (i) implies that p acts trivially on 0-cycles of $X \times X$, i.e. $p_* A^{2n}(X \times X) = 0$. The Bloch–Srinivas argument [11] then implies that there exists a rational equivalence $p = \gamma$ in $A^{2n}((X \times X) \times (X \times X))$, where γ is a cycle supported on $X \times X \times D$, for some divisor $D \subset X \times X$. It follows that

$$\wedge^2 H^n(X) = p_*(H^n(X) \otimes H^n(X)) \subset H^{2n}(X \times X)$$

is supported on the divisor D. In particular, we see that $\wedge^2 H^{n,0}(X, \mathbb{C}) \subset H^{2n,0}(X \times X, \mathbb{C})$ is (supported on a divisor and hence) zero. This proves (ii). □

Remark 2.2. We have actually proven more than the implication from (i) to (ii). We have proven a special instance of the generalized Hodge conjecture: for any variety X satisfying the assumptions of Lemma 2.1, the sub Hodge structure $\wedge^2 H^n(X) \subset H^{2n}(X \times X)$ is supported on a divisor. This implication was already observed by Voisin [72, Corollary 3.5.1].

2.2 Finite-dimensional motives. We refer to [37], [1], [30], [33], [51] for the definition of finite-dimensional motive. An essential property of varieties with finite-dimensional motives is given by the nilpotence theorem.

Theorem 2.3 (Kimura [37] Proposition 7.2 (ii)). *Let X be a smooth projective variety of dimension n with finite-dimensional motive. Let $\Gamma \in A^n(X \times X)_{\mathbb{Q}}$ be a correspondence which is numerically trivial. Then there exists $N \in \mathbb{N}$ such that $\Gamma^{\circ N} = 0 \in A^n(X \times X)_{\mathbb{Q}}$.*

Actually, the nilpotency (for all powers of X) could serve as an alternative definition of finite-dimensional motives, as shown by a result of Jannsen [33, Corollary 3.9]. Conjecturally, any variety has finite-dimensional motive, see [37]. We are still far from knowing this, but at least there are quite a few non-trivial examples.

Remark 2.4. The following varieties have finite-dimensional motives: varieties dominated by products of curves (which is the case for Fermat hypersurfaces) and abelian varieties, see [37]; $K3$ surfaces with Picard number 19 or 20, see [52]; surfaces not of general type with vanishing geometric genus, see [27, Theorem 2.11]; Godeaux surfaces, see [27]; certain surfaces of general type with $p_g = 0$, see [71], [4], [53]; Hilbert schemes of surfaces known to have finite-dimensional motives, see [17]; generalized Kummer varieties, see [73, Remark 2.9(ii)]; 3-folds with nef tangent bundle, see [32] (an alternative proof is given in [66, Example 3.16]); 4-folds with nef tangent bundle, see [31]; log-homogeneous varieties in the sense of [13] (this follows from [31, Theorem 4.4]); certain 3-folds of general type, see [65, Section 8]; varieties of dimension ≤ 3 rationally dominated by products of curves, see [66, Example 3.15]; varieties X with $A^i_{AJ}(X) = 0$ for all i, see [64, Theorem 4]; and products of varieties with finite-dimensional motives, see [37].

Remark 2.5. It is a (somewhat embarrassing) fact that all examples known so far of finite-dimensional motives happen to be in the tensor subcategory generated by Chow motives of curves (i.e. they are "motives of abelian type" in the sense of [66]). That is, the finite-dimensionality conjecture is still open for any motive *not* generated by curves; on the other hand, there exist many motives not generated by curves, cf. [19, 7.6].

2.3 The Lefschetz standard conjecture and (co-)niveau filtrations. Let X be a smooth projective variety of dimension n and $h \in H^2(X,\mathbb{Q})$ the class of an ample line bundle. By the hard Lefschetz theorem the map

$$L^{n-i}\colon H^i(X,\mathbb{Q}) \to H^{2n-i}(X,\mathbb{Q})$$

obtained by cupping with h^{n-i} is an isomorphism, for any $i < n$. One of the standard conjectures, also known as the Lefschetz standard conjecture $B(X)$, asserts that the inverse isomorphism is algebraic:

Conjecture 2.6. *Given a smooth projective variety X, the class $h \in H^2(X,\mathbb{Q})$ of an ample line bundle, and an integer $0 \le i < n$, the isomorphism*

$$(L^{n-i})^{-1}\colon H^{2n-i}(X,\mathbb{Q}) \xrightarrow{\cong} H^i(X,\mathbb{Q})$$

is induced by a correspondence.

We recall the following filtration which, via Proposition 3.3, will play a central role in our criterion (Theorem 4.1) to check Conjecture 1.2.

Definition 2.7 (Coniveau filtration [10]). Let X be a quasi-projective variety. The *coniveau filtration* on cohomology and on homology is defined by

$$N^cH^i(X,\mathbb{Q}) = \sum \operatorname{Im}(H^i_Y(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})),$$
$$N^cH_i(X,\mathbb{Q}) = \sum \operatorname{Im}(H_i(Z,\mathbb{Q}) \to H_i(X,\mathbb{Q})),$$

where Y (respectively Z) runs over all subvarieties of X of codimension $\ge c$ (respectively of dimension $\le i-c$), and $H^i_Y(X,\mathbb{Q})$ denotes the cohomology with support along Y.

Remark 2.8. It is known that $B(X)$ holds for the following varieties: for curves, surfaces, abelian varieties by [38], [39], for threefolds not of general type by [60], for hyperkähler varieties of $K3^{[n]}$-type by [15], for n-dimensional varieties X which have $A_i(X)$ supported on a subvariety of dimension $i+2$ for all $i \le \frac{n-3}{2}$ by [62, Theorem 7.1], for n-dimensional varieties X which have $H_i(X) = N^{\lfloor i/2 \rfloor}H_i(X)$ for all $i > n$ by [64, Theorem 4.2], and for products and hyperplane sections of any of these by [38], [39] (in particular it holds for projective hypersurfaces, a fact that we will use).

For smooth projective varieties X over $\mathbb{C}$, the standard conjecture $B(X)$ implies the standard conjecture $D(X)$, i.e. homological and numerical equivalence coincide on X and $X \times X$; see [38], [39].

Friedlander, and independently Vial, introduced the following variant of the coniveau filtration:

Definition 2.9 (Niveau filtration, see [22], [23] [63]). Let X be a smooth projective variety. The *niveau filtration* on homology is defined as

$$\widetilde{N}^jH_i(X) = \sum_{\Gamma \in A_{i-j}(Z\times X)} \operatorname{Im}(H_{i-2j}(Z) \xrightarrow{\Gamma_*} H_i(X)),$$

where the union runs over all smooth projective varieties Z of dimension $i-2j$, and all correspondences $\Gamma \in A_{i-j}(Z \times X)$. The niveau filtration on cohomology is defined as

$$\widetilde{N}^cH^iX := \widetilde{N}^{c-i+n}H_{2n-i}X.$$

Remark 2.10. In [22], [23], the niveau filtration $\widetilde{N}^*$ is called the "correspondence filtration".

The relation between the standard conjecture $B(X)$ and the niveau and coniveau filtrations is made clear in the following.

Remark 2.11. The niveau filtration is included in the coniveau filtration $\widetilde{N}^jH^i(X) \subset N^jH^i(X)$. These two filtrations are expected to coincide; indeed, one can show that the two filtrations coincide if and only if the Lefschetz standard conjecture is true for all varieties; see [22, Proposition 4.2], [63, Proposition 1.1].

Using the truth of the Lefschetz standard conjecture in degree ≤ 1, it can be checked that the two filtrations coincide in a certain range: one has $\widetilde{N}^jH^i(X) = N^jH^iX$ for all $j \ge (i-1)/2$, see [63, page 415 "Properties"]. In particular $\widetilde{N}^1H^3(X) = N^1H^3(X)$ and $\widetilde{N}^2H^4(X) = N^2H^4(X)$.

The following "refined Künneth decomposition" and "refined Chow–Künneth decomposition" are very useful:

Theorem 2.12 (Vial [63]). *Let X be a smooth projective variety of dimension $n \leq 5$. Assume that $B(X)$ holds. Then there exist algebraic cycles $\pi_{i,j}$ on $X \times X$ and a decomposition of the diagonal*

$$\Delta_X = \sum_{i,j} \pi_{i,j} \quad \text{in } H^{2n}(X \times X),$$

where the $\pi_{i,j}$'s are mutually orthogonal idempotents. The correspondence $\pi_{i,j}$ acts on $H^*(X)$ as a projector on $\mathrm{Gr}^j_{\widetilde{N}} H^i(X)$. Moreover, $\pi_{i,j}$ can be chosen to factor over a variety of dimension $i - 2j$, i.e. for each $\pi_{i,j}$ there exist a smooth projective variety $Z_{i,j}$ of dimension $i - 2j$ and correspondences $\Gamma_{i,j} \in A^{n-j}(Z_{i,j} \times X)$, $\Psi_{i,j} \in A^{i-j}(X \times Z_{i,j})$ such that $\pi_{i,j} = \Gamma_{i,j} \circ \Psi_{i,j}$ in $H^{2n}(X \times X)$.

Proof. This is a special case of [63, Theorem 1]. Indeed, as mentioned in loc. cit., varieties X of dimension ≤ 5 such that $B(X)$ holds satisfy condition (*) of loc. cit. □

Under the extra hypothesis of finite-dimensionality of the motive the conclusion can be proved at the level of Chow groups.

Theorem 2.13 (Vial [63]). *Let X be a smooth projective variety of dimension $n \leq 5$. Assume that X has finite-dimensional motive and that $B(X)$ holds. Then there exists a decomposition of the diagonal*

$$\Delta_X = \sum_{i,j} \Pi_{i,j} \quad \text{in } A^n(X \times X),$$

where the $\Pi_{i,j}$'s are mutually orthogonal idempotents lifting the $\pi_{i,j}$ of Theorem 2.12. Moreover, $\Pi_{i,j}$ can be chosen to factor over a variety of dimension $i - 2j$, i.e. for each $\Pi_{i,j}$ there exist a smooth projective variety $Z_{i,j}$ of dimension $i - 2j$ and correspondences $\Gamma_{i,j} \in A^{n-j}(Z_{i,j} \times X)$, $\Psi_{i,j} \in A^{i-j}(X \times Z_{i,j})$ such that $\Pi_{i,j} = \Gamma_{i,j} \circ \Psi_{i,j}$ in $A^n(X \times X)$.

Proof. This is a special case of [63, Theorem 2]. Indeed, X as in Theorem 2.13 satisfies conditions (*) and (**) of loc. cit. □

Remark 2.14. Let X be as in Theorem 2.12. Note that Conjecture $B(X)$ implies in particular that the $\pi_{i,j}$ are algebraic, cf. [39, Theorem 4.1, item (3)].

Remark 2.15. Let X be as in Theorem 2.13. Then, as in [40], one can define the "most transcendental part" of the motive of X by setting $t_n(X) := (X, \Pi_{n,0}, 0) \in \mathcal{M}_{\mathrm{rat}}$. The fact that $t_n(X)$ is well-defined up to isomorphism follows from [35, Theorem 7.7.3] and [63, Proposition 1.8]. For $n = 2$, $t_n(X)$ coincides with the "transcendental part" $t_2(X)$ constructed for any surface in [35].

3 $\widetilde{N}^1$-maximal varieties

Let X be a smooth projective n-dimensional variety. Then $H^n(X)$ is a polarized Hodge structure, and the niveau $N^1 := N^1 H^n(X)$ is a Hodge substructure. It follows from the semisimplicity of the category of polarizable pure Hodge structures, see [18, 4.2.3] and also [72, Theorem 2.22], that the Hodge substructure N^1 of the polarized Hodge structure $H^n(X, \mathbb{Q})$ induces a splitting with respect to the Lefschetz intersection pairing related to a choice of a polarization, namely

$$H^n(X, \mathbb{Q}) = N^1 \oplus (N^1)^\perp.$$

Definition 3.1. The "transcendental cohomology" is the orthogonal complement

$$H^n_{tr}(X) := (N^1)^\perp \subset H^n(X, \mathbb{Q}).$$

Remark 3.2. Note that $H^n_{tr}(X)$ is isomorphic to the graded piece $\mathrm{Gr}^0_{N^\cdot} H^n(X)$ (which is a priori only a quotient of $H^n(X)$).

Provided that the generalized Hodge conjecture is true, $H^n_{tr}(X)$ is the smallest Hodge substructure $V \subset H^n(X,\mathbb{Q})$ for which $V_{\mathbb{C}}$ contains $H^{n,0}$.

Proposition 3.3. *Let X be a smooth projective n-fold. The following are equivalent:*

(i) $\dim H^n_{tr}(X) = 2p_g(X)$;
(ii) *the subspace $H^{n,0}\oplus H^{0,n} \subset H^n(X,\mathbb{C})$ is defined over $\mathbb{Q}$, and $H^n(X)\cap F^1 = N^1H^n(X)$;*
(iii) $\dim N^1H^n(X) = \sum_{i,j>0,\, i+j=n} h^{i,j}(X)$;
(iv) *the subspace $\bigoplus_{i,j>0,\, i+j=n} H^{i,j} \subset H^n(X,\mathbb{C})$ is defined over $\mathbb{Q}$, and $H^n(X)\cap F^1 = N^1H^n(X)$.*

Proof. Obviously, (i)⇔(iii). The equivalence (ii)⇔(iv) is obtained using the polarization on $H^n(X,\mathbb{Q})$. Indeed, suppose $V\subset H^n(X,\mathbb{Q})$ is a subspace such that $V_{\mathbb{C}} = H^{n,0}\oplus H^{0,n}$. Then $V\subset H^n(X,\mathbb{Q})$ is a Hodge substructure. As mentioned above, a Hodge substructure V of the polarizable Hodge structure $H^n(X,\mathbb{Q})$ induces a splitting

$$H^n(X,\mathbb{Q}) = V\oplus V^{\perp},$$

where both V and $V^\perp$ are Hodge substructures. The subspace $V^\perp$ has $(V^\perp)_{\mathbb{C}} = \bigoplus_{i,j>0,\, i+j=n} H^{i,j}$. The rest is clear: (i)⇒(ii) because (i) forces $(H^n_{tr}(X))_{\mathbb{C}}$ (which always contains $H^{n,0}\oplus H^{0,n}$) to be equal to $H^{n,0}\oplus H^{0,n}$, and hence also $(N^1H^n(X))_{\mathbb{C}} = \sum_{i,j>0,\, i+j=n} H^{i,j}(X)$. Similarly, (ii)⇒(i): if $V\subset H^n(X,\mathbb{Q})$ is such that $V_{\mathbb{C}} = H^{n,0}\oplus H^{0,n}$, then both V and $H^n_{tr}(X)$ are the smallest Hodge substructure of $H^n(X,\mathbb{Q})$ containing $H^{n,0}$; as such, they are equal. □

Definition 3.4. A smooth projective n-dimensional variety satisfying the equivalent conditions of Proposition 3.3 is called N^1-*maximal*.

Definition 3.5. A smooth projective n-dimensional variety X is called $\widetilde{N}^1$-*maximal* if it is N^1-maximal and there is equality $N^1H^n(X) = \widetilde{N}^1H^n(X)$.

Remark 3.6. Proposition 3.3 is inspired by [6, Proposition 1], where a similar result is proven for surfaces. A surface with $\dim H^2_{tr}(S) = 2p_g(S)$ is called a ρ-maximal surface.

In dimension $n\le 3$, the notions of N^1-maximality and $\widetilde{N}^1$-maximality coincide, in view of Remark 2.11.

Remark 3.7. While looking for examples of N^1-maximal Calabi–Yau 3-folds we realised that the notion of N^1-maximality was already considered (under a different name) in [47, Remarks, p. 48, item 3], via the characterization (ii) of Proposition 3.3.

Remark 3.8. Let X be an N^1-maximal n-fold. The equality $H^n(X,\mathbb{Q})\cap F^1 = N^1H^n(X,\mathbb{Q})$ means that X satisfies a strong (i.e. non-amended) version of the generalized Hodge conjecture.

4 A general result

The following result gives sufficient conditions ensuring that a Calabi–Yau n-fold verifies Voisin's conjecture 1.2:

Theorem 4.1. *Let X be a smooth projective variety of dimension $n\le 5$ with $h^{i,0}(X) = 0$ for $0<i<n$ and $p_g(X) = 1$. Assume moreover that*

(i) *X is rationally dominated by a variety X' of dimension n, that X' has finite-dimensional motive and that $B(X')$ is true;*
(ii) *X is $\widetilde{N}^1$-maximal;*
(iii) $\widetilde{N}^1H^i(X) = H^i(X)$ *for* $0<i<n$;
(iv) *X is rationally dominated by a variety X'' of dimension n and the Hodge conjecture is true for $X''\times X''$.*

Then any $a, a'\in A^n_{hom}(X)$ satisfy $a\times a' = (-1)^n a'\times a$ in $A^{2n}(X\times X)$.

Remark 4.2. Note that all hypotheses are satisfied in dimension 1.

Remark 4.3. Note that we need only a special instance of the Hodge conjecture for $X'' \times X''$, namely the algebraicity of the Hodge substructure $\wedge^2 H^n_{tr}(X'')$. Also (as pointed out to us by the referee), we actually only need assumption (iv) in case the dimension n is even. Indeed, for odd n the $\mathbb{Q}$-vector space $\wedge^2 H^n_{tr}(X'')$ is generated by the class of the refined Chow–Künneth projector $\Pi_{n,0}$.

Let $\iota\colon X \times X \to X \times X$ denote the involution exchanging the two factors. We consider the correspondence

$$\Lambda := \frac{1}{2}(\Delta_{X\times X} + (-1)^{n+1}\Gamma_\iota) \ \in A^{2n}(X^4),$$

where $\Delta_{X\times X} \subset X^4$ denotes the diagonal of $(X \times X) \times (X \times X)$ and Γ_ι denotes the graph of the involution ι. Note that Λ is idempotent. To prove Theorem 4.1 we must check that

$$\Lambda_* \operatorname{Im}(A^n_{\mathrm{hom}}(X) \otimes A^n_{\mathrm{hom}}(X) \to A^{2n}(X \times X)) = 0.$$

We need to modify Λ a bit, as follows. Let $\Psi \in A^n(X' \times X)$ denote the closure of the graph of the dominant rational map ψ from X' to X. We know that

$$\Psi_* \Psi^* = d \cdot \mathrm{id}\colon A^n(X) \to A^n(X), \tag{1}$$

where d is the degree of Ψ. Set $\Pi_{n,0} := \frac{1}{d}\Psi \circ \Pi^{X'}_{n,0} \circ {}^t\Psi$ where Ψ is as above and $\Pi^{X'}_{n,0}$ is given by Vial's result Theorem 2.12, thanks to the finite-dimensionality of the motive of X' plus $B(X')$. By (1) combined with the idempotence of $\Pi^{X'}_{n,0}$ we have

$$(\Pi_{n,0})_* \circ (\Pi_{n,0})_* = (\Pi_{n,0})_* : A^n(X) \to A^n(X). \tag{2}$$

Hence, up to dividing by a constant, we may assume that $(\Pi_{n,0})$ acts as an idempotent on 0-cycles on X. We finally introduce the correspondence

$$\Lambda_{tr} := \Lambda \circ (\Pi_{n,0} \times \Pi_{n,0}) \ \in A^{2n}(X^4),$$

where the $\Pi_{n,0}$ are as above; see [72, Section 4.3.5.2] for a similar construction. Note that Λ_{tr} depends on the choice of $\Pi_{n,0}$. The key point is the following:

Claim 4.4. *Λ_{tr} acts as an idempotent on 0-cycles, i.e.*

$$(\Lambda_{tr} \circ \Lambda_{tr})_* = (\Lambda_{tr})_* : A_0(X \times X) \to A_0(X \times X).$$

Proof of Claim 4.4. Note that Λ is an idempotent. Moreover, by Equation (2) also $\Pi_{n,0}$ acts as an idempotent on 0-cycles. Write

$$\begin{aligned}(\Lambda_{tr} \circ \Lambda_{tr})_* &:= \frac{1}{4}[(\Delta_{X\times X} + (-1)^{n+1}\Gamma_\iota) \circ (\Pi_{n,0} \times \Pi_{n,0}) \circ (\Delta_{X\times X} + (-1)^{n+1}\Gamma_\iota) \circ (\Pi_{n,0} \times \Pi_{n,0})]_* \\ &= [(\Lambda \circ \Lambda) \circ (\Pi_{n,0} \times \Pi_{n,0}) \circ (\Pi_{n,0} \times \Pi_{n,0})]_* = \Lambda_*(\Pi_{n,0} \times \Pi_{n,0})_* = (\Lambda_{tr})_*\end{aligned}$$

where the second equality follows from the fact that Λ and $\Pi_{n,0}$ commute (a fact that can either be checked by hand, or deduced from the commutativity between Γ_ι and $\Pi_{n,0}$, which in turn follows from [37, Lemma 3.4]), while the third equality follows from Equation (2). □

We prove some intermediate results.

Lemma 4.5. *In the situation of Theorem* 4.1*, the correspondence Λ_{tr} acts on the cohomology as a projector on the subspace $\wedge^2 H^n_{tr}(X) \subset H^{2n}(X \times X)$.*

Proof. First we observe that $\Pi_{n,0} \times \Pi_{n,0}$ acts as projector onto $H^n_{tr}(X) \otimes H^n_{tr}(X)$. For $\beta, \beta' \in H^n_{tr}(X)$ we have

$$(\Delta_{X\times X} + \Gamma_\iota)_*(\beta \otimes \beta') = \beta \otimes \beta' + (-1)^{n+1}\beta' \otimes \beta \ \in H^{2n}(X \times X).$$

This shows that an element in $(\Lambda_{tr})_* H^*(X \times X)$ can be written as a sum of tensors of type

$$\beta \otimes \beta' + (-1)^{n+1}\beta' \otimes \beta$$

with $\beta, \beta' \in H^n_{tr}(X)$. Since the cup-product map $H^n(X) \otimes H^n(X) \to H^{2n}(X)$ is $(-1)^{n^2}$-commutative, tensors of this type correspond exactly to elements of

$$\{b \in \mathrm{Im}(H^n_{tr}(X) \otimes H^n_{tr}(X) \to H^{2n}(X \times X)) \mid \iota_*(b) = -b\}.$$

Thus $(\Lambda_{tr})_* H^*(X \times X) \cong \wedge^2 H^n_{tr}(X) \subset H^{2n}(X \times X)$. □

Remark 4.6. Just to fix ideas, let us suppose for a moment that X and X' coincide, so that $\Pi_{n,0}$ (and hence Λ_{tr}) is idempotent. In this case, Λ_{tr} defines the Chow motive $\mathrm{Sym}^2\, t_n(X) \in \mathcal{M}_{\mathrm{rat}}$ in the language of [37, Definition 3.5], where $t_n(X)$ is the "transcendental motive" $(X, \Pi_{n,0}, 0)$ as in Remark 2.15.

The next lemma ensures that Λ and Λ_{tr} have the same action on the 0-cycles that we are interested in. This is the only place in the proof where we need the full force of hypothesis (iii).

Lemma 4.7. *In the situation of Theorem* 4.1, *let*

$$A^{(n,n)} := \mathrm{Im}(A^n(X) \otimes A^n(X) \xrightarrow{\times} A^{2n}(X \times X)) \subset A^{2n}(X \times X)$$

and let

$$A^{(2,2)} := \mathrm{Im}(A^2_{AJ}(X) \otimes A^2_{AJ}(X) \xrightarrow{\times} A^4(X \times X)) \subset A^4(X \times X)$$

(*where* $\times$ *denotes the map sending* $a \otimes a'$ *to* $a \times a'$). *Then for any choice of* $\Pi_{n,0}$ *as in Theorem* 2.12 *we have*

$$(\Lambda_{tr})_{*|A^{(n,n)}} = \Lambda_{*|A^{(n,n)}} \quad \textit{and} \quad (\Lambda_{tr})_{*|A^{(2,2)}} = \Lambda_{*|A^{(2,2)}}.$$

Proof. The point is that according to Theorem 2.12 there is a decomposition

$$\Delta_X = \Pi_{n,0} + \sum_{(i,j)\neq(n,0)} \Pi_{i,j} \quad \text{in } A^n(X \times X).$$

We claim that the components $\Pi_{i,j}$ with $(i,j) \neq (n,0)$ do not act on $A^n(X)$:

$$(\Pi_{i,j})_* A^n(X) = 0 \quad \text{for all } (i,j) \neq (n,0).$$

Indeed, $\Pi_{i,j}$ may be chosen to factor over a variety Z of dimension $i - 2j$ (by Theorem 2.12). Hence the action of $\Pi_{i,j}$ on $A^n(X)$ factors as follows:

$$(\Pi_{i,j})_* : A^n(X) \to A^{i-j}(Z) \to A^j(X).$$

Our hypotheses imply that any $\Pi_{i,j}$ different from $\Pi_{n,0}$ has $j > 0$. Thus, the group in the middle is 0 (for dimension reasons), and the claim is proven.

We now consider the diagonal $\Delta_{X\times X}$ of the self-product $X \times X$. There is a decomposition

$$\Delta_{X\times X} = \sum_{i,j,i',j'} \Pi_{i,j} \times \Pi_{i',j'} \quad \text{in } A^{2n}(X^4).$$

Let $a, a' \in A^n(X)$. Using the claim, we find that

$$(\Pi_{i,j} \times \Pi_{i',j'})_*(a \times a') = (\Pi_{i,j})_*(a) \times (\Pi_{i',j'})_*(a') = 0 \quad \text{for } (i,j,i',j') \neq (n,0,n,0).$$

It follows that

$$a \times a' = (\Delta_{X\times X})_*(a \times a') = (\Pi_{n,0} \times \Pi_{n,0})_*(a \times a') \quad \text{in } A^{2n}(X \times X),$$

which proves the $A^{(n,n)}$ statement.

The second statement of Lemma 4.7 is proven similarly: we claim that the components $\Pi_{i,j}$ with $(i,j) \neq (n,0)$ do not act on $A^2_{AJ}(X)$. This claim follows from the factorization

$$(\Pi_{i,j})_* : A^2_{AJ}(X) \to A^{2+i-j-n}_{AJ}(Z) \to A^2_2(X),$$

where $\dim Z = i - 2j$ (one readily checks that for $j > 0$ the middle group vanishes in all cases). □

We now have all the ingredients for the

Proof of Theorem 4.1. (For a related conjecture, the argument that follows was hinted at in [40, Remark 35].)

Consider the correspondence $\Lambda_{tr} \in A^{2n}(X^4)$. By Lemma 4.5 it acts on $H^*(X \times X)$ by projecting onto the 1-dimensional subspace $\wedge^2 H^n_{tr}(X) \subset H^{2n}(X \times X)$. This implies that there is a containment

$$\Lambda_{tr} \in (\wedge^2 H^n_{tr}(X)) \otimes (\wedge^2 H^n_{tr}(X)) \subset H^{4n}(X^4).$$

The 1-dimensional subspace $\wedge^2 H^n_{tr}(X)$ is contained in $(H^n_{tr}(X) \otimes H^n_{tr}(X)) \cap F^n$. The dominant map $X'' \dashrightarrow X$ induces a surjection $H^n_{tr}(X'') \to H^n_{tr}(X)$, thus classes in $\wedge^2 H^2_{tr}(X)$ come from Hodge classes in $H^n_{tr}(X'') \to H^n_{tr}(X)$. Using the truth of the Hodge conjecture for $X'' \times X''$, these classes are algebraic.

By hypothesis (iv), this subspace is algebraic, i.e. there is a codimension n subvariety $P \subset X \times X$ such that $\wedge^2 H^n_{tr}(X)$ is supported on P. This implies that $\Lambda_{tr} = \gamma$ in $H^{4n}(X^4)$, where γ is a cycle supported on $P \times P \subset X^4$. In other words, we have $\Lambda_{tr} - \gamma \in A^{2n}_{\text{hom}}(X^4)$.

Recall that $\Psi \in A^n(X' \times X)$ denotes the closure of the graph of the dominant rational map ψ from X' to X. The correspondence

$$\Gamma := ({}^t\Psi \times {}^t\Psi) \circ (\Lambda_{tr} - \gamma) \circ (\Psi \times \Psi) \ \in A^{2n}((X')^4)$$

is homologically trivial (because the factor in the middle is homologically trivial). Using finite-dimensionality and Theorem 2.3, we know there exists $N \in \mathbb{N}$ such that $\Gamma^{\circ N} = 0$ in $A^{2n}((X')^4)$. In particular, this implies that

$$(\Psi \times \Psi) \circ \Gamma^{\circ N} \circ ({}^t\Psi \times {}^t\Psi) = 0 \quad \text{in } A^{2n}(X^4).$$

Developing this expression, and applying the result to 0-cycles, and repeatedly using relation (1), we obtain

$$((\Lambda_{tr})^{\circ N})_* = (Q_1 + Q_2 + \cdots + Q_N)_* : \ A^{2n}(X \times X) \to A^{2n}(X \times X),$$

where each Q_j is a composition of Λ_{tr} and γ in which γ occurs at least once. Since Λ_{tr} is an idempotent, this simplifies to

$$(\Lambda_{tr})_* = (Q_1 + Q_2 + \cdots + Q_N)_* : \ A^{2n}(X \times X) \to A^{2n}(X \times X).$$

The correspondence γ acts trivially on $A^{2n}(X \times X)$ for dimension reasons, and so the Q_j likewise act trivially on $A^{2n}(X \times X)$. It follows that

$$(\Lambda_{tr})_* = (Q_1 + \cdots + Q_N)_* = 0 : \ A^{2n}(X \times X) \to A^{2n}(X \times X).$$

By Lemma 4.7 this ends the proof of Theorem 4.1. □

Remark 4.8. The above proof is somehow indirect as we prove the statement for the auxiliary correspondence Λ_{tr} and then check that its action on $A^n_{\text{hom}}(X) \otimes A^n_{\text{hom}}(X)$ coincides with that of Λ.

Remark 4.9. Hypothesis (i) of Theorem 4.1 may be weakened as follows: it suffices that there exists X' of dimension ≤ 5 such that X' has finite-dimensional motive and $B(X')$ is true, and there exists a correspondence from X' to X inducing a surjection $A^i(X') \twoheadrightarrow A_0(X)$. The argument is similar.

Remark 4.10. We have seen (Remark 3.6) that n-dimensional manifolds with $\dim H^n_{tr}(X) = 2$ are higher-dimensional analogues of ρ-maximal surfaces. In [41, Proposition 5] it is shown that surfaces S with finite-dimensional motive and $\dim H^2_{tr}(S) = 2$ (i.e. $p_g = 1$ and S is ρ-maximal) verify Voisin's conjecture. Theorem 4.1 is a higher-dimensional analogue of this result.

Remark 4.11. Following Voisin's approach [68] one can extend the analysis above to 0-cycles on higher products of X with itself. In this direction we get the following.

Theorem 4.12. *Let X be a smooth projective variety of dimension n less than or equal to* 5. *Assume further that $h^{i,0}(X) = 0$ for $0 < i < n$ and that $p_g(X) \leq 2$. Suppose moreover that*

(1) *X is rationally dominated by a variety X', that X' has finite dimensional motive and that $B(X')$ is true;*

(2) *the dimension of $H^n_{tr}(X)$ is at most* 4;

(3) $\tilde{N}^1 H^i(X) = H^i(X)$ *for* $0 < i < n$;

(4) *the Hodge conjecture is true for* X^4.

Then any $a_1, a_2, a_3, a_4 \in A^n_{\mathrm{hom}}(X)$ *satisfy*

$$\sum_{\sigma \in \mathfrak{S}_4} \varepsilon(\sigma)\sigma^*(a_1 \times a_2 \times a_3 \times a_4) = 0 \quad \text{in } A^{4n}(X \times X \times X \times X).$$

Proof. The proof closely follows that of Theorem 4.1. In that situation, we took into account $\Lambda^2(H^n_{tr}(X))$ and then described a generator of it via an explicit cycle that is induced by a correspondence. In this situation, it is possible to give a generator of the 1-dimensional space $\Lambda^4(H^n_{tr}(X))$. The rest of the proof is similar to that of Theorem 4.1. □

Note that Theorem 4.12 is not optimal in all cases, since for $p_g = 2$ one expects relations in $X \times X \times X$ and not in $X \times X \times X \times X$.

Conjecturally, any variety X with $h^{2,0}(X) = 0$ should have $A^2_{AJ}(X) = 0$ (this would follow from the Bloch–Beilinson conjectures, or from a strong form of Murre's conjectures). We cannot prove this for all varieties with $p_g(X) > 0$ (such as the Fermat sextic fourfold). However, the above argument at least gives a weaker statement concerning $A^2_{AJ}(X)$:

Proposition 4.13. *Let X be as in Theorem* 4.1. *Then for any* $a, a' \in A^2_{AJ}(X)$ *we have* $a \times a' = -a' \times a$ *in* $A^4(X \times X)$.

Proof. This is really the same argument as for Theorem 4.1. We have proven that there is a rational equivalence

$$\Lambda_{tr} = (\Lambda_{tr})^{\circ N} = Q_1 + Q_2 + \cdots + Q_N \quad \text{in } A^6(X^4),$$

where each Q_j is a composition of Λ_{tr} and γ in which γ occurs at least once. The correspondence γ does not act on $A^4(X \times X)$ for dimension reasons (it factors over $A^4(P)$ where $\dim P = 3$), and so the Q_j do not act on $A^4(X \times X)$. It follows that

$$(\Lambda_{tr})_* = (Q_1 + \cdots + Q_N)_* = 0\colon\ A^4(X \times X) \to A^4(X \times X).$$

On the other hand, we know from Lemma 4.7 that

$$\Lambda_* = (\Lambda_{tr})_* = 0\colon\ \mathrm{Im}(A^2_{AJ}(X) \otimes A^2_{AJ}(X) \to A^4(X \times X)) \to A^4(X \times X).$$

This means that for any $a, a' \in A^2_{AJ}(X)$ we have $\Lambda_*(a \times a') = a \times a' + a' \times a = 0$ in $A^4(X \times X)$. □

5 Applications

In this section we apply our general result to some Calabi–Yau varieties X with dimensions between 2 and 5. First, we give new examples of ρ-maximal surfaces. Then we focus on dimension 3, where we give examples of different types. In some cases we prove Voisin's Conjecture as stated in (1.2); in other ones we get the generalization of it on $X \times X \times X \times X$ that appears in Theorem 4.12. Remarkably, one can often study the dimensions of the $H^n_{tr}(F)$ for a Fermat-type hypersurface F in certain weighted projective spaces by looking at the (topological) mirror of F. Finally, the conjecture is proved in dimension 4 for the Fermat sextic fourfold and in dimension 5 for some Calabi–Yau varieties studied in [16].

5.1 Examples of dimension 2 ?.

Remark 5.1. Many examples of surfaces satisfying the conditions (i), (ii, (iii) of Theorem 4.1 can be found in [5]. Indeed (as explained to us by Roberto Pignatelli), the "duals" (cf. [5, Section 9]) of the 14 families in [5, Table 2] are ρ-maximal surfaces with $p_g = 1$ and $q = 0$. Being rationally dominated by a product of curves, these surfaces have finite-dimensional motives. We do not know whether condition (iv) holds for these surfaces, so we are not sure whether Theorem 4.1 applies to these surfaces.

5.2 Examples of dimension 3 of Fermat type: weak version. Let us consider some examples of Calabi–Yau 3-folds. Recall that in dimension 3 the notions of N^1-maximality and $\tilde{N}^1$-maximality coincide by Remark 3.6. One of the examples is the Fermat quintic F_5 in 4-dimensional projective space, which we work out in full detail. We also consider other Fermat type 3-folds in weighted projective spaces (for the basics on weighted projective spaces see e.g. [21]).

A different example is taken in [61] and is a small resolution Y' of a complete intersection Y of type $(2,2,2,2)$ in 7-dimensional projective space. For the Fermat type examples, we show that $\dim H^3_{tr} = 4$; in the latter example we do not know whether the dimension of $H^3_{tr}(Y')$ is 2 or 4. If it were 2, we could apply our main result and get another example for which Voisin’s conjecture holds. If it is 4, as in the case of F_5, we can still deduce something interesting, namely a weak version of Voisin’s conjecture thanks to Theorem 4.12.

We start by collecting some useful facts.

Lemma 5.2. *Every Fermat hypersurface $\{\sum x_i^d = 0\} \subset \mathbb{P}^n$ has finite-dimensional motive.*

Proof. A Fermat hypersurface is rationally dominated by curves by the Katsura–Shioda inductive structure, see [57], [59, Section 1]. The analysis of the indeterminacy locus shows, cf. [27], that this implies that its motive is finite-dimensional. □

Theorem 5.3 (Shioda [57] Theorem IV). *Let X be a Fermat hypersurface of degree $d \le 20$. Then the Hodge conjecture is true for X^r for all $r \in \mathbb{N}$.*

Theorem 5.4 (Shioda [58]). *Let X be a Fermat threefold of degree $d \le 10$. Then the generalized Hodge conjecture is true for X.*

Proof. This is [58, §3 point (13)]. (This has recently been generalized to Fermat 3-folds of arbitrary degree d, see [36], but we do not need this generalization here.) □

Consider now the Fermat quintic hypersurface

$$X := \{x_0^5 + \cdots + x_4^5 = 0\} \subset \mathbb{P}^4.$$

(Later in the paper we also denote the Fermat quintic hypersurface by F_5.) Its Hodge numbers are

$$h^{2,1}(X) = 101, \quad h^{1,1}(X) = 1 = h^{3,0}(X).$$

Its “mirror” $\hat{X}$ has been constructed explicitly in [26; 14] as follows. Inside the quotient $(\mathbb{Z}/5\mathbb{Z})^5/\mathrm{diag}$ of $(\mathbb{Z}/5\mathbb{Z})^5$ under the natural diagonal action, consider the subgroup

$$G := \{(a_0, \dots, a_4) \mid \textstyle\sum_i a_i = 0\}.$$

This subgroup G, which is abstractly isomorphic to $(\mathbb{Z}/5\mathbb{Z})^3$, acts on X; by [45, Proposition 4] and [54, Proposition 2] the quotient X/G possesses a Calabi–Yau resolution $\hat{X}$, in other words we have the following diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow p \\ \hat{X} & \xrightarrow{f} & X' := X/G. \end{array}$$

Notice that the automorphisms $\sigma \in G$ satisfy

$$\sigma^* = \mathrm{id}\colon H^{3,0}(X) \to H^{3,0}(X).$$

The variety $\hat{X}$ turns out to be *the mirror* of X, see e.g. [48; 69] for more explanations and details (the analogous construction and the same result hold for any smooth member of the Dwork pencil). In particular its Hodge numbers are

$$h^{1,1}(X) = 101, \quad h^{2,1}(X) = 1 = h^{3,0}(X).$$

First of all, as observed in Remark 2.8, X verifies $B(X)$ (because it is a projective hypersurface) and has finite-dimensional motive by Lemma 5.2.

We note that X' is a quotient variety X/G for a finite group G. As such, there is a well-defined theory of correspondences with rational coefficients for X' (this is because X' has $A^*(X') \cong A_{3-*}(X')$ where A_* denotes Chow groups and A^* denotes operational Chow cohomology; see [24, Example 17.4.10], [24, Example 16.1.13]).

We denote by $\Gamma := {}^t\Gamma_f \circ \Gamma_p \in A^3(X \times \hat{X})$ the natural correspondence from X to $\hat{X}$.

Zero-cycles on X and $\hat{X}$ can be related as follows:

Proposition 5.5. *There is an isomorphism of Chow motives $\Gamma\colon t_3(X) \cong t_3(\hat{X})$ in $\mathcal{M}_{\mathrm{rat}}$, with inverse given by $\frac{1}{d}\,{}^t\Gamma$, where d is the order of G. In particular, the homomorphisms*

$$f^*p_*\colon A^3(X) \to A^3(\hat{X}) \quad \text{and} \quad p^*f_*\colon A^3(\hat{X}) \to A^3(X)$$

are isomorphisms.

Proof. As we have seen, X satisfies $B(X)$ and has finite-dimensional motive. Moreover, the generalized Hodge conjecture holds for X by [58]. The proposition now follows from the proof of [40, Corollary 29(i)]. □

Thanks to Proposition 5.5, much information can be transported from X to $\hat{X}$, and vice versa. For example, if $B(X)$ holds then $B(\hat{X})$ holds, because

$$h(\hat{X}) = t_3(\hat{X}) \oplus h(C) \oplus \bigoplus_j \mathbb{L}(m_j) \quad \text{in } \mathcal{M}_{\mathrm{rat}},$$

where C is a (not necessarily connected) curve. Likewise, if X has finite-dimensional motive then $\hat{X}$ has finite-dimensional motive. Alternatively, $B(\hat{X})$ can be proven by invoking the main result of [60], and the finite-dimensionality of the motive of $\hat{X}$ can also be derived from [66, Example 3.15] and the fact that $\hat{X}$ is rationally dominated by a product of curves (as X is).

Lemma 5.6. *Let X be the Fermat quintic in $\mathbb{P}^4$. Then the dimension of $H^3_{tr}(X)$ is 4.*

Proof. Take the order 5 automorphism that permutes the coordinates of $\mathbb{P}^4$. This descends to X and commutes with the elements of the group G of order 125. Therefore, there exists an order 5 automorphism of the mirror $\hat{X}$ acting on the 4-dimensional space of degree 3 rational cohomology. This space splits into four eigenspaces of such an automorphism, namely

$$H^3(\hat{X}, \mathbb{Q}) = V(\eta) \oplus V(\eta^2) \oplus V(\eta^3) \oplus V(\eta^4),$$

where η is a primitive fifth root of unity. Up to renaming the primitive root of unity, we can assume that $H^{3,0}(\hat{X}) \oplus H^{0,3}(\hat{X}) \simeq V(\eta) \oplus V(\eta^4)$, which is not defined over the field of rational numbers. Therefore, by Proposition 3.3 we have that $\dim H^3_{tr}(\hat{X}) = 4$. As the isomorphism of Hodge structures induced by Γ yields an isomorphism between $H^3_{tr}(\hat{X})$ and $H^3_{tr}(X)$, the lemma is proved. □

Proposition 5.7. *The hypotheses of Theorem 4.12 hold for the following Calabi–Yau 3-folds:*

(1) *the Fermat quintic F_5 and its mirror;*
(2) *the Fermat hypersurface $x_0^8+x_1^8+x_2^8+x_3^8+x_4^2 = 0$ in the weighted projective space $\mathbb{P}(1^4, 4)$ and its mirror;*
(3) *the Fermat hypersurface $x_0^{10}+x_1^{10}+x_2^{10}+x_3^5+x_4^2 = 0$ in the weighted projective space $\mathbb{P}(1^3, 2, 5)$ and its mirror;*
(4) *the Fermat hypersurface $x_0^8+x_1^8+x_2^4+x_3^4+x_4^4 = 0$ in the weighted projective space $\mathbb{P}(1^2, 2^3)$ and its mirror.*

Proof. The claim follows for the Fermat quintic due to Lemma 5.6 and the fact that Fermat hypersurfaces have finite-dimensional motives by Lemma 5.2. For Examples (2), (3), (4), note that they are dominated by Fermat hypersurfaces; the hypotheses (i) and (iv) thus follow from Lemma 5.2 and Theorem 5.3. The fact that $\dim H^3_{tr}(X) \le 4$ is established in [34, Examples 5.3, (c), (d) and Table 4]. As for the mirror partners, one can directly check that the hypotheses of Theorem 4.12 are satisfied. □

Remark 5.8. Note that the N^1-maximality is also connected to modularity conditions. For instance, Hulek and Verrill in [29] investigate Calabi–Yau threefolds over the field of rational numbers that contain birational ruled elliptic surfaces S_j for $j = 1, \dots, b$, where b is the dimension of $H^{1,2}(X)$. As they show, this is equivalent to the N^1-maximality. Under these assumptions, the L-function of X factorizes as a product of the L-functions of the base elliptic curves of the birational ruled surfaces and the L-function of the weight 4 modular form associated with the 2-dimensional Galois representation given by the kernel U of the exact sequence

$$0 \to U \to H^3_{\text{et}}(\overline{X}, \mathbb{Q}_l) \to \bigoplus H^3_{\text{et}}(\overline{S}_j, \mathbb{Q}_l) \to 0.$$

In [29], Section 3, examples of this type of Calabi–Yau varieties are given; however, we do not know whether they have finite dimensional motive.

5.3 Example of dimension 3 of Fermat type: strong version. The main result of this subsection is the following.

Proposition 5.9. *Let X be the hypersurface* $\{[x_0 : x_1 : x_2 : x_3 : x_4] \mid x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_4^3 = 0\}$ *in the weighted projective space* $\mathbb{P}^4(1, 1, 1, 1, 2)$. *Then Conjecture* 1.2 *holds for X.*

Proof. It is easy to check that X is a smooth Calabi–Yau variety. Moreover, it can be realized as a degree 3 finite covering of $\mathbb{P}^3$ branched over the Fermat sextic surface. As such, X has an order 3 automorphism, say τ. This also shows that is rationally dominated by a product of curves; hence it has finite-dimensional motive. It remains to prove the N^1-maximality stated in Theorem 4.1. This is proven in [47, Section 8.3.1, Example 1], and also follows readily from [34, Example 5.3, (b)]; we propose a more direct proof:

Note that X can be thought of as the quotient of the degree 6 Fermat threefold $\{Y_1^6 + Y_2^6 + Y_3^6 + Y_4^6 + Y_5^6 = 0\}$ in 4-dimensional projective space by the action of the group generated by the automorphism $[Y_1 : Y_2 : Y_3 : Y_4 : Y_5] \mapsto [Y_1 : Y_2 : Y_3 : Y_4 : -Y_5]$.

The Hodge numbers of X are given by $(h^{1,1}(X), h^{1,2}(X)) = (1, 103)$. As explained in [34], the (topological) mirror of X can be described as follows. Take the group

$$\widehat{G} := \{(\varepsilon_6^{i_0}, \varepsilon_6^{i_1}, \varepsilon_6^{i_2}, \varepsilon_6^{i_3}, \varepsilon_3^{i_4}) : i_0 + i_1 + i_2 + i_3 + 2i_4 \equiv 0 \bmod 6\}/H,$$

where H is a diagonal copy of $\mathbb{Z}/6\mathbb{Z}$ that acts trivially on the weighted projective space $\mathbb{P}(1, 1, 1, 1, 2)$. We consider the polynomials

$$\sum_{I=(i_0,i_1,i_2,i_3,i_4)} C_I x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4} + \lambda x_0 x_1 x_2 x_3 x_4 \tag{3}$$

where λ varies in $\mathbb{A}^1$, the sum ranges over all solutions of the equation $i_0 + i_1 + i_2 + i_3 + 2i_4 \equiv 0 \bmod 6$ and the C_I are generic complex numbers. The vanishing of these polynomials defines a pencil of varieties X'_λ in $\mathbb{P}(1, 1, 1, 1, 2)$ that is $\widehat{G}$-invariant. Note that the members of it are smooth for a generic choice of λ because they do not contain the singular point of the weighted projective space. A mirror family of X can be found analogously to that of the mirror Fermat quintic by taking the quotient of the pencil (3) by the group $\widehat{G}$ and then taking a crepant resolution. We denote by $\widehat{X}$ a crepant resolution of X'_0.

Now we consider the order four automorphism τ of $\mathbb{P}(1, 1, 1, 1, 2)$ given by $[x_0, x_1, x_2, x_3, x_4] \mapsto [x_1, x_2, x_3, x_4, x_0]$. An easy computation shows that τ belongs to the normalizer of $\widehat{G}$ in the group of automorphisms of $\mathbb{P}(1, 1, 1, 1, 2)$. Moreover, there exist complex numbers C_I such that X'_0 is invariant with respect to τ. Finally, for such a choice the fixed locus of $\widehat{G}$ is invariant with respect to the τ-action because τ normalizes $\widehat{G}$. Since τ permutes the homogeneous coordinates of $\mathbb{P}(1, 1, 1, 1, 2)$, it extends to all the members of the mirror family, which by definition means that τ is maximal. Moreover, a direct computation shows that any λ is mapped to itself. The space of invariants of $H^{1,2}(X)$ with respect to the $\widehat{G}$-action is thus one-dimensional; hence τ induces the identity on $H^{1,2}(\widehat{X}) \oplus H^{2,1}(\widehat{X})$. It remains to understand the action induced by τ on $H^{3,0}(\widehat{X}) \oplus H^{0,3}(\widehat{X})$. For this purpose, we recall that a generator of $H^{3,0}(\widehat{X})$ is a 3-form on X that is invariant with respect to $\widehat{G}$; recall that $\widehat{X}$ is a crepant resolution of $X'_0 = X/\widehat{G}$. More precisely, this 3-form can be described as a ratio in which the denominator is $\widehat{G}$-invariant by definition and the numerator is given as follows:

$$\begin{aligned} &x_0 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 - x_1 dx_0 \wedge dx_2 \wedge dx_3 \wedge dx_4 + x_2 dx_0 \wedge dx_1 \wedge dx_3 \wedge dx_4 \\ &-x_3 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_4 + 2x_4 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

It is easy to check that this polynomial is mapped to its opposite by the induced action of τ. Therefore, the action on the group $H^{3,0}(\widehat{X}) \oplus H^{0,3}(\widehat{X})$ is the opposite of the identity.

To recap, the action of $\widehat{\tau}$ on the space $H^3(\hat{X}, \mathbb{Q})$ induces a splitting into two eigenspaces of dimension two, one with eigenvalue $+1$ and one with eigenvalue -1. The second eigenspace has strictly positive Hodge level, and so (using the truth of the generalized Hodge conjecture for $\hat{X}$, which follows from Theorem 5.4 as $\hat{X}$ is rationally dominated by a degree 6 Fermat hypersurface in $\mathbb{P}^4$) the second eigenspace lies in N^1. This shows the N^1-maximality for the Calabi–Yau threefold $\widehat{X}$ and accordingly, for X because their H^3_{tr}'s are isomorphic via an isomorphism of Hodge structures. □

Remark 5.10. This example is not new; yet the proof of the N^1-maximality is more geometric than those in [34] and [47]. In the former reference, the authors prove the maximality by describing two Fermat motives.

5.4 Examples of dimension 3 of Borcea–Voisin type: strong version. Let E be the elliptic curve given by the equation $y^2 = x^3 - 1$. This curve admits an order three automorphism $h(x, y) = (\omega x, y)$, where ω is a primitive third root of unit. Now take S to be a $K3$ surface with an order three automorphism g such that the second cohomology group with rational coefficients splits as the transcendental part T_S and the Neron Severi group such that $H^{2,0}(S) \subseteq T_S$ and the rank of $NS(S)$ is 20. Moreover, the Neron Severi group coincides with the subspace of invariant classes of $H^2(S, \mathbb{Q})$ with respect to the action of g. In particular g is antisymplectic. Such a K3 surface exists as shown in [7, p. 280].

The product $S \times E$ admits the order three automorphism $g \times h$. We may assume that the action of g on the period of S is given by multiplication by ω^2 (if not, just take the inverse of g). Note that the fixed point locus of g consists of isolated points and (smooth) rational curves.

Denote by X a resolution of the (singular) quotient $S \times E$ by the group generated by the automorphism $g \times h$. By the description of the fixed locus of $g \times h$, the third cohomology group of X with rational coefficients is the invariant part of $H^3(S \times E, \mathbb{Q})$, which is isomorphic to $H^2(S, \mathbb{Q}) \otimes H^1(E, \mathbb{Q})$. To prove the N^1-maximality, we check the equivalent condition that $H^{3,0}(S \times E) \oplus H^{0,3}(S \times E)$ is defined over the field of rational numbers. By the Künneth formula we have

$$H^{3,0}(S \times E) \oplus H^{0,3}(S \times E) \simeq H^{2,0}(S) \otimes H^{1,0}(E) \oplus H^{0,2}(S) \otimes H^{0,1}(E),$$
$$H^{2,1}(S \times E) \oplus H^{1,2}(S \times E) \simeq H^{1,1}(S) \otimes H^{1,0}(E) \oplus H^{1,1}(S) \otimes H^{0,1}(E) \oplus H^{0,2}(S) \otimes H^{1,0}(E) \oplus H^{2,0}(S) \oplus H^{0,1}(E).$$

The space $H^{3,0}(S \times E) \oplus H^{0,3}(S \times E)$ is defined over the rational field because it can be defined as the subspace of invariants with respect to the action of the isomorphism $(g \times h)^*$ on $H^3(S \times E, \mathbb{Q})$. Indeed, the action of this isomorphism is trivial on $H^{2,0}(S) \otimes H^{1,0}(E) \oplus H^{0,2}(S) \otimes H^{0,1}(E)$. As for $H^{2,1}(S \times E) \oplus H^{1,2}(S \times E)$, the action is by multiplication by $\omega, \omega^2, \omega^2, \omega$ on $H^{1,1}(S) \otimes H^{1,0}(E)$, $H^{1,1}(S) \otimes H^{0,1}(E)$, $H^{0,2}(S) \otimes H^{1,0}(E)$, $H^{2,0}(S) \oplus H^{0,1}(E)$ respectively, because the action of g on $H^{1,1}(S)$ is trivial.

As for hypothesis (iv) of the criterion, we observe that any K3 surface with Picard number 20 is either a Kummer surface, or rationally dominated by a Kummer surface. Then, we just need to know the Hodge conjecture for $A \times A \times E \times E$, where A is an abelian surface and E an elliptic curve; this is known e.g. by [46, Theorem 0.1, item (iv)].

5.5 The Fermat 4-fold: strong version. We already know that every Fermat hypersurface $\{\sum x_i^d = 0\} \subset \mathbb{P}^n$ has finite-dimensional motive. As the Lefschetz standard conjecture holds for hypersurfaces and the hypothesis $\widetilde{N}^1 H^3(X) = H^3(X)$ also holds for a 4-dimensional hypersurface, in order to prove Theorem 4.1 we are left with the $\widetilde{N}^1$-maximality.

Proposition 5.11. *The Fermat sextic fourfold is $\widetilde{N}^1$-maximal.*

Proof. We will use that

(a) the Fermat sextic surface $S \subset \mathbb{P}^3$ is ρ-maximal, see [6, Corollary 1], and

(b) the Fermat sextic 4-fold $X \subset \mathbb{P}^5$ is $\widetilde{N}^2$-maximal, i.e. $\widetilde{N}^2 H^4(X) \otimes \mathbb{C} = H^{2,2}(X)$, see [49, Corollary 15.11.1].

Consider the dominant rational morphism $\varphi : S \times S \dashrightarrow X$. It yields a surjective morphism of Hodge structures

$$\varphi_* : H^4_{tr}(S \times S) \twoheadrightarrow H^4_{tr}(X). \tag{4}$$

Now $H^4_{tr}(S \times S) \subset H^2_{tr}(S) \times H^2_{tr}(S)$. By item (a) above $H^2_{tr}(S) \otimes \mathbb{C} = H^{2,0}(S) \oplus H^{0,2}(S)$. This together with (4), implies that

$$(H^4_{tr}(X) \otimes \mathbb{C}) \subset H^{4,0}(X) \oplus H^{2,2}(X) \oplus H^{0,4}(X).$$

By item (b) we see that there exists a non-empty Zariski open $\tau\colon U \subset X$ (defined as the complement of the span of the codimension 2 cycle classes in $H^4(X, \mathbb{Q})$) such that $H^{2,2}(X)$ maps to 0 under the restriction map

$$\tau^*\colon H^4(X, \mathbb{C}) \to H^4(U, \mathbb{C}).$$

This implies that

$$\tau^*(H^4_{tr}(X) \otimes \mathbb{C}) \subset \tau^*(H^{4,0}(X) \oplus H^{0,4}(X)),$$

and so the restriction $\tau^*(H^4_{tr}(X) \otimes \mathbb{C})$ has dimension at most 2. On the other hand, by definition of $H^4_{tr}(\ )$ the map $\tau^*\colon H^4_{tr}(X) \to H^4(U)$ is an injection. Therefore, we conclude that $\dim(H^4_{tr}(X) \otimes \mathbb{C}) = 2$, i.e. X is N^1-maximal.

To establish the $\widetilde{N}^1$-maximality, it remains to show that the inclusion $\widetilde{N}^1 H^4(X) \subset N^1 H^4(X)$ is an equality. Here, we again use the dominant rational map φ. The indeterminacy of the map φ is resolved by the blow-up $\widetilde{S \times S}$ with center $C \times C$ (where $C \subset S$ is a curve). It thus suffices to prove the equality

$$\widetilde{N}^1 H^4(\widetilde{S \times S}) = N^1 H^4(\widetilde{S \times S}).$$

The blow-up formula gives an isomorphism

$$H^4(\widetilde{S \times S}) = H^4(S \times S) \oplus H^2(C \times C),$$

and the second summand is entirely contained in $\widetilde{N}^1$. It thus suffices to prove the equality

$$\widetilde{N}^1 H^4(S \times S) \stackrel{??}{=} N^1 H^4(S \times S). \tag{5}$$

This readily follows from the N^1-maximality of S: indeed, there is a decomposition $H^2(S) = T \oplus N$, where $T := H^2_{tr}(S)$ is such that $T \otimes \mathbb{C} = H^{2,0} \oplus H^{0,2}$. This induces a decomposition

$$H^4(S \times S) = T \otimes T \oplus N \otimes T \oplus T \otimes N \oplus N \otimes N \oplus H^0(S) \otimes H^4(S) \oplus H^4(S) \otimes H^0(S).$$

All summands except the first one are obviously contained in $\widetilde{N}^1$ (because $D \times S$ satisfies the standard conjecture B, for any divisor $D \subset S$). As for the first summand, we note that

$$(T \otimes T)_{\mathbb{C}} \subset H^{4,0} \oplus H^{2,2} \oplus H^{0,4},$$

and so

$$N^1(T \otimes T) = (T \otimes T) \cap F^2 = N^2(T \otimes T) = \widetilde{N}^2(T \otimes T),$$

since the Hodge conjecture is true for $S \times S$ by [57, Theorem IV]. This proves the equality (5), and hence the $\widetilde{N}^1$-maximality of X is established. □

To finish we observe that all the hypotheses of Theorem 4.1 are satisfied for a Fermat sextic fourfold, hence Conjecture 1.2 holds for it.

5.6 Examples of dimension 5.

Proposition 5.12 (Cynk–Hulek [16]). *Let E be an elliptic curve with an order 3 automorphism, and let n be a positive integer. There exists a Calabi–Yau variety X of dimension n which is rationally dominated by E^n, and which has* $\dim H^n(X) = 2$ *if n is even, and* $\dim H^n_{tr}(X) = 2$ *if n is odd.*

Proof. This is [16, Theorem 3.3]. The construction is also explained in [28, Section 5.3]. □

Proposition 5.13. *Let X be a Calabi–Yau variety as in Proposition* 5.12*, of dimension $n \leq 5$. Then conjecture* 1.2 *is true for X.*

Proof. We check that the conditions of Theorem 4.1 are satisfied. Point (i) is obvious, as X is rationally dominated by a product of curves. Point (ii) is taken care of by Proposition 5.12. Point (iii) is proven (in a more general set-up) in [42, Proof of Corollary 4.1]. Point (iv) holds since the Hodge conjecture is known for self-products of elliptic curves E^r; see [58]. □

6 Questions

Question 6.1. Let F_d denote the Calabi–Yau Fermat hypersurface of degree d in $\mathbb{P}^{d-1}$, i.e. the hypersurface defined by $x_0^d + x_1^d + \cdots + x_{d-1}^d = 0$. The variety F_d is $\widetilde{N}^1$-maximal for $d = 4$ and for $d = 6$. Are these the only two values of d for which F_d is $\widetilde{N}^1$-maximal?

We suspect that this might be the case (by analogy with the ρ-maximality of Fermat surfaces in $\mathbb{P}^3$: as remarked in [6], the only ρ-maximal Fermat surfaces are in degree 4 and 6), but we have no proof.

Question 6.2. Let $\{X_\lambda\}$ denote the Dwork pencil of Calabi–Yau quintic threefolds given by

$$x_0^5 + x_1^5 + \cdots + x_4^5 + \lambda x_0 x_1 x_2 x_3 x_4 = 0.$$

As we have seen, the central fibre X_0 has $\dim H^3_{tr}(X_0) = 4$. Are there values of λ where $\dim H^3_{tr}(X_\lambda)$ drops to 2? Are these values dense in $\mathbb{P}^1$?

Also, can one somehow prove finite-dimensionality of the motive for non-zero values of λ? (This seems to be difficult: as noted in [34, Remark 4.3], the varieties X_λ are *not* dominated by a product of curves outside of $\lambda = 0$.)

Acknowledgements: We wish to thank Giuseppe Ancona, Lie Fu, Bert van Geemen, Hossein Movasati, Roberto Pignatelli and Charles Vial for useful and stimulating exchanges related to this paper. We warmly thank Claire Voisin for pointing out a mistake in a previous version of the paper, and the referee for several helpful and highly pertinent remarks.

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