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MONOTONICITY AND TOTAL BOUNDEDNESS IN SPACES OF "MEASURABLE" FUNCTIONS

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Cordially dedicated to Professor Paolo de Lucia on the occasion of his 80th birthday anniversary

(*Communicated by Anna De Simone*)

ABSTRACT. We define and study the moduli $d(x, \mathcal{A}, D)$ and $i(x, \mathcal{A}, D)$ related to monotonicity of a given function x of the space $L_0(\Omega)$ of real-valued "measurable" functions defined on a linearly ordered set Ω. We extend the definitions to subsets X of $L_0(\Omega)$, and we use the obtained quantities, $d(X)$ and $i(X)$, to estimate the Hausdorff measure of noncompactness $\gamma(X)$ of X. Compactness criteria, in special cases, are obtained.

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1. Introduction

Measures of noncompactness and some other quantities which measure the lack of a given property for functions or sets of functions have a relevant role in the theory of real-valued functions and in nonlinear analysis (among a large literature, we cite [1, 3, 4, 6, 7, 10, 12, 13] and references therein). Moreover for applications it is very useful to have precise formulas or estimates for the Hausdorff measure of noncompactness. In the frame of the theory of measures of noncompactness some quantities related to monotonicity of functions have been considered by some authors in [2,8,9], for real-valued functions defined and bounded on a compact interval of the reals. The main aim of this paper is to consider similar quantities in a more general setting. Given $(\Omega, \leq)$ a linearly ordered nonempty set, we deal with the space $L_0(\Omega)$ of real-valued functions defined on Ω, which depends on an algebra $\mathcal{F}$ in the power set $\mathcal{P}(\Omega)$ of Ω and on a submeasure $\eta\colon \mathcal{P}(\Omega) \to [0, +\infty]$. For a particular η the space $L_0(\Omega)$ coincides with the space $\mathcal{B}(\Omega)$ that is the closure of all simple functions with respect to the topology of uniform convergence. We recall that in [5] (see also [3, 15]) for a subset X of $L_0(\Omega)$ the quantities $\omega(X)$ and $\sigma(X)$, which measure, respectively, the lack of equi-measurability and the lack of equi-quasiboundedness of X, are used to estimate the Hausdorff measure of noncompactness $\gamma(X)$ of X. In this paper, for a given function x in the space $L_0(\Omega)$, we introduce and study the modulus of $\mathcal{A}$-decrease and the modulus of $\mathcal{A}$-increase, related to monotonicity of the function x. Then for subsets $X \subseteq L_0(\Omega)$, we define the quantities $d(X)$ and $i(X)$ and we use them to estimate the lack of equi-measurability $\omega(X)$. Precisely, we prove $\frac{1}{2}\max\{d(X), i(X)\} \leq \omega(X) \leq (d + i)(X)$, and that the estimates are sharp (Example 2 shows the result for the right estimate). As a consequence we estimate the Hausdorff measure of noncompactness $\gamma(X)$ of X. In particular, for equi-quasibounded subsets X of $L_0(\Omega)$ we find

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$\frac{1}{4}\max\{d(X),i(X)\} \le \gamma(X) \le (d+i)(X)$. Focusing our attention on $\mathcal{B}(\Omega)$, we obtain, for subsets $X \subseteq \mathcal{B}(\Omega)$ the precise formula $\omega(X) = \frac{1}{2}\max\{d(X),i(X)\}$, which for bounded subsets gives us $\gamma(X) = \frac{1}{4}\max\{d(X),i(X)\}$. When Ω is a topological space, being the topology that induced by the order, we apply our results to the space of functions bounded and continuous on Ω. We recall that other quantities related to monotonicity have been introduced in [9] for functions in the space $B(I)$ of all real-valued functions defined and bounded on a compact interval I of the reals, and the quantities $d_0(X)$ and $i_0(X)$ have been defined for bounded subsets $X \subseteq B(I)$. We notice that the space $B(\Omega)$ coincides with $\mathcal{B}(\Omega)$ for $\mathcal{F} = \mathcal{P}(\Omega)$, therefore in this paper we obtain results that do not depend on the compactness of the set Ω. Finally, we compare, in the space $C(I)$ of all real-valued functions defined and continuous on I, our quantities with those of [9]. We improve some results from [9]. In particular, for a bounded subset $X \subseteq C(I)$, we obtain $\gamma_{C(I)}(X) = \frac{1}{4}\max\{d_0(X),i_0(X)\}$, where $\gamma_{C(I)}$ is the Hausdorff measure of noncompactness in $C(I)$.

2. Preliminaries

We begin by defining the space $L_0(\Omega)$. We assume that $(\Omega,\le)$ is a nonempty linearly ordered set and $\eta\colon \mathcal{P}(\Omega) \to [0,+\infty]$ a submeasure on the power set $\mathcal{P}(\Omega)$ of Ω. We set $\inf\emptyset := +\infty$. Let $x \in \mathbb{R}^\Omega$, the space of all real-valued functions defined on Ω. Then $\|x\|_0 := \inf\{a > 0 : \eta(\{t \in \Omega : |x(t)| \ge a\}) \le a\}$ defines a group seminorm on the space $\mathbb{R}^\Omega$. Let $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ be an algebra. A function $h \in \mathbb{R}^\Omega$ is called *$\mathcal{F}$-simple* if there are $m_1,\dots,m_n \in \mathbb{R}$ such that $h(\Omega) = \{m_1,\dots,m_n\}$ and $h^{-1}(m_i) \in \mathcal{F}$, for $i = 1,\dots,n$. The space $L_0(\Omega) := L_0(\mathcal{F},\Omega,\eta)$ of *measurable functions* is the closure of the set of all $\mathcal{F}$-simple functions in $(\mathbb{R}^\Omega, \|\cdot\|_0)$. Set $\|x\|_\infty := \sup_{t\in\Omega}|x(t)|$, the space $\mathcal{B}(\Omega) := \mathcal{B}(\mathcal{F},\Omega)$ is the closure of all $\mathcal{F}$-simple functions in $(\mathbb{R}^\Omega, \|\cdot\|_\infty)$. As $\|x\|_0 \le \|x\|_\infty$, for all $x \in \mathbb{R}^\Omega$, we always have $\mathcal{B}(\Omega) \subseteq L_0(\Omega)$. Moreover we find $\mathcal{B}(\Omega) = L_0(\Omega)$ for $\eta = \eta_\infty$, where $\eta_\infty(G) = 0$ if $G = \emptyset$ and $\eta_\infty(G) = +\infty$ if G is a nonempty set in $\mathcal{P}(\Omega)$. Unless specified otherwise, $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ will be any fixed algebra. Clearly $\mathcal{B}(\mathcal{F},\Omega) \subseteq \mathcal{B}(\mathcal{P}(\Omega),\Omega)$. We recall that $\mathcal{B}(\mathcal{P}(\Omega),\Omega)$ coincides with the space $B(\Omega)$ of all real-valued functions defined and bounded on Ω and equipped with the supremum norm. Whenever we consider Ω equipped with the topology induced by the order, we denote by $BC(\Omega)$ the space of all real-valued functions defined, bounded and continuous on Ω. So we have $BC(\Omega) \subseteq B(\Omega) = \mathcal{B}(\mathcal{P}(\Omega),\Omega)$.

Now, for a subset X of $L_0(\Omega)$ and $\varepsilon > 0$ arbitrarily fixed, we denote by $\mathcal{M}_\varepsilon$ the set of all multifunctions $F_\varepsilon\colon X \to \mathcal{P}(\Omega)$ such that $\eta(F_\varepsilon(x)) \le \varepsilon$ for all $x \in X$. We will denote by Π the family of all finite partitions $\mathcal{A} = \{A_1,\dots,A_n\}$ of Ω in $\mathcal{F}$, the trivial partitition will be denoted by $\{\Omega\}$. Given a subset A of Ω we denote by χ_A the characteristic function of A in Ω. Recall that a map φ defined on the family of all nonempty subsets of a pseudometric space E taking values in $[0,+\infty]$ is a *measure of noncompactness* in E in the sense of [6] if φ satisfies the following properties:

$$\varphi(X) = 0 \text{ if and only if } X \subseteq E \text{ is a totally bounded set;} \tag{1}$$

$$\varphi(X) = \varphi(\overline{X}), \text{ for every } X \subseteq E \text{ where } \overline{X} \text{ is the closure of } X; \tag{2}$$

$$\varphi(X_1 \cup X_2) = \max\{\varphi(X_1),\varphi(X_2)\}, \text{ for every } X_1,X_2 \subseteq E. \tag{3}$$

When E is a Banach space a measure of noncompactness satisfies additional properties, for which we refer to [7].

Given $X \subseteq E$ the *Hausdorff measure of noncompactness* $\gamma_E(X)$ of X is the infimum of all $\varepsilon > 0$ such that X has a finite ε-net in E. In the following we will denote $\gamma_{L_0(\Omega)}$ simply by γ.

For $x \in L_0(\Omega)$, $\mathcal{A} = \{A_1, \ldots, A_n\} \in \Pi$ and $D \subseteq \Omega$, we define

$$\omega(x, \mathcal{A}, D) := \max_{i=1}^{n} \sup\{|x(s) - x(t)| : t, s \in A_i \setminus D\},$$

and we call it the *modulus of $\mathcal{A}$-equi-measurability of x* (let us observe that in the above definition we can always consider $t \le s$). For $X \subseteq L_0(\Omega)$, $\mathcal{A} \in \Pi$, $\varepsilon > 0$ and $F_\varepsilon \in \mathcal{M}_\varepsilon$ we set $\omega(X, \mathcal{A}, F_\varepsilon) := \sup\{\omega(x, \mathcal{A}, F_\varepsilon(x)) : x \in X\}$.
Then the quantity $\omega(X)$ which appears in [5] can be defined as follows

$$\omega(X) := \inf\{\varepsilon > 0 : \text{ there are } \mathcal{A} \in \Pi \text{ and } F_\varepsilon \in \mathcal{M}_\varepsilon \text{ such that } \omega(X, \mathcal{A}, F_\varepsilon) \le \varepsilon\}.$$

The set X is said to be *equi-measurable* if $\omega(X) = 0$, and *equi-quasibounded* (*bounded* for subsets $X \subseteq \mathcal{B}(\Omega)$) if $\sigma(X) = 0$, where as in [5] we set

$$\sigma(X) := \inf\{\varepsilon > 0 : \text{ there is } a \ge 0 \text{ such that } \eta(\{t \in \Omega : |x(t)| \ge a\}) \le \varepsilon, \text{ for all } x \in X\}.$$

We recall the following estimates for the Hausdorff measure of noncompactness (cfr. [5: Theorem 2.1] and [5: p. 579])

$$\max\left\{\sigma(X), \frac{1}{2}\omega(X)\right\} \le \gamma(X) \le \omega(X) + \sigma(X), \quad \text{if} \quad X \subseteq L_0(\Omega) \tag{4}$$

and

$$\gamma(X) \le \frac{1}{2}\omega(X) + \sigma(X), \quad \text{if} \quad X \subseteq \mathcal{B}(\Omega). \tag{5}$$

As a consequence if $\sigma(X) = 0$, we have $\frac{1}{2}\omega(X) \le \gamma(X) \le \omega(X)$ if $X \subseteq L_0(\Omega)$, and the precise formula $\gamma(X) = \frac{1}{2}\omega(X)$ if $X \subseteq \mathcal{B}(\Omega)$.

3. Results in the space $L_0(\Omega)$

Let x be a function in $L_0(\Omega)$, $\mathcal{A} = \{A_1, \ldots, A_n\} \in \Pi$ and $D \subseteq \Omega$ be given, then we define

$$d(x, \mathcal{A}, D) := \max_{i=1}^{n} \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in A_i \setminus D,\ t \le s\},$$

and

$$i(x, \mathcal{A}, D) := \max_{i=1}^{n} \sup\{|x(s) - x(t)| - [x(t) - x(s)] : t, s \in A_i \setminus D,\ t \le s\}.$$

We call the above quantities, respectively, the *modulus of $\mathcal{A}$-decrease* and the *modulus of $\mathcal{A}$-increase* of x. We observe that if $\mathcal{A}' \in \Pi$ is a partition finer than $\mathcal{A}$, then $d(x, \mathcal{A}', D) \le d(x, \mathcal{A}, D)$ and also $i(x, \mathcal{A}', D) \le i(x, \mathcal{A}, D)$.

Proposition 1. *Let $x \in L_0(\Omega)$, $\mathcal{A} = \{A_1, \ldots, A_n\} \in \Pi$ and $D \subseteq \Omega$ be arbitrarily fixed. Then $d(x, \mathcal{A}, D) = 0$ if and only if x is nondecreasing on each $A_i \setminus D$, and $i(x, \mathcal{A}, D) = 0$ if and only if x is nonincreasing on each $A_i \setminus D$.*

Proof. If x is nondecreasing on $A_i \setminus D$, for $i = 1, \ldots, n$, then it follows by the definition $d(x, \mathcal{A}, D) = 0$. To prove the converse implication, suppose $d(x, \mathcal{A}, D) = 0$ and assume on the contrary that x is strictly decreasing on $A_i \setminus D$ for some $i \in \{1, \ldots, n\}$. Then there are $t, s \in A_i \setminus D$ with $t < s$ and $x(t) > x(s)$. Then

$$|x(s) - x(t)| - [x(s) - x(t)] = 2[x(t) - x(s)] > 0,$$

hence $d(x, \mathcal{A}, D) > 0$, a contradiction. In the same way follows the assert for the modulus of $\mathcal{A}$-increase of x. $\square$

As first step, using the moduli of $\mathcal{A}$-decrease and $\mathcal{A}$-increase we obtain a formula for the modulus of $\mathcal{A}$-equi-measurability $\omega(x, \mathcal{A}, D)$ of x.

Lemma 1. *Let $x \in L_0(\Omega)$, $\mathcal{A} \in \Pi$ and $D \subseteq \Omega$ be arbitrarily fixed. Then*

$$\omega(x,\mathcal{A},D) = \frac{1}{2}\max\{d(x,\mathcal{A},D), i(x,\mathcal{A},D)\}.$$

Proof. Let $\mathcal{A} = \{A_1,\ldots,A_n\}$ and, for a given index i, let $t,s \in A_i \setminus D$ with $t \le s$. Then, on the one hand, we have $|x(s)-x(t)| - [x(s)-x(t)] \le 2|x(s)-x(t)| \le 2\omega(x,\mathcal{A},D)$, and $|x(s)-x(t)| - [x(t)-x(s)] \le 2|x(s)-x(t)| \le 2\omega(x,\mathcal{A},D)$. Hence we get

$$d(x,\mathcal{A},D) \le 2\omega(x,\mathcal{A},D), \tag{6}$$

$$i(x,\mathcal{A},D) \le 2\omega(x,\mathcal{A},D), \tag{7}$$

and consequently $\frac{1}{2}\max\{d(x,\mathcal{A},D), i(x,\mathcal{A},D)\} \le \omega(x,\mathcal{A},D)$. On the other hand, we have

$$\begin{aligned} 2|x(s)-x(t)| &= \max\{|x(s)-x(t)| - [x(s)-x(t)],\ |x(s)-x(t)| - [x(t)-x(s)]\} \\ &\le \max\{d(x,\mathcal{A},D), i(x,\mathcal{A},D)\}, \end{aligned}$$

so that $\omega(x,\mathcal{A},D) \le \frac{1}{2}\max\{d(x,\mathcal{A},D), i(x,\mathcal{A},D)\}$, as desired. □

We generalize in a natural manner the quantities related to monotonicity of single functions to quantities related to monotonicity of sets of functions. Given a subset X of $L_0(\Omega)$ and $\varepsilon > 0$ we define

$$\begin{aligned} d(X,\mathcal{A},F_\varepsilon) &:= \sup\{d(x,\mathcal{A},F_\varepsilon(x))\colon x \in X\}, \text{ where } F_\varepsilon \in \mathcal{M}_\varepsilon, \\ i(X,\mathcal{A},G_\varepsilon) &:= \sup\{i(x,\mathcal{A},G_\varepsilon(x))\colon x \in X\}, \text{ where } G_\varepsilon \in \mathcal{M}_\varepsilon, \end{aligned}$$

and

$$(d+i)(X,\mathcal{A},H_\varepsilon) := d(X,\mathcal{A},H_\varepsilon) + i(X,\mathcal{A},H_\varepsilon), \text{ where } H_\varepsilon \in \mathcal{M}_\varepsilon.$$

Next we set

$$d(X) := \inf\{\varepsilon > 0 : \text{ there exist } \mathcal{A} \in \Pi \text{ and } F_\varepsilon \in \mathcal{M}_\varepsilon \text{ such that } d(X,\mathcal{A},F_\varepsilon) \le \varepsilon\},$$

and analogously we define the quantities $i(X)$ and $(d+i)(X)$.

The maps d, i and $(d+i)$ satisfy properties (2) and (3) of a measure of noncompactness. Property (3) is immediate, we prove (2).

Proposition 2. *Let $X \subseteq L_0(\Omega)$. Then, $d(X) = d(\overline{X})$, $i(X) = i(\overline{X})$ and $(d+i)(X) = (d+i)(\overline{X})$.*

Proof. Let $y \in \overline{X}$. Given $k \in \mathbb{N}$, find $x_k \in X$ such that $\|y - x_k\|_0 < 1/k$. Then let $D_k \subseteq \Omega$ such that $\eta(D_k) \le 1/k$ and $|y(t) - x_k(t)| < 1/k$ for every $t \in \Omega \setminus D_k$. Let $\alpha > d(X)$. Find $\mathcal{A} = \{A_1,\ldots,A_n\} \in \Pi$ and $F_\alpha \in \mathcal{M}_\alpha$ such that $d(x,\mathcal{A},F_\alpha(x)) \le \alpha$ for each $x \in X$. In particular, we have for each k

$$\max_{i=1,\ldots,n} \sup_{\substack{t,s \in A_i \setminus F_\alpha(x_k) \\ t \le s}} |x_k(s) - x_k(t)| - [x_k(s) - x_k(t)] \le \alpha.$$

Now for any k set $F_{\alpha+1/k}(y) = F_\alpha(x_k) \cup D_k$, then $\eta(F_{\alpha+1/k})(y) \le \alpha + 1/k$. Moreover for $t,s \in A_i \setminus F_{\alpha+1/k}(y)$, $t \le s$, we have

$$\begin{aligned} |y(s)-y(t)| - [y(s)-y(t)] &\le 2|y(s)-x_k(s)| + 2|x_k(t)-y(t)| + |x_k(s)-x_k(t)| - [x_k(s)-x_k(t)] \\ &< \alpha + \frac{4}{k}. \end{aligned}$$

Therefore we have $d(y,\mathcal{A},F_{\alpha+1/k}(y)) < \alpha + 4/k$. By the arbitrariness of α and k, we infer $d(\overline{X}) \le d(X)$. We conclude $d(\overline{X}) = d(X)$. In the same way we can prove $i(\overline{X}) = i(X)$ and $(d+i)(\overline{X}) = (d+i)(X)$. □

We remark that the maps d and i are not measures of noncompactness in $L_0(\Omega)$. Indeed, if X in $L_0(\Omega)$ is a non-totally bounded set consisting of nondecreasing, respectively nonincreasing, functions, then $d(X) = 0$, respectively $i(X) = 0$, so the maps d and i do not satisfy property (1) of a measure of noncompactness.
The following is a corollary of Proposition 1.

Corollary 1. *Let $X \subseteq L_0(\Omega)$, $\mathcal{A} = \{A_1, \dots, A_n\} \in \Pi$, $\varepsilon > 0$ and $F_\varepsilon \in \mathcal{M}_\varepsilon$. Then $d(X, \mathcal{A}, F_\varepsilon) = 0$ if and only if any function $x \in X$ is nondecreasing on each $A_i \setminus F_\varepsilon(x)$, and $i(X, \mathcal{A}, F_\varepsilon) = 0$ if and only if any function $x \in X$ is nonincreasing on each $A_i \setminus F_\varepsilon(x)$.*

Moreover, as a consequence of Lemma 1 we get the following result.

Proposition 3. *Let $X \subseteq L_0(\Omega)$, $\mathcal{A} = \{A_1, \dots, A_n\} \in \Pi$, $\varepsilon > 0$ and $F_\varepsilon \in \mathcal{M}_\varepsilon$. Then*

$$\frac{1}{2} \max\{d(X, \mathcal{A}, F_\varepsilon), i(X, \mathcal{A}, F_\varepsilon)\} \le \omega(X, \mathcal{A}, F_\varepsilon).$$

The following example shows that Corollary 1 does not generalize to the case $d(X) = 0$ or $i(X) = 0$.

***Example* 1.** Assume $\Omega = [0, +\infty[$. Let $\mathcal{F}$ be the σ-algebra of Lebesgue measurable sets in Ω and $\eta|_{\mathcal{F}}$ the Lebesgue measure. Let $X \subseteq L_0(\Omega)$ be the set consisting of the two functions x_1, which we suppose strictly increasing and bounded, and $x_2 = -x_1$, which will be a strictly decreasing and bounded function. Then $\omega(X) = 0$. Hence by Proposition 3 we have $d(X) = i(X) = 0$. Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\} \in \Pi$, $\varepsilon > 0$ and $F_\varepsilon \in \mathcal{M}_\varepsilon$ be arbitrarily fixed, then clearly we have $d(X, \mathcal{A}, F_\varepsilon) > 0$ and $i(X, \mathcal{A}, F_\varepsilon) > 0$. So $d(X) = 0$, respectively $i(X) = 0$, does not imply that all functions x from the set X are nondecreasing, respectively nonincreasing, on the sets $A_i \setminus F_\varepsilon(x)$ for $i = 1, 2, \dots, n$.

Theorem 1. *Let $X \subseteq L_0(\Omega)$. Then*

$$\frac{1}{2} \max\{d(X), i(X)\} \le \omega(X) \le (d + i)(X). \tag{8}$$

P r o o f. Let $\alpha > \omega(X)$. Find $\mathcal{A} \in \Pi$ and $F_\alpha \in \mathcal{M}_\alpha$ such that $\omega(x, \mathcal{A}, F_\alpha(x)) \le \alpha$ for each $x \in X$. Then, by Lemma 1, for each $x \in X$, we have

$$\frac{1}{2} \max\{d(x, \mathcal{A}, F_\alpha), i(x, \mathcal{A}, F_\alpha)\} \le \alpha,$$

which implies

$$\frac{1}{2} \max\{d(X), i(X)\} \le \omega(X).$$

To prove the right inequality let $\alpha > (d + i)(X)$. Choose $\mathcal{A} = \{A_1, \dots, A_n\} \in \Pi$, $F_\alpha \in \mathcal{M}_\alpha$ such that $(d + i)(x, \mathcal{A}, F_\alpha) \le \alpha$ for each $x \in X$. Now if $i \in \{1, \dots, n\}$ and $t, s \in A_i \setminus F_\alpha$ with $t \le s$ we have, for all $x \in X$, $|x(s) - x(t)| - [x(s) - x(t)] \le \alpha$ and $|x(s) - x(t)| - [x(t) - x(s)] \le \alpha$, and consequently $|x(s) - x(t)| \le \alpha/2$. Having in mind that $\eta(F_\alpha(x) \le \alpha$, we obtain $\omega(X) \le \alpha$, and so the proof is complete. □

Both the estimates of Theorem 1 are sharp. For the left inequality it is a consequence of our subsequent Theorem 2 where we prove that in $\mathcal{B}(\Omega)$ the precise formula holds, the optimality of the right inequality will be shown in Example 2.

From Theorem 1 and (4) we find the following estimates for the Hausdorff measure of noncompactness.

Corollary 2. *Let $X \subseteq L_0(\Omega)$, then*

$$\max\Big\{\sigma(X), \frac{1}{4}d(X), \frac{1}{4}i(X)\Big\} \le \gamma(X) \le (d+i)(X) + \sigma(X)$$

In particular, if X is equi-quasibounded we have

$$\frac{1}{4}\max\{d(X),\ i(X)\} \le \gamma(X) \le (d+i)(X).$$

Remark 1. We observe that given $x \in L_0(\Omega)$, $\mathcal{A} \in \Pi$ and $D \subseteq \Omega$, using (6) and (7), we can write $\frac{1}{4}(d+i)(x,\mathcal{A},D) \le \omega(x,\mathcal{A},D)$. Consequently, having in mind (8), for $X \subseteq L_0(\Omega)$ we obtain

$$\frac{1}{4}(d+i)(X) \le \omega(X) \le (d+i)(X). \tag{9}$$

In such a way we estimate $\omega(X)$ by means of the quantity $(d+i)(X)$, but the left inequality of (9) will not be sharp anymore.

Then (9) and (4) for equi-quasibounded subsets $X \subseteq L_0(\Omega)$ give us

$$\frac{1}{8}(d+i)(X) \le \gamma(X) \le (d+i)(X). \tag{10}$$

From the inequalities (10) we also have that $(d+i)$ is a measure of noncompactness in the family of all equi-quasibounded subsets of $L_0(\Omega)$.

The following example shows that in (8) the equality on the right may hold, so that the estimate $\omega(X) \le (d+i)(X)$ is sharp. The example is adapted from [14], where the value of the lack of equi-measurability of the set $K_c(\Omega)$ of all Lebesgue-measurable real-valued functions defined on Ω such that $0 \le f \le c$ almost everywhere on Ω is calculated, repairing a gap of [3: Example 2.3].

***Example* 2.** Assume $\Omega = [0,1]$. Let $\mathcal{F}$ be the σ-algebra of all Lebesgue measurable subsets of $[0,1]$ and $\eta_{|\mathcal{F}}$ the Lebesgue measure. Then $L_0(\Omega)$ is the space of all Lebesgue measurable real-valued functions. Given $\mathcal{A} = \{A_1,\dots,A_n\} \in \Pi$, let $\{A_{i,0},\ A_{i,1}\}$ be a partition of A_i in $\mathcal{F}$, for each $i = 1,\dots,n$, such that $\mu(A_{i,0}) = \mu(A_{i,1}) = \frac{1}{2}\mu(A_i)$. Then we define the simple function $h_{\mathcal{A}}$ by setting

$$h_{\mathcal{A}} = \frac{1}{2}\sum_{i=1}^{n}\chi_{A_{i,1}}.$$

We claim that

$$\operatorname{osc}(h_{\mathcal{A}},\mathcal{A},\varepsilon) = \begin{cases} \frac{1}{2} & \text{if } \ 0 \le \varepsilon < \frac{1}{2} \\ 0 & \text{if } \ \frac{1}{2} \le \varepsilon, \end{cases}$$

where $\operatorname{osc}(h_{\mathcal{A}},\mathcal{A},\varepsilon) := \inf\limits_{\substack{D_\varepsilon \subseteq \Omega \\ \eta(D_\varepsilon)\le\varepsilon}} \max\limits_{i=1}^{n} \sup\{|h_{\mathcal{A}}(s) - h_{\mathcal{A}}(t)| : t,s \in A_i \setminus D_\varepsilon\}$. Observe that $\operatorname{osc}(h_{\mathcal{A}},\mathcal{A},0) \le \max\limits_{i=1}^{n} \operatorname{diam}(h_{\mathcal{A}}(A_i)) = \frac{1}{2}$ and moreover taking $D_0 = \cup_{i=1}^{n} A_{i,0}$, we find

$$\operatorname{osc}(h_{\mathcal{A}},\mathcal{A},\frac{1}{2}) \le \max_{i=1}^{n} \operatorname{diam}(h_{\mathcal{A}}(A_i \setminus D_0)) = 0.$$

Since the function $\varepsilon \to \operatorname{osc}(h_{\mathcal{A}},\mathcal{A},\varepsilon)$ is decreasing on $[0,+\infty[$, we obtain

$$\operatorname{osc}(h_{\mathcal{A}},\mathcal{A},\varepsilon) \le \frac{1}{2} \quad \text{if} \quad 0 \le \varepsilon < \frac{1}{2} \tag{11}$$

and

$$\operatorname{osc}(h_{\mathcal{A}},\mathcal{A},\varepsilon) = 0 \quad \text{if} \quad \varepsilon \ge \frac{1}{2}.$$

To prove our claim we show that in (11) the equality holds. Assume by contradiction that there are $0 \le \bar{\varepsilon} < \frac{1}{2}$ and $D_{\bar{\varepsilon}} \subseteq \Omega$ such that

$$\operatorname{osc}(h_{\mathcal{A}}, \mathcal{A}, \varepsilon) \le \max_{i=1}^{n} \sup\{|h_{\mathcal{A}}(s) - h_{\mathcal{A}}(t)| : t, s \in A_i \setminus D_{\bar{\varepsilon}}\} < \frac{1}{2},$$

without loss of generality we can replace $D_{\bar{\varepsilon}}$ with its measurable envelope, so we can assume $D_{\bar{\varepsilon}} \in \mathcal{F}$. As $\eta(D_{\bar{\varepsilon}}) < \frac{1}{2}$ we can write $\sum_{i=1}^{n} \eta(A_i \cap D_{\bar{\varepsilon}}) < \frac{1}{2}$, then there is at least one element, that we fix, A_l of $\mathcal{A}$ such that

$$\eta(A_l \cap D_{\bar{\varepsilon}}) < \frac{1}{2}\eta(A_l). \tag{12}$$

Observe that we cannot have $\eta(A_{l,0} \setminus D_{\bar{\varepsilon}}) > 0$ and $\eta(A_{l,1} \setminus D_{\bar{\varepsilon}}) > 0$, because it would imply $\sup\{|x(s)-x(t)| : t, s \in A_l \setminus D_{\bar{\varepsilon}})\} = \frac{1}{2}$. So we have $\eta(A_{l,0} \setminus D_{\bar{\varepsilon}}) = 0$ or $\eta(A_{l,1} \setminus D_{\bar{\varepsilon}}) = 0$. Assuming $\eta(A_{l,0} \setminus D_{\bar{\varepsilon}}) = 0$, since $\eta(A_{l,0} \setminus D_{\bar{\varepsilon}}) = \eta(A_{l,0}) - \eta(A_{l,0} \cap D_{\bar{\varepsilon}})$, we have $\eta(A_{l,0}) = \eta(A_{l,0} \cap D_{\bar{\varepsilon}})$. Then we find

$$\eta(A_l \cap D_{\bar{\varepsilon}}) \ge \eta(A_{l,0} \cap D_{\bar{\varepsilon}}) = \frac{1}{2}\eta(A_l),$$

which contradicts (12), and this proves our claim.

Now we set $X = \{h_{\mathcal{A}} : \ \mathcal{A} \in \Pi\}$. From what we have proved we infer $\omega(X) = \frac{1}{2}$. Therefore looking at the right inequality of (8) we can write

$$\frac{1}{2} = \omega(X) \le (d+i)(X)$$

On the other hand for $h_{\mathcal{A}}$ arbitrarily taken in X, if $\mathcal{A} = \{A_1, \ldots, A_n\}$, we set $D_0 = \cup_{i=1}^{n} A_{i,0}$, according to our previous notation. Then $d(h_{\mathcal{A}}, \{\Omega\}, D_0) = i(h_{\mathcal{A}}, \{\Omega\}, D_0) = 0$, having in mind that $\eta(D_0) = \frac{1}{2}$ it follows $(d+i)(X) \le \frac{1}{2}$. This implies $\omega(X) = (d+i)(X)$, as desired.

4. Results in the space $\mathcal{B}(\Omega)$

In the space $\mathcal{B}(\Omega)$, being $\mathcal{B}(\Omega) = L_0(\Omega)$ for $\eta = \eta_\infty$, the expressions of all the quantities we have considered in the paper up to now can be rewritten in a simpler manner (due to the fact that no exceptional sets will appear in any of the definitions). For a function $x \in \mathcal{B}(\Omega)$ and $\mathcal{A} = \{A_1, \ldots, A_n\} \in \Pi$, the modulus of $\mathcal{A}$-equi-measurability of x is given by $\omega(x, \mathcal{A}) = \max_{i=1}^{n} \sup\{|x(s) - x(t)| : t, s \in A_i\}$, and for a subset $X \subseteq \mathcal{B}(\Omega)$ we have $\omega(X, \mathcal{A}) = \sup\{\omega(x, \mathcal{A}) : x \in X\}$ and

$$\omega(X) = \inf\{\varepsilon > 0 : \text{ there exists } \mathcal{A} \in \Pi \text{ such that } \omega(X, \mathcal{A}) \le \varepsilon\}.$$

Analogously, we have $d(x, \mathcal{A}) = \max_{i=1}^{n} \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in A_i, t \le s\}$, $d(X, \mathcal{A}) = \sup\{d(x, \mathcal{A}) : x \in X\}$, and

$$d(X) = \inf\{\varepsilon > 0 : \text{ there exists } \mathcal{A} \in \Pi \text{ such that } d(X, \mathcal{A}) \le \varepsilon\}.$$

In the same way we can write $i(x, \mathcal{A})$, $i(X, \mathcal{A})$ and $i(X)$. Notice that in this setting $(d+i)(X) = d(X) + i(X)$.

We observe that given a function $x \in \mathcal{B}(\Omega)$ and $\mathcal{A} = \{A_1, \ldots, A_n\} \in \Pi$, then x is nondecreasing, respectively nonincreasing, on A_i, for $i = 1, \ldots, n$, if and only $d(x, \mathcal{A}) = 0$, respectively $i(x, \mathcal{A}) = 0$. In particular the latter holds for $\mathcal{A} = \{\Omega\}$, therefore we have that the function x is nondecreasing, respectively nonincreasing, on Ω, if and only $d(x, \mathcal{A}) = 0$, respectively $i(x, \mathcal{A}) = 0$, for every partition $\mathcal{A} \in \Pi$.

The following is the main result of this section.

Theorem 2. *Let X be a subset of $\mathcal{B}(\Omega)$. Then*

$$\omega(X) = \frac{1}{2}\max\{d(X), i(X)\}. \tag{13}$$

Proof. By Theorem 1 we have $\frac{1}{2}\max\{d(X), i(X)\} \le \omega(X)$. To prove the reverse inequality, reasoning as in Theorem 1 , let $\alpha > d(X)$ and $\beta > i(X)$ and choose $\mathcal{A} \in \Pi$ such that $d(x,\mathcal{A}) \le \alpha$ and $i(x,\mathcal{A}) \le \beta$ for each $x \in X$. Then by Lemma 1, for every $x \in X$ we get

$$\omega(x,\mathcal{A}) = \frac{1}{2}\max\{d(x,\mathcal{A}),\ i(x,\mathcal{A})\} \le \frac{1}{2}\max\{\alpha,\ \beta\},$$

from which the desired inequality follows. □

Having in mind (5) we obtain the following corollary.

Corollary 3. *Let $X \subseteq \mathcal{B}(\Omega)$, then*

$$\max\Big\{\sigma(X), \frac{1}{4}d(X), \frac{1}{4}i(X)\Big\} \le \gamma(X) \le \sigma(X) + \frac{1}{4}\max\{d(X),\ i(X)\}.$$

In particular, if X is bounded we have

$$\gamma(X) = \frac{1}{4}\max\{d(X),\ i(X)\}. \tag{14}$$

Equality (14) gives us a precise formula for the Hausdorff measure of noncompactness of bounded subsets X of $\mathcal{B}(\Omega)$, and shows that $\max\{d(X), i(X)\}$ is a measure of noncompactness in the sense of [7] in the family of all bounded subsets of $\mathcal{B}(\Omega)$.

We recall, assuming Ω equipped with the topology induced by the order, that the Bartle criterion of compactness (see [11]) states that a bounded subset X of $BC(\Omega)$ is totally bounded if and only if $\omega(X) = 0$. We observe that Theorem 2 together with formula (14) contains a Bartle type criterion of compactness in the space $\mathcal{B}(\Omega)$, which can be formulated as follows: A bounded subset X of $\mathcal{B}(\Omega)$ is totally bounded if and only if $d(X) = i(X) = 0$. As a consequence, if Ω is a compact space, we obtain an Ascoli-Arzelà type theorem: A bounded subset X of $C(\Omega)$ is compact if and only if $d(X) = i(X) = 0$.

We conclude this section by considering $(\Omega, \le)$ a *linear continuum*. We prove that if the value of the modulus of $\mathcal{A}$-decrease, respectively the value of the modulus of $\mathcal{A}$-increase, of a function $x \in C(\Omega)$ is null then the function x is nondecreasing, respectively nonincreasing, on Ω, a stronger result compared with the one we have obtained in Proposition 1 in spaces $L_0(\Omega)$. We recall that the linearly order set $(\Omega, \le)$ is a *linear continuum* if the following two conditions are satisfied:

(i) for every $t, s \in \Omega$ with $t \le s$ there exists $u \in \Omega$ such that $t \le u \le s$;

(ii) every nonempty subset of Ω that is bounded from above has the least upper bound in Ω.

A linear continuum Ω is connected with respect to the topology induced by the order and so are intervals in Ω, moreover every closed interval in Ω is compact and the intermediate value Theorem holds.

Theorem 3. *Assume Ω to be a linear continuum and let $x \in C(\Omega)$. If there is $\mathcal{A} \in \Pi$ such that $d(x,\mathcal{A}) = 0$, respectively $i(x,\mathcal{A}) = 0$, then x is nondecreasing, respectively nonincreasing, on Ω.*

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ and assume $d(x,\mathcal{A}) = 0$, then x is nondecreasing on each A_i. We have to show that x is nondecreasing on Ω. Assume the contrary, and let $t_1 < t_2$ such that $x(t_2) < x(t_1)$. Set $y_1 = (x(t_1) + x(t_2))/2$. Since x is continuous and Ω is connected, by the intermediate value Theorem, we find $s_1 \in\,]t_1, t_2[$ such that $y_1 = x(s_1)$. Now we consider $y_2 = (y_1 + x(t_1))/2$ and find $s_2 \in\,]t_1, s_1[$ such that $y_2 = x(s_2)$. Then we repeat the same argument $(n-1)$-times by setting $y_{i+1} = (y_i + x(t_1))/2$ and finding $s_{i+1} \in\,]t_1, s_i[$ such that $y_{i+1} = x(s_{i+1})$.

Setting $t_1 = s_n$ and $t_2 = s_0$, we obtain $n+1$ points s_i such that, $s_n < s_{n-1} < \cdots < s_1 < s_0$ and $x(s_0) < x(s_1) < \cdots < x(s_{n-1}) < x(s_n)$. Considering that at least two of the points s_i, say s_p, s_q with $s_p < s_q$, must belong to one of the sets $A_1, A_2, \ldots, A_n$, say A_l, we have $s_p, s_q \in A_l$ with $s_p < s_q$ and $x(s_p) > x(s_q)$. This contradicts the hypothesis and completes the proof. □

Corollary 4. *Assume Ω to be a linear continuum. Let $x \in C(\Omega)$ and $\mathcal{A} \in \Pi$. Then $d(x, \mathcal{A}) = 0$, respectively $i(x, \mathcal{A}) = 0$. if and only if the function x is nondecreasing, respectively nonincreasing, on Ω.*

Corollary 5. *Assume Ω to be a linear continuum. Let X be a subset of $C(\Omega)$ and $\mathcal{A} \in \Pi$. Then $d(X, \mathcal{A}) = 0$, respectively $i(X, \mathcal{A}) = 0$, if and only if all functions from the set X are nondecreasing, respectively nonincreasing, on Ω.*

5. Comparison of d and i with quantities d_0 and i_0

Let I be a real compact interval. In [9] for $x \in B(I)$ the modulus of decrease and the modulus of increase of the function x have been introduced by setting, for a given $\varepsilon > 0$,

$$d(x, \varepsilon) = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in I, t \le s, s - t \le \varepsilon\}$$

and

$$i(x, \varepsilon) = \sup\{|x(s) - x(t)| - [x(t) - x(s)] : t, s \in [a, b], t \le s, s - t \le \varepsilon\},$$

and results of monotonicity for functions in $B(I)$ have been proved. Furthermore, for a bounded subset X of $B(I)$ the quantities $d_0(X) = \lim_{\varepsilon \to 0} \sup\{d(x, \varepsilon) : x \in X\}$ and $i_0(X) = \lim_{\varepsilon \to 0} \sup\{i(x, \varepsilon) : x \in X\}$ have been considered. Recall that, for a bounded subset X of $C(I)$, $\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon)$, where $\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\}$ and $\omega(x, \varepsilon) = \sup\{|x(s) - x(t)| : t, s \in I, |s - t| \le \varepsilon\}$, being $\omega(x, \varepsilon)$ the modulus of continuity of x. In [9: Theorem 3.8] the following estimates for $\omega_0(X)$ have been proved for bounded subsets $X \subseteq C(I)$

$$\frac{1}{4}(d_0(X) + i_0(X)) \le \omega_0(X) \le \frac{1}{2}(d_0(X) + i_0(X)),$$

hence, due to the formula $\gamma_{C(I)}(X) = \frac{1}{2}\omega_0(X)$ (see [7]) the map $d_0 + i_0$ is a measure of noncompactness in $C(I)$ equivalent to the Hausdorff measure of noncompactness. Here we have the following result.

Theorem 4. *Let X be a bounded subset of $C(I)$. Then*

$$\omega_0(X) = \frac{1}{2} \max\{d_0(X), i_0(X)\}. \tag{15}$$

P r o o f. The inequality $\frac{1}{2} \max\{d_0(X), i_0(X)\} \le \omega_0(X)$ has been already pointed out in [9: p. 180 (3.1) and (3.2)]. We prove the reverse inequality. Let $\alpha > d_0(X)$ and $\beta > i_0(X)$. Find $\varepsilon > 0$ such that $d(X, \varepsilon) = \sup\{d(x, \varepsilon) : x \in X\} < \alpha$ and $i(X, \varepsilon) = \sup\{i(x, \varepsilon) : x \in X\} < \beta$. Hence for all $x \in X$, $t, s \in I$, with $t \le s$ and $s - t \le \varepsilon$ we have $|x(s) - x(t)| - [x(s) - x(t)] < \alpha$ and $|x(s) - x(t)| - [x(t) - x(s)] < \beta$, which imply $2|x(s) - x(t)| < \max\{\alpha, \beta\}$. We have found $2\omega(X, \varepsilon) \le \max\{\alpha, \beta\}$, it follows $2\omega_0(X) \le \max\{\alpha, \beta\}$ which, for the arbitrariness of α, β, completes the proof. □

As a consequence for bounded subsets X of $C(I)$ we obtain the equality $\gamma_{C(I)}(X) = \frac{1}{4} \max\{d_0(X), i_0(X)\}$. The next examples show that $d_0 + i_0$ fails to be a measure of noncompactness in $B([0, 1])$ (compact case) and as well in $C([1, +\infty)[)$ (non-compact case).

***Example* 3.** Let $x \in B([0,1])$ be the function defined by setting

$$x(t) = \begin{cases} \frac{1}{2} & \text{if} \quad 0 \le t < \frac{1}{3}, \\ 1 & \text{if} \quad \frac{1}{3} \le t < \frac{2}{3}, \\ 0 & \text{if} \quad \frac{2}{3} \le t \le 1. \end{cases}$$

Then $\omega(x,\varepsilon) = 1$ for any arbitrary $\varepsilon > 0$, so that $\omega_0(\{x\}) = 1$, $d_0(\{x\}) = 2$ and $i_0(\{x\}) = 1$. Hence $d_0(\{x\}) + i_0(\{x\}) \ne 0$ for a singleton set in $B([0,1])$.

***Example* 4.** Let $x \in C([1,+\infty[)$ be the function defined in the following way

$$x(t) = \begin{cases} 0 & \text{if} \quad t = 1 + \frac{1}{2}, \dots, \sum\limits_{k=1}^{2n} \frac{1}{k}, \dots, \\ 1 & \text{if} \quad t = 1, 1 + \frac{1}{2} + \frac{1}{3}, \dots, \sum\limits_{k=1}^{2n+1} \frac{1}{k}, \dots, \\ \text{to be linear} & \text{if} \quad t \in \left[\sum\limits_{k=1}^{n} \frac{1}{k}, \sum\limits_{k=1}^{n+1} \frac{1}{k} \right] \quad \text{for } n = 1, 2, \dots. \end{cases}$$

Then $\omega(x,\varepsilon) = 1$ for any arbitrary $\varepsilon > 0$, so that $\omega_0(\{x\}) = 1$, it can be calculated $d_0(\{x\}) = 2$ and $i_0(\{x\}) = 2$. Therefore, again $d_0(\{x\}) + i_0(\{x\}) \ne 0$ for a singleton set in $C([1,+\infty[)$.

Finally we compare, for bounded subsets X of $C(I)$, the quantities $d(X)$ and $i(X)$, introduced in the general setting of spaces $L_0(\Omega)$, with the quantities $d_0(X)$ and $i_0(X)$. We recall that for bounded subsets $X \subseteq C(I)$ from [5: Proposition 5.1] we have

$$\omega(X) \le \omega_0(X) \le 2\omega(X). \tag{16}$$

We need the following lemma.

Lemma 2. *Let x be a function in $C(I)$ a A a nonempty subset of I. Set*

$$\alpha = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in A,\ t \le s\},$$

and

$$\beta = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in \bar{A},\ t \le s\}.$$

Then $\alpha = \beta$.

Proof. Clearly $\alpha \le \beta$. To prove $\beta \le \alpha$, suppose on the contrary $\alpha < \beta$. Then there exist $t, s \in \bar{A}$ such that $t < s$ and

$$|x(s) - x(t)| - [x(s) - x(t)] \ge \delta > \alpha.$$

Observe that at least one of the two points belongs to $\bar{A} \setminus A$. We now examine all possible cases. The first $s \in A$ and $t \in \bar{A} \setminus A$, the second $t \in A$ and $s \in \bar{A} \setminus A$, and the last case $s, t \in \bar{A} \setminus A$.

Assume $s \in A$ and $t \in \bar{A} \setminus A$ and choose a sequence $\{t_n\}$ in A such that $t_n \to t$ and $t_n < s$. Given $\varepsilon > 0$ choose n such that $|x(t_n) - x(t)| \le \varepsilon$. Then

$$\begin{aligned} \alpha < \delta &\le |x(s) - x(t)| - [x(s) - x(t)] \\ &\le |x(s) - x(t_n)| - [x(s) - x(t_n)] + |x(t_n) - x(t)| - [x(t_n) - x(t)] \le \alpha + 2\varepsilon. \end{aligned}$$

Assume $t \in A$ and $s \in \bar{A} \setminus A$ and choose a sequence $\{s_n\}$ in A such that $s_n \to s$ and $s_n < t$. Given $\varepsilon > 0$ choose n such that $|x(s_n) - x(s)| \le \varepsilon$. Then

$$\begin{aligned} \alpha < \delta &\le |x(s) - x(t)| - [x(s) - x(t)] \\ &\le |x(s) - x(s_n)| - [x(s) - x(s_n)] + |x(s_n) - x(t)| - [x(s_n) - x(t)] \le \alpha + 2\varepsilon. \end{aligned}$$

Assume $t, s \in \bar{A} \setminus A$ and choose two sequences $\{t_n\}$ and $\{s_n\}$ in A such that $t_n \to t,\ s_n \to s$ and $t_n < s_n$. Given $\varepsilon > 0$ choose n such that $\max\{|x(t_n) - x(t)|, |x(s_n) - x(s)|\} \leq \varepsilon$. Then

$$\begin{aligned}\alpha < \delta &\leq |x(s) - x(t)| - [x(s) - x(t)] \\ &\leq |x(s) - x(s_n)| - [x(s) - x(s_n)] + |x(s_n) - x(t_n)| - [x(s_n) - x(t_n)] \\ &\quad + |x(t_n) - x(t)| - [x(t_n) - x(t)] \leq \alpha + 4\varepsilon.\end{aligned}$$

As ε was arbitrary, we have a contradiction in each of the cases. □

The same conclusion of Lemma 2 holds if $\alpha = \sup\{|x(s) - x(t)| - [x(t) - x(s)] : t, s \in A,\ t \leq s\}$, and $\beta = \sup\{|x(s) - x(t)| - [x(t) - x(s)] : t, s \in \bar{A},\ t \leq s\}$. Hence by Lemma 2 we have the following result.

Lemma 3. *Let $x \in C(I)$ and $\mathcal{A} = \{A_1, \ldots, A_n\} \in \Pi$. Then $d(x, \mathcal{A}) = d(x, \bar{\mathcal{A}})$ and $i(x, \mathcal{A}) = i(x, \bar{\mathcal{A}})$, where $\bar{\mathcal{A}} = \{\bar{A}_1, \ldots, \bar{A}_n\}$.*

Proposition 4. *Let X be a bounded subset of $C(I)$, then the following inequalities hold:*

(i) $d(X) \leq d_0(X) \leq d(X) + \max\{d(X), i(X)\}$,

(ii) $i(X) \leq i_0(X) \leq i(X) + \max\{d(X), i(X)\}$.

Proof. (i) Assume $I = [a, b]$. Let $\alpha > d_0(X)$. Then there exists $\varepsilon > 0$ such that for every $x \in X$ we have $|x(s) - x(t)| - [x(s) - x(t)] \leq \alpha$ for $t, s \in [a, b]$ with $t \leq s$ and $s - t \leq \varepsilon$. Choose $n \in \mathbb{N}$ such that $(b - a)/n \leq \varepsilon$, and let $\mathcal{A} = \{I_1, \ldots, I_n\} \in \Pi$ be a partition of $[a, b]$ consisting of n pairwise disjoint intervals I_i each of length $(b - a)/n$. Then we have $d(x, \mathcal{A}) \leq \alpha$. Therefore we obtain $d(X, \mathcal{A}) \leq \alpha$ and $d(X) \leq \alpha$, which by the arbitrariness of α implies $d(X) \leq d_0(X)$.
If $d(X) \geq i(X)$ the right inequality follows from (13), (15) and (16).
Assume $d(X) < i(X)$. To prove $d_0(X) \leq d(X) + i(X)$ assume by contradiction $d(X) + i(X) < d_0(X)$ and choose α, β and δ such that $d(X) < \beta$, $i(X) < \delta$ and

$$d(X) + i(X) < \beta + \delta < \alpha < d_0(X).$$

Choose $\mathcal{A} = \{A_1, \ldots, A_n\} \in \Pi$ such that $d(X, \mathcal{A}) < \beta$ and $i(X, \mathcal{A}) < \delta$. Hence, for all $x \in X$, we have $d(x, \mathcal{A}) < \beta$ and $i(x, \mathcal{A}) < \delta$, that is

$$\max_{i=1}^{n} \sup\{|x(s) - x(t)| - [x(s) - x(t)], t, s \in A_i, t \leq s\} < \beta$$

and

$$\max_{i=1}^{n} \sup\{|x(s) - x(t)| - [x(t) - x(s)], t, s \in A_i, t \leq s\} < \delta.$$

On the other hand being $d_0(X) > \alpha$, for each $n \in \mathbb{N}$ there exist $x_n \in X$ and $t_n, s_n \in [a, b]$ with $|s_n - t_n| \leq \frac{1}{n}$ and $t_n < s_n$ such that

$$|x_n(s_n) - x_n(t_n)| - [x_n(s_n) - x_n(t_n)] \geq \alpha.$$

Choose a subsequence of $\{s_n\}$, still denoted by $\{s_n\}$, which converges to a point $z \in [a, b]$, then $\{t_n\}$ will converge to the same z. If there is $i \in \{1, \ldots, n\}$ such that $t_n \in A_i$ and $s_n \in A_i$ for all $n \in \mathbb{N}$, then

$$\alpha \leq |x_n(s_n) - x_n(t_n)| - [x_n(s_n) - x_n(t_n)] \leq d(x_n, \mathcal{A}) \leq d(X, \mathcal{A}) \leq d(X) < \beta < \alpha,$$

which is a contradiction. Otherwise, we can assume that $t_n \in A_i$ and $s_n \in A_j$ for all $n \in \mathbb{N}$, for some $i, j \in \{1, \ldots, n\}$, so that $z \in \overline{A_i} \cap \overline{A_j}$. Using Lemma 3 we obtain

$$\begin{aligned}\alpha &\leq |x_n(s_n) - x_n(t_n)| - [x_n(s_n) - x_n(t_n)] \\ &\leq |x_n(s_n) - x_n(z)| + |x_n(z) - x_n(t_n)| - [x_n(s_n) - x_n(t_n)] \\ &= |x_n(s_n) - x_n(z)| - [x_n(s_n) - x_n(z)] + |x_n(z) - x_n(t_n)| - [x_n(z) - x_n(t_n)] \leq \beta + \delta,\end{aligned}$$

a contradiction which completes the proof.

(ii) The second part of the theorem can be proved in the same way. □

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