# Stochastic Response of Beams Equipped with Tuned Mass Dampers Subjected to Poissonian Loads

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ABSTRACT:

This contribution deals with the vibrational response of Euler-Bernoulli beams equipped with tuned mass dampers, subjected to random moving loads. The theory of generalised functions is used to capture the discontinuities of the response variables at the positions of the tuned mass dampers, which involves deriving exact complex eigenvalues and eigenfunctions from a characteristic equation built as the determinant of a 4 x 4 matrix, regardless of the number of tuned mass dampers. Building pertinent orthogonality conditions for the deflection eigenfunctions, the stochastic responses, under Poissonian white noise, are evaluated. In a numerical application, a beam with multiple tuned mass dampers, acted upon by random moving loads, is considered

## 1. INTRODUCTION

The dynamic analysis of Euler-Bernoulli beams is a quintessential engineering problem and the cornerstone upon which the solutions of many problems in the field are based. In recent years, thanks to advances in computer technology required to solve increasingly complex problems and necessitated by significant leaps in the transport industry, the dynamic response of structures which can be typically modelled as a discontinuous Euler-Bernoulli beam, such as rail or road bridges, has become increasingly important.

The analysis of these types of multi-span beam often focuses on moving loads which simulate the effects of traffic loading which these structures would be subjected to. Additionally, this type of forcing action is assumed because beams subjected to moving loads have greater maximum deflections and maximum moments than beams subjected to static loads. As ever faster locomotives are designed, this analysis becomes ever more important due to the danger of the locomotive reaching the beam's critical velocity; the velocity at which serious damage is caused to the beam.

While the vast majority of these studies focus on deterministic solutions to known loading, a number of studies consider the effect that a series of random moving loads have on the beam. In these studies, there is a general assumption made by many authors, in which the arrivals of the forces acting upon the beam are considered to follow a Poissonian distribution. Poissonian loading is a special case of random loading in which a series of impulses arrives at independent random times and with random magnitudes. Traffic loading (particularly considering road traffic, although this is also applicable to rail traffic) can be described as a Poissonian process as the magnitude of the forces is random as are their arrival times.

Ricciardi [1] proposed a method of modelling the forcing action as a filtered Poisson process, this is obtained by finding "the response of a linear undamped oscillator excited by a Poisson white noise process"

Currently when calculating the dynamic response of a Euler-Bernoulli beam to any forcing action, there are only a handful of solution methods which can be used to obtain accurate results; namely: computer models such as the finite element method, or the classical numerical method. This paper aims to expand on work conducted by [2] in which a novel numerical method was developed to find the response of a continuous system equipped with tuned mass dampers, to a series of moving loads.

This paper will study the response of a continuous Euler-Bernoulli beam, equipped with tuned mass dampers, subjected to a series of random moving loads. The method developed in [2] will be applied and expanded upon, this method considers complex eigenfunctions caused by localised damping, in this case from the tuned mass dampers, for a beam which is subjected to a series of moving loads. By extending this to consider a series of random moving loads with random magnitudes and arrival times, the results should more closely reflect the response of a road bridge subjected to normal traffic.

## 2. PROBLEM STATEMENT

Consider a beam carrying just one tuned mass damper (TMD) although, using the method proposed in this paper, j number of spring-masses could be attached without changing the equation of motion.

Using the proposed formulation, supports, lumped masses, TMDs, and a number of other attachments which do not cause localised rotation are assumed to be a shear discontinuity acting on a specific point, this allows the equation of motion to take the form [2]:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\overline{m}\frac{\partial^2 w(x,t)}{\partial t^2}+R(x,t)=f(x,t) \qquad (1)$$

Where: $EI$ is the flexural rigidity, $\overline{m}$ is the mass per unit length, $w(x,t)$ is the transversal displacement response of the beam in the space and time domains, $R(x,t)$ is a generalised function [3] used to account for the discontinuities and $f(x,t)$ is the forcing action. In this case, the generalised function represents a single TMD [4] which can be modelled as a shear discontinuity:

$$R(x,t)=\sum_{j=1}^{N}-P_j(t)\;\delta(x-x_j) \qquad (2)$$

Where: $-P_j(t)$ is the shear reactionary force, and $\delta(x-x_j)$ is a Dirac's delta function ensuring that the reactionary force is acting only at point $x_j$, the location of the TMD.

### 2.1. Free Vibration

Let the equation of motion, in free vibration, be expressed as:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\overline{m}\frac{\partial^2 w(x,t)}{\partial t^2}+R(x,t)=0 \qquad (3)$$

At this point the separable variables approach is generally applied, where:

$$w(x,t)=\psi(x)g(t) \qquad (4)$$

Where $g(t)$ can also be defined as: $g(t)=e^{i\omega t}$

Here, the space domain term $\psi(x)$ and the time domain term- $g(t)$ are split; this allows for the solution of the beam's eigenvalues and eigenfunctions, however, in order to account for the effects of the reactionary forces in the space domain, $R(x,t)$ must be Fourier transformed [5]:

$$P_j(\omega) = -\sum_{j=1}^{N} \varphi_j(\omega)\delta\left(x - x_j\right) \qquad (5)$$

Where $\delta\left(x - x_j\right)$ is a Dirac's delta function specifying the application point of the force as before and [6]:

$$\varphi_j(\omega) = -K_{TMD\,j}(\omega)\ \psi(x_j) \qquad (6)$$

Where $\psi(x_j)$ is the eigenfunction of deflection, and $K_{TMD\,j}(\omega)$ is the frequency dependent stiffness of the spring-mass attachment given by [7]:

$$K_{\mathrm{TMD}\,j}(\omega) = \frac{(k_{\mathrm{TMD\,j}} + \mathrm{i}\omega\ c_{TMD\,j})\ \mathrm{M}_{TMD\,j}\ \ \omega^2}{\mathrm{M}_{TMD\,j}\ \ \omega^2 - (k_{\mathrm{TMD\,j}} + \mathrm{i}\omega\ c_{TMD\,j})} \qquad (7)$$

Where: $\mathrm{M}_{TMD\,j}$ is the magnitude of the j$^{th}$ TMD, $k_{\mathrm{TMD\,j}}$ the spring stiffness of the j$^{th}$ TMD, $c_{TMD\,j}$ is the damping coefficient of the dashpot, which forms part of the j$^{th}$ TMD, and $\omega$ is the natural frequency.

## 3. EIGENSOLUTION

The exact modes of vibration can be found by applying the separable variables method, equation 4, allowing the transversal displacement $w(x,t)$, rotation $\theta(x,t)$, bending moment $m(x,\tau)$, and shear $Q(x,t)$, to be expressed as:

$$w(x,t) = \psi(x)e^{\mathrm{i}\omega t}\ ;\ \theta(x,t) = \vartheta(x)e^{\mathrm{i}\omega t}$$
$$m(x,\tau) = \mu(x)e^{\mathrm{i}\omega\tau}\ ;\ Q(x,t) = \chi(x)e^{\mathrm{i}\omega t} \qquad (8)$$

This gives the four eigenfunctions of the response variables $\psi(x)$ deflection, $\vartheta(x)$ rotation, $\mu(x)$ bending moment, and $\chi(x)$ shear. These eigenfunctions are related in the following manner:

$$\vartheta(x) = \frac{d\psi(x)}{dx};\ \frac{d\vartheta(x)}{dx} = -\frac{\mu(x)}{EI} \qquad (9)$$

$$\chi(x) = \frac{d\mu(x)}{dx};$$

$$\frac{d\chi(x)}{dx} + \sum_{j=1}^{N} -P_j(t)\ \delta(x - x_j) + \sigma^2\psi(x) = 0$$

From these relations, the free vibration of the beam can then be expressed in terms of the first eigenfunction in only the space domain:

$$\frac{d^4\psi(x)}{dx^4} + \sum_{j=1}^{N} P_j(\omega) - \sigma^2\psi(x) = 0 \qquad (10)$$

Where $\sigma^2 = (\omega^2 \overline{m} L^4)/EI$ .

Eq. (7) shows that the attached TMD reactionary force's relation to the deflection at point $x_j$ depends solely on the frequency dependent term concerning the spring stiffness and the attached mass.

In Eq. (5) the variable $\varphi_j(\omega)$ at $x_j$ is an unknown due to the term containing the eigenfunction of deflection and the frequency dependence of the term which is currently unknown. Therefore, the matrix approach is applied:

Let $\mathbf{Y}(x)$ be a vector of the response variables of the eigenfunctions:

$$\mathbf{Y}(x) = \left[\psi(x)\quad \vartheta(x)\quad \mu(x)\quad \chi(x)\right]^T \qquad (11)$$

Following the approach proposed by [8] the unknown $\varphi_j(\omega)$ can be obtained as a linear function of the $4\times1$ vector $\boldsymbol{c}$ which is composed of the four integration constants found from the solution of the homogeneous equation. This leads to the following closed analytical expression of $\mathbf{Y}(x)$:

$$\mathbf{Y}(x) = \tilde{\mathbf{Y}}(x)\boldsymbol{c} \qquad (12)$$

Where $\tilde{\mathbf{Y}}(x)$ is a $4\times4$ matrix given by:

$$\tilde{\mathbf{Y}}(x) = \mathbf{\Omega}(x) + \sum_{j=1}^{N} \mathbf{J}\left(x, x_j\right) P_j(\omega) \qquad (13)$$

Where:

$$\mathbf{\Omega}(x) = \begin{bmatrix} \Omega_{\psi_1} & \Omega_{\psi_2} & \Omega_{\psi_3} & \Omega_{\psi_4} \\ \Omega_{\vartheta_1} & \Omega_{\vartheta_2} & \Omega_{\vartheta_3} & \Omega_{\vartheta_4} \\ \Omega_{\mu_1} & \Omega_{\mu_2} & \Omega_{\mu_3} & \Omega_{\mu_4} \\ \Omega_{\chi_1} & \Omega_{\chi_2} & \Omega_{\chi_3} & \Omega_{\chi_4} \end{bmatrix} \quad (14)$$

and [9]:

$$\mathbf{J}(x, x_j) = \left[ J_{\psi}^{(p)} \quad J_{\vartheta}^{(p)} \quad J_{\mu}^{(p)} \quad J_{\chi}^{(p)} \right]^T \quad (15)$$

Where, for the sake of clarity, the matrix $\mathbf{\Omega}(x)$ is constructed from terms contained in the general solution of the homogeneous equation in the first row and then, following the derivative method used to relate the eigenfunctions, the subsequent rows are constructed:

$$\Omega_{\psi_1}(x) = e^{-\sigma x} \qquad \Omega_{\psi_2}(x) = e^{\sigma x}$$
$$\Omega_{\psi_3}(x) = \cos(\sigma x) \qquad \Omega_{\psi_4}(x) = \sin(\sigma x)$$
$$\Omega_{\vartheta_1}(x) = -\sigma e^{-\sigma x} \qquad \Omega_{\vartheta_2}(x) = \sigma e^{\sigma x}$$
$$\Omega_{\vartheta_3}(x) = -\sigma \sin(\sigma x) \qquad \Omega_{\vartheta_4}(x) = \sigma \cos(\sigma x)$$
$$\Omega_{\mu_1}(x) = -\sigma^2 e^{-\sigma x} \qquad \Omega_{\mu_2}(x) = \sigma^2 e^{\sigma x}$$
$$\Omega_{\mu_3}(x) = \sigma^2 \cos(\sigma x) \qquad \Omega_{\mu_4}(x) = \sigma^2 \sin(\sigma x)$$
$$\Omega_{\chi_1}(x) = \sigma^3 e^{-\sigma x} \qquad \Omega_{\chi_2}(x) = -\sigma^3 e^{\sigma x}$$
$$\Omega_{\chi_3}(x) = -\sigma^3 \sin(\sigma x) \qquad \Omega_{\chi_4}(x) = \sigma^3 \cos(\sigma x) \quad (16)$$

From here, the boundary conditions of the beam can be enforced which leads to the solution below:

$$\mathbf{B}_{4\times4}\mathbf{c}_{4\times1} = \mathbf{0}_{4\times1} \quad (17)$$

Where $\mathbf{B}$ is a matrix constructed from enforcing the boundary conditions on the matrix $\tilde{\mathbf{Y}}(x)$.

The characteristic equation can then be built as the determinant of the $4\times4$ matrix $\mathbf{B}$:

$$\det(\mathbf{B}_{4\times4}) = 0 \quad (18)$$

At this point the non-trivial solutions of $\boldsymbol{c}$ are found and exact closed form expressions can be built for the beam's eigenfunctions. As there is a damping element in the model, complex modes must be considered.

## 4. ORTHOGONALITY CONDITIONS

The orthogonality conditions are then built following the method presented in [10] to derive the particular impulse response function of this beam.

Firstly the equation of motion in free vibration in the form shown below is considered:

$$\frac{d^4\psi_k(x)}{dx^4} - \sigma_k^{\,2}\psi_k(x) - \sum_{j=1}^{N} K_{TMD\ j}(\omega_k)\psi_k(x_j) = 0 \quad (19)$$

Considering modes $m$ and $n$, multiplying the equation of motion at mode $m$ by $\psi_n(x)$ and at mode $n$ by $\psi_m(x)$ and then integrating between 0 and L with respect to x:

$$\int_0^L \frac{d^2\psi_m(x)}{dx^2}\frac{d^2\psi_n(x)}{dx^2}dx - \sigma_m^{\,2}\int_0^L \psi_{mn}(x)\,dx + \sum_{j=1}^{N} K_{TMD\ j}(\omega_m)\psi_{mn}(x_j) = 0 \quad (20)$$

Where: $\psi_{mn}(x) = \psi_m(x)\psi_n(x)$

$$\int_0^L \frac{d^2\psi_n(x)}{dx^2}\frac{d^2\psi_m(x)}{dx^2}dx - \sigma_n^{\,2}\int_0^L \psi_{nm}(x)\,dx + \sum_{j=1}^{N} K_{TMD\ j}(\omega_n)\psi_{nm}(x_j) = 0 \quad (21)$$

Integrating by parts and subtracting Eq. (21) from Eq. (20) then yields the first orthogonality condition:

$$(\sigma_m^{\,2} - \sigma_n^{\,2})\int_0^L \psi_{nm}(x)\,dx + \sum_{j=1}^{N}[K_{TMD\ j}(\omega_n) - K_{TMD\ j}(\omega_m)]\psi_{nm}(x_j) = 0 \quad (22)$$

The second orthogonality condition is then found by multiplying Eq. (20) by $\sigma_n$ and Eq. (21) by $\sigma_m$ and then subtracting Eq. (21) from Eq. (20):

$$\sigma_{m-n}\int_0^L \frac{\partial^2\psi_{mn}(x)}{\partial x^2}dx + \sigma_m\sigma_n(\sigma_{m-n})\int_0^L \psi_{nm}(x)\,dx + [\sigma_m K_{TMD\ j}(\omega_n) - \sigma_n K_{TMD\ j}(\omega_m)]\psi_{nm}(x_j) = 0 \quad (23)$$

Where $\frac{\partial^2\psi_{mn}(x)}{\partial x^2} = \frac{\partial^2\psi_m(x)}{\partial x^2}\frac{\partial^2\psi_n(x)}{\partial x^2}$ and $\sigma_{m-n} = (\sigma_m - \sigma_n)$

## 5. FORCED VIBRATIONS

These orthogonality conditions are then used to derive the beam's response to arbitrary loading. This is accomplished by using the complex modal superposition principle as defined by [10] where the complex modal impulse response function is used. This leads to [2]:

$$w(x,t)=\sum_{k=1}^{\infty}\psi_k(x)\frac{2}{i\,\Xi_k\,\omega_k}\int_0^t f(\tau)\,e^{i\omega_k(\tau-t)}d\tau \qquad (24)$$

Where: $f(\tau)$ is the moving load and $\Xi_k$ is the effect that the beam's mass and the attached TMD have on the beam's response:

$$\Xi_k=2\int_0^L \overline{m}(x)[\psi_k(x)]^2dx+\sum_j^N TMD_j\ [\psi_k(x_J)]^2 \qquad (25)$$

Where

$$TMD_j=\frac{M_{TMD\ j}[2\,(k_{TMD\ j}+i\omega_k c_{TMD\ j})^2-i\omega_k{}^3M_{TMD\ j}c_{TMD\ j}]}{[(k_{TMD\ j}+i\omega_k c_{TMD\ j})-M_{TMD\ j}\omega_k{}^2]^2}$$

For a moving load, the response equation takes the following form:

$$w(x,t)=\sum_{k=1}^{\infty}\left(\psi_k(x)\int_0^t e^{i\omega_k(\tau-t)}d\tau\right)\frac{2\int_0^L\psi_k(x)\delta(x-V_0\ \tau)\,dx}{i\,\Xi_k\,\omega_k} \qquad (26)$$

Where: $\delta(\cdot)$ is a Dirac's delta function and $V_0$ is the velocity of the load.

Further, when considering multiple loads traversing the beam, the effects of preceding loads must also be accounted for:

$$w(x,t)=\sum_{k=1}^{\infty}\psi_k(x)\frac{2\int_0^L\psi_k(x)\delta(x-V_0\ \tau)\,dx}{i\,\Xi_k\,\omega_k}\left(\int_{\tau_L^0}^t e^{i\omega_k(\tau-t)}d\tau-\int_{\tau_L^E}^t e^{i\omega_k(\tau-t)}d\tau\right) \qquad (27)$$

Where: $\tau_L^0$ and $\tau_L^E$ denote the start and end times of the L[th] load.

Due to the presence of complex conjugate pairs, Eq. (27) can revert to the following real form [2]:

$$w(x,t)=\mathrm{Re}\left[\sum_{k=1}^{\infty}\psi_k(x)\frac{2\int_0^L\psi_k(x)\delta(x-V_0\ \tau)\,dx}{i\,\Xi_k\,\omega_k}\left(\int_{\tau_L^0}^t e^{i\omega_k(\tau-t)}d\tau-\int_{\tau_L^E}^t e^{i\omega_k(\tau-t)}d\tau\right)\right] \qquad (28)$$

## 6. POISSONIAN WHITE NOISE PROCESSES

A Poissonian white noise process is a type of delta-correlated process [11], which is most commonly defined as [1]:

$$S_P(t)=\sum_{P=1}^{N(t)}Y_P\delta(t-T_P) \qquad (29)$$

Where $N(t)$ is a counting function giving the number of impulses in the time interval (0,t), $Y_P$ is the random amplitude of the forcing action, and $\delta(t-T_P)$ is a series of Dirac delta impulses [12] occurring at independent random times ($T_P$) following a Poissonian distribution.

When considering a moving load, this characterisation of the Poissonian load must be altered, it is also assumed that the loads will have a constant and equal velocity:

$$S_P(t)=\sum_{P=1}^{N(t)}Y_P\delta\left[x-(t-T_P)V_0\right]W(t-T_P,t_L) \qquad (30)$$

Where $\delta\left[x-(t-T_P)V_0\right]$ is a modification of the Dirac delta function from equation (29) in which moving loads arriving at random times with random amplitudes are considered, $t_L$ is the time taken for the load to traverse the beam, length divided by the velocity of the moving load, $L/V_0$ and $W(t-t_p,t_L)$ is a window function which removes the force after it has traversed the beam; here $U(\cdot)$ is a unit step function: $W(t-T_P,t_L)=U(\tau)[1-U(\tau-t_L)]$.

Following the method proposed by [1] [13] the Poisson process is filtered to ensure that it is applicable to the beam's characteristics, this filtering causes Eq. (30) to take the following form:

$$S_k\left(t\right)=\sum_{P=1}^{N(t)}Y_P\psi_k\left(V_0\,\tau\right)W(t-T_P,t_L) \tag{31}$$

Substituting this into the original equation of motion gives:

$$EI\,\frac{\partial^4 w(x,t)}{\partial x^4}+\bar{m}\frac{\partial^2 w(x,t)}{\partial t^2}+R(x,t)=$$

$$=\sum_{\mathrm{P}=1}^{N(t)}Y_P\psi_k(V_0\ t)W(t-T_P,t_L) \tag{32}$$

Considering the theory of separable variable, this can also take the following form in the time domain:

$$\ddot{g}_k\left(t\right)+\omega_k^{\,2}g_k(t)=\frac{2}{\Xi_k}S_k(t) \tag{33}$$

Where $S_k(t)$ is the random forcing action at the k$^{th}$ mode, this takes the form:

$$S_k\left(t\right)=\sum_{P=1}^{N(t)}Y_P\,\psi_k\left(V_0\,\tau\right)W(\tau,t_L) \tag{34}$$

## 7. NUMERICAL APPLICATION

Numerical results were then obtained for the beam shown in figure 2; this beam has a length of 100m, its mass per unit length is 12000kg/m, and its flexural rigidity is 400GPa. It was subjected to a series of moving loads following a Poissonian distribution moving at a constant speed of 25m/s, which was simulated using the Monte-Carlo method where 2000 samples were generated for each of the 5 modes considered. The mean and the standard deviations of both the displacement and the velocity of the resultant vibrations are shown in figures 3-6 below.

## 8. CONCLUSIONS

This paper presents a novel method to calculate the dynamic response to deterministic loads and moving random loads of beams with any number of discontinuities, this method was then extended to consider random moving loads which follow a Poissonian distribution; arriving at random times and with random magnitudes, but with a constant velocity. The proposed solution yields significant computational advantages with respect to the classical modal superposition method which requires the beam to be divided into segments at the location of each discontinuity, this results in a matrix of dimensions $4(n+1)\mathrm{x}4(n+1)$ for n number of attachments where the computational time required to find the determinant of this matrix, and to then build the characteristic equation, can be prohibitive.

This was achieved through the use of the theory of generalised functions and the modal superposition principle, this allowed novel closed-form expressions for the beam's eigenfunctions to be derived by Fourier transforming the equation of motion. From this the characteristic equation could be found as the determinant of a 4x4 matrix for n number of attachments, saving computational time. From this characteristic equation, through the use of appropriate orthogonality conditions, the beam's impulse response function can be used to find closed-form expressions of the time domain response which can then be calculated through the use of Duhamel's integral for m number of modes.

A Monte-Carlo simulation was then used to find the standard deviation and the mean response of the beam to a series of random loads. This consisted of solving Eq. 27 two thousand times for each mode, ten thousand simulations in total, with

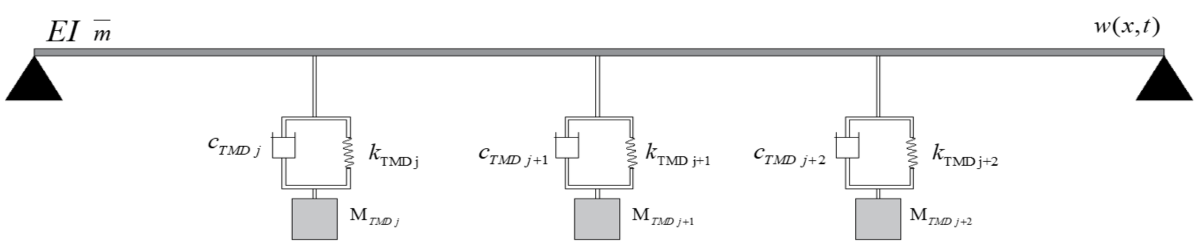


*Figure 1 - Continuous Beam fitted with three TMDs*

a sampling rate of 1000 samples per second for 15 seconds.

The numerical results section clearly shows a reduction in the standard deviation of displacement when a TMD is fitted although there does not appear to be any significant improvement using three TMDs vs only one. No such improvement can be seen in the mean of these results whereas the maximum velocity is greater when considering the standard deviation in the case with three TMDs although this could be due to the lower displacements involved.

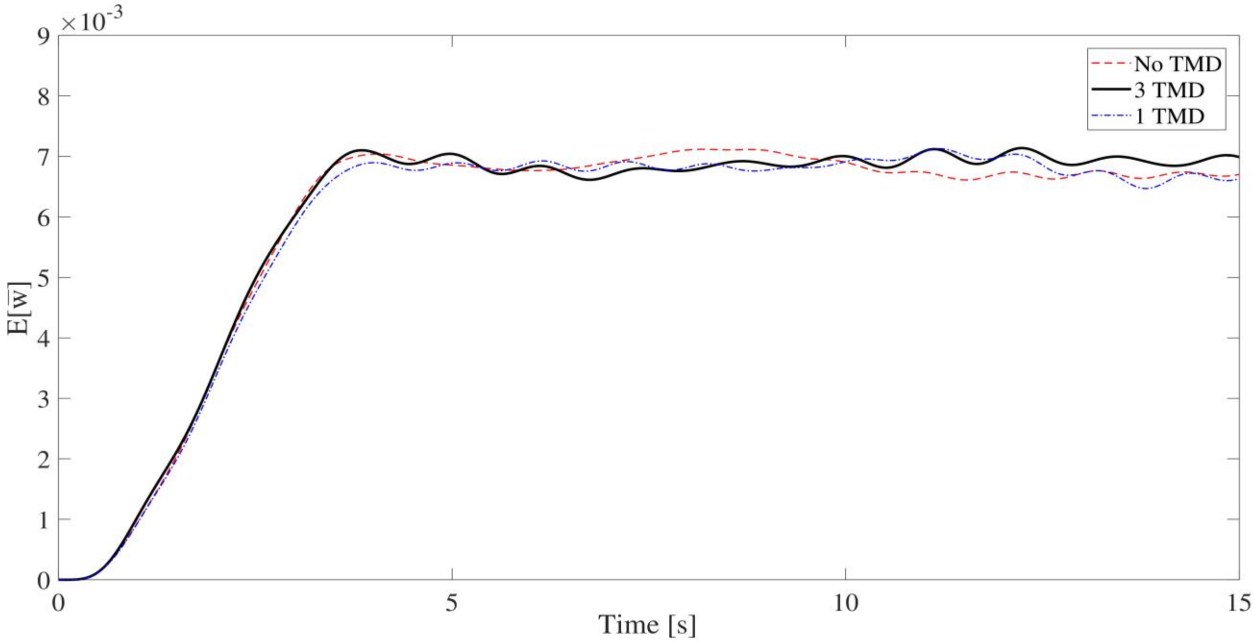


*Figure 2 - Mean Displacement*

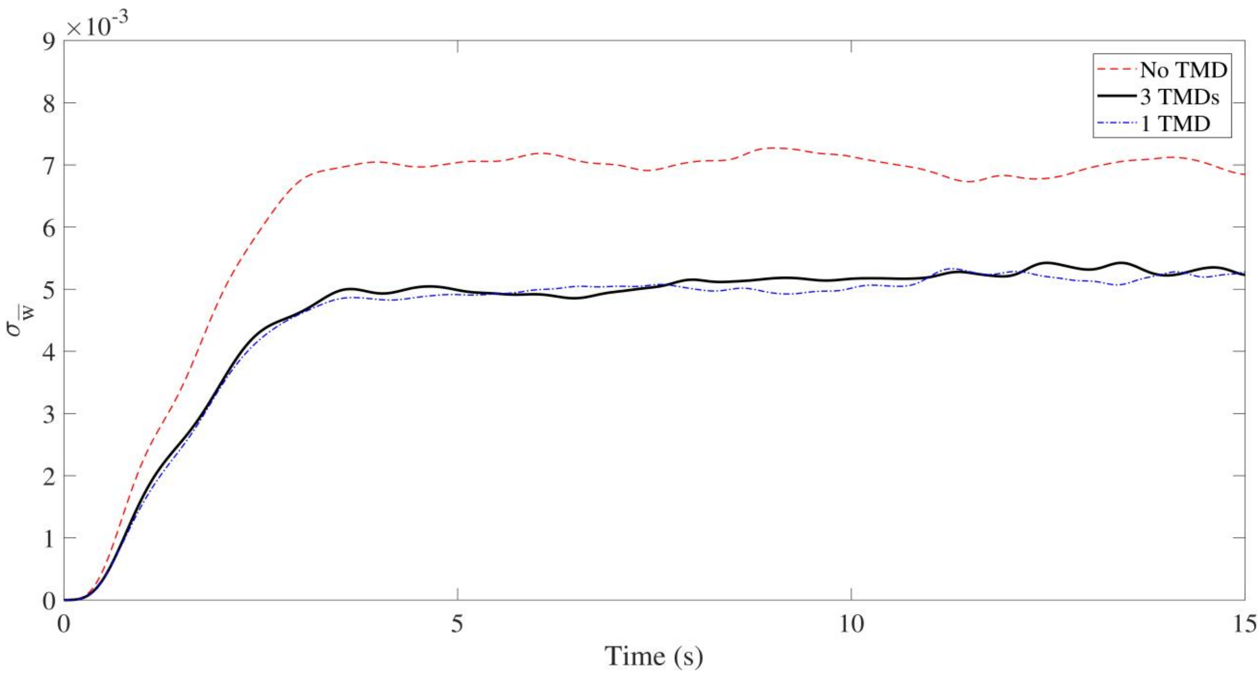


*Figure 3 - Standard Deviation of Displacement*

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