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NON ABSOLUTELY CONVERGENT INTEGRALS OF FUNCTIONS TAKING VALUES IN A LOCALLY CONVEX SPACE

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ABSTRACT. Properties of McShane and Kurzweil-Henstock integrable functions taking values in a locally convex space are considered and the relations with other integrals are studied. A convergence theorem for the Kurzweil-Henstock integral is given.

1. Introduction. In this paper we continue the investigation of the McShane and the Kurzweil-Henstock integrals for functions defined on a compact interval of the real line and taking values in a locally convex space. In [**12**] the McShane and Kurzweil-Henstock integrals for functions taking values in a locally convex space were introduced and some properties of the integrals were considered.

When the range is a Banach space, a measurable and Pettis integrable function f is McShane integrable, see [**9**, Theorem 17]. We prove that the same result holds for functions whose range is a Hausdorff locally convex topological vector space (Theorem 3), if we consider the measurability by seminorm instead of measurability.

In Section 3 we study some properties of the McShane and Pettis integrals.

In Section 4 we establish relations between the McShane, the Pettis and the Kurzweil-Henstock integrals.

In Section 5 we prove a convergence theorem for the Kurzweil-Henstock integral.

2. Definitions and notations. Let X be a Hausdorff locally convex topological vector space (briefly a locally convex space) with

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its topology $\mathcal{T}$ and topological dual X^*. $\mathcal{P}(X)$ denotes a family of $\mathcal{T}$-continuous seminorms on X so that the topology is generated by $\mathcal{P}(X)$. For $p \in \mathcal{P}(X)$, let $V_p = \{x \in X : p(x) \leq 1\}$, so that V_p^0, the polar of V_p in X^*, is a *weak**-closed, absolutely convex equicontinuous set in X^*. For a set E of the real numbers $|E|$ and χ_E denote respectively the Lebesgue outer measure and the characteristic function of E. $\mathcal{M}$ denotes the family of all Lebesgue measurable subsets of $[0,1]$. The word "measurable" as well as the expression "almost everywhere," abbreviated as a.e., always refer to the Lebesgue measure. An *interval* is a compact subinterval of $\mathbf{R}$. A collection of intervals is called *nonoverlapping* if their interiors are disjoint. A *partition* P in $[0,1]$ is a collection $\{(I_i, t_i) : i = 1, \dots, s\}$, where $I_1, \dots, I_s$ are nonoverlapping subintervals of $[0,1]$ and $t_1, \dots, t_s \in [0,1]$. Given a set $E \subset \mathbf{R}$, we say that P is

(i) a partition *in* E if $\cup_{i=1}^s I_i \subset E$;

(ii) a partition *of* E if $\cup_{i=1}^s I_i = E$;

(iii) a *Perron* partition if $t_i \in I_i$, $i = 1, \dots, s$.

Given $f : [0,1] \to X$ and a partition $P = \{(I_i, t_i) : i = 1, \dots, s\}$ in $[0,1]$, we set

$$\sigma(f, P) = \sum_{i=1}^{s} |I_i| \, f(t_i).$$

A *gauge* δ on $E \subset [0,1]$ is a positive function on E. For a given gauge δ on E a partition $P = \{(I_i, t_i) : i = 1, \dots, s\}$ in $[0,1]$ is called δ-*fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$.

A function $f : [0,1] \to X$ is called

(a) *strongly measurable* if there exists a sequence $(f_n)_n$ of simple functions such that $f_n(t) \to f(t)$ a.e.;

(b) *measurable by seminorm* if for any $p \in \mathcal{P}(X)$ there exists a sequence $(f_n^p)_n$ of simple functions such that $\lim_{n\to\infty} p(f_n^p(t) - f(t)) = 0$ a.e.;

(c) *weakly-measurable* if the function $x^* f$ is measurable for every $x^* \in X^*$.

We recall the following definitions, see [**2**, Definition 2.4].

Definition 1. A function $f : [0,1] \to X$ is said to be strongly (or Bochner) integrable if there exists a sequence $(f_n)_n$ of simple functions such that

(i) $f_n(t) \to f(t)$ a.e.;

(ii) $p(f(t) - f_n(t)) \in L^1([0,1])$ for each $n \in \mathbf{N}$ and $p \in \mathcal{P}(X)$, and $\lim_{n\to\infty} \int_0^1 p(f(t) - f_n(t))\, dt = 0$ for each $p \in \mathcal{P}(X)$;

(iii) $\int_A f_n$ converges in X for each measurable subset A of $[0,1]$.

In this case we put $(B)\int_A f = \lim_{n\to\infty} \int_A f_n$.

Definition 2. A function $f : [0,1] \to X$ is said to be integrable by seminorm if for any $p \in \mathcal{P}(X)$ there exists a sequence $(f_n^p)_n$ of simple functions such that

(i) $\lim_{n\to\infty} p(f_n^p(t) - f(t)) = 0$ a.e.;

(ii) $p(f(t) - f_n^p(t)) \in L^1([0,1])$ for each $n \in \mathbf{N}$ and $p \in \mathcal{P}(X)$, and $\lim_{n\to\infty} \int_0^1 p(f(t) - f_n^p(t))\, dt = 0$ for each $p \in \mathcal{P}(X)$;

(iii) for each measurable subset A of $[0,1]$ there exists an element $y_A \in X$ such that $\lim_{n\to\infty} p(\int_A f_n^p(t) - y_A) = 0$ for every $p \in \mathcal{P}(X)$.

We then put $\int_A f = y_A$.

Clearly a Bochner integrable function is integrable by seminorm, and the two definitions coincide in a Banach space.

Definition 3. A function $f : [0,1] \to X$ is said to be Pettis integrable if $x^* f$ is Lebesgue integrable on $[0,1]$ for each $x^* \in X^*$ and for every $E \in \mathcal{M}$ there is a vector $\nu_f(E) \in X$ such that $x^*(\nu_f(E)) = \int_E x^* f(t)\, dt$ for all $x^* \in X^*$.

The set function $\nu_f : \mathcal{M} \to X$ is called the indefinite Pettis integral of f. It is known that ν_f is a countably additive vector measure, continuous with respect to the Lebesgue measure (in the sense that if $|E| = 0$ then $\nu_f(E) = 0$).

We recall the definition of McShane and Kurzweil-Henstock integral, see [**12**, Definition 4].

Definition 4. A function $f : [0,1] \to X$ is said to be McShane integrable, respectively Kurzweil-Henstock integrable, (briefly McS-integrable, respectively KH-integrable) on $[0,1]$, if there exists a vector $w \in X$ satisfying the following property: given $\varepsilon > 0$ and $p \in \mathcal{P}(X)$ there exists a gauge δ_p on $[0,1]$ such that for each δ_p-fine partition, respectively Perron partition, $P = \{(I_i, t_i) : i = 1, \dots, s\}$ of $[0,1]$, we have

$$p\left(\sigma(f, P) - w\right) < \varepsilon.$$

We denote by $McS([0,1], X)$, respectively $KH([0,1], X)$, the family of all McS-integrable, respectively KH-integrable, functions on $[0,1]$, and we set $w = (McS)\int_0^1 f$, respectively $w = (KH)\int_0^1 f$. If $f : [0,1] \to X$ is McS-integrable, respectively KH-integrable, and if $0 \le a < b \le 1$, then the function $f\chi_{[a,b]}$ is McS-integrable, respectively KH-integrable, see [**12**, Lemma 1]. Moreover every Bochner integrable function is McS-integrable and the two integrals coincide, see [**12**, Corollary 1].

To simplify the notation, in the following we write $|x^*| \le p$ instead of $|x^*(x)| \le p(x)$ for each $x \in X$, and we denote by X_p^* the set $\{x^* \in X^* : |x^*| \le p\}$. We recall that a seminorm $p \in P(X)$ is called *representable* if

$$p(x) = \sup_{X_p^*} |x^*(x)| \tag{1}$$

for all $x \in X$. If (1) holds for all $p \in \mathcal{P}(X)$, the space X is said to be *representable by seminorm.* If a space X is separable by seminorm, then it is representable by seminorm, see [**7**, p. 185].

3. Properties of McShane and Pettis integrable functions. In this section we extend to locally convex spaces some results known for Banach spaces.

From now on X will be a complete locally convex space.

We need the following lemmata.

Lemma 1 [**15**, Lemma 7]. *There exists a positive McS-integrable function $\varphi : [0,1] \to (0, \infty)$ and a gauge δ_φ such that $0 \le \sigma(\varphi, P) \le 1$ for every δ_φ-fine partition P of $[0,1]$.*

Lemma 2 [**12**, Lemma 2]. *Let $f : [0,1] \to X$ be an McS-integrable function. Then to each $\varepsilon > 0$ and to each $p \in \mathcal{P}(X)$ there corresponds a gauge δ_p such that*

$$p\bigg(\sum_{i=1}^{s}\bigg(|I_i|\,f(t_i) - (McS)\int_{I_i} f\bigg)\bigg) < \varepsilon$$

for each partition δ_p-fine $P = \{(I_i, t_i) : i = 1, \ldots, s\}$ in $[0,1]$.

The following proposition when the range is a Banach space, has been proved in [**9**, Theorem 15] with a different technique.

Proposition 1. *Let $(E_n)_n$ be a sequence of disjoint measurable sets in $[0,1]$ and let $(x_n)_n$ be a sequence in X and let $f : [0,1] \to X$ be defined by*

$$f(t) = \sum_n x_n \chi_{E_n}(t).$$

If the series $\sum_n |E_n| x_n$ is unconditionally convergent, then the function f is McS-integrable on $[0,1]$ and

$$(McS)\int_0^1 f = \sum_n |E_n| x_n.$$

Proof. The function f is countably valued, so $f([0,1])$ is separable and without loss of generality we can restrict our domain to $f([0,1])$. Therefore for every $p \in \mathcal{P}(X)$ and $y \in X$, we have $p(y) = \sup_{\{|x^*| \leq p\}} |x^*(y)|$. Fix $p \in \mathcal{P}(X)$ and $\varepsilon > 0$. Since the series $\sum_{n=1}^{\infty} |E_n| x_n$ is unconditionally convergent there is a natural number N such that for $n > N$

$$\sum_{k=n}^{\infty} |x^*(x_k)|\,|E_k| < \frac{\varepsilon}{4} \tag{2}$$

uniformly with respect to $x^* \in X_p^*$, [**14**, p. 120]. Moreover for each natural number n the function $f_n = \sum_{k=1}^{n} x_k \chi_{E_k}$ is Bochner integrable, therefore it is McS-integrable and $(McS)\int_0^1 f_n = \sum_{k=1}^n |E_k| x_k$. So by Lemma 2 there is a gauge $\delta_p^n = \delta_n$ such that

$$p\bigg(\sum_{i=1}^{s}\bigg(|I_i|\, f_n(t_i) - (McS)\int_{I_i} f_n\bigg)\bigg) < \frac{\varepsilon}{2^{n+1}} \tag{3}$$

for each δ_n-fine partition $P = \{(I_i, t_i) : i = 1, \dots, s\}$ in $[0,1]$. Also $\lim_{n\to\infty}(McS)\int_0^1 f_n = \sum_{n=1}^{\infty} |E_n| x_n$ and $(f_n)_n$ converges to f pointwise. Thus for every $t \in [0,1]$ there exists $n(t) \ge N$ such that if $k > n(t)$

$$p(f_k(t) - f(t)) < \frac{\varepsilon}{4}\,\varphi(t) \tag{4}$$

where $\varphi(t)$ is an McS-integrable function satisfying Lemma 1. Let $g_k(t) = x_k \chi_{E_k}$. Define $\delta_\varphi(t)$ related to the real valued function $\varphi(t)$ as in Lemma 1 and put $\delta_p(t) = \min\{\delta_{n(t)}(t), \delta_\varphi(t)\}$. If $P = \{(I_i, t_i) : i = 1, \dots, s\}$ is a δ_p-fine partition of $[0,1]$, we have

$$\begin{aligned}
&p\bigg(\sigma(f,P) - \sum_{n=1}^{\infty} |E_n| x_n\bigg)\\
&\quad = p\bigg(\sum_{i=1}^{s} |I_i|\, f(t_i) - \sum_{n=1}^{\infty} |E_n| x_n\bigg)\\
&\quad = p\bigg(\sum_{i=1}^{s}\bigg\{\sum_{k=1}^{\infty} |I_i|\, g_k(t_i) - \sum_{k=1}^{\infty}(McS)\int_{I_i} g_k\bigg\}\bigg)\\
&\quad \le p\bigg(\sum_{i=1}^{s}\bigg\{\sum_{k=n(t_i)+1}^{\infty} |I_i|\, g_k(t_i)\bigg\}\bigg)\\
&\qquad + p\bigg(\sum_{i=1}^{s}\bigg\{\sum_{k=1}^{n(t_i)} |I_i|\, g_k(t_i) - \sum_{k=1}^{n(t_i)}(McS)\int_{I_i} g_k\bigg\}\bigg)\\
&\qquad + p\bigg(\sum_{i=1}^{s}\sum_{k=n(t_i)+1}^{\infty}(McS)\int_{I_i} g_k\bigg)\\
&\quad = A_1 + A_2 + A_3,
\end{aligned} \tag{5}$$

where the definitions for A_j are obvious. We estimate each A_j. Applying (4) and Lemma 1 to A_1, we get

$$\begin{aligned} A_1 &\le \sum_{i=1}^{s} p\bigg(\sum_{k=n(t_i)+1}^{\infty} |I_i|\, g_k(t_i)\bigg) \le \sum_{i=1}^{s} |I_i|\, p\bigg(\sum_{k=n(t_i)+1}^{\infty} g_k(t_i)\bigg) \\ &< \sum_{i=1}^{s} |I_i|\, \frac{\varepsilon}{4}\, \varphi(t_i) = \frac{\varepsilon}{4}\, \sigma(\varphi, P) \le \frac{\varepsilon}{4}. \end{aligned} \tag{6}$$

For estimating A_2, let $r = \max\{n(t_1), \dots, n(t_s)\}$. By (3) we obtain

$$\begin{aligned} A_2 &= p\bigg(\sum_{i=1}^{s} \bigg\{|I_i|\, f_{n(t_i)}(t_i) - (McS)\int_{I_i} f_{n(t_i)}\bigg\}\bigg) \\ &= p\bigg(\sum_{k=1}^{r} \sum_{\{i:n(t_i)=k\}} \bigg\{|I_i|\, f_{n(t_i)}(t_i) - (McS)\int_{I_i} f_{n(t_i)}\bigg\}\bigg) \\ &\le \sum_{k=1}^{r} p\bigg(\sum_{\{i:n(t_i)=k\}} \bigg\{|I_i|\, f_{n(t_i)}(t_i) - (McS)\int_{I_i} f_{n(t_i)}\bigg\}\bigg) \\ &\le \sum_{k=1}^{r} \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{2}. \end{aligned} \tag{7}$$

For A_3, by (2) we get

$$\begin{aligned} A_3 &\le \sup_{\{|x^*|\le p\}} \sum_{i=1}^{s} \sum_{k=n(t_i)+1}^{\infty} \int_{I_i} |x^* g_k| \\ &\le \sup_{\{|x^*|\le p\}} \sum_{i=1}^{s} \sum_{k=N+1}^{\infty} \int_{I_i} |x^* g_k| \\ &\le \sup_{\{|x^*|\le p\}} \sum_{k=N+1}^{\infty} \int_0^1 |x^* g_k| \\ &\le \sup_{\{|x^*|\le p\}} \sum_{k=N+1}^{\infty} |x^*(x_k)|\, |E_k| < \frac{\varepsilon}{4}. \end{aligned} \tag{8}$$

Applying (6), (7) and (8) in (5) we obtain

$$p\biggl(\sigma(f,P)-\sum_{n=1}^{\infty}|E_n|x_n\biggr)\le\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon,$$

and the assertion follows. □

In the sequel we shall use the fact that every separated convex space is a projective limit of normed spaces, see [**13**, p. 86]. For each continuous seminorm p on the convex space X, $p^{-1}(0)$ is a vector subspace and p defines a norm on $X/p^{-1}(0)$. X_p is the associated Banach space, namely the completion of the normed linear space $X/p^{-1}(0)$ and π_p is the canonical mapping of X into X_p. Then X is the projective limit of the spaces X_p by the canonical mapping π_p of X onto X_p. For a function $f:[0,1]\to X$ and for each $p\in\mathcal{P}(X)$, define the function $f_p:[0,1]\to X_p$ by

$$f_p(t)=(\pi_p\circ f)(t)=\pi_p(f(t))$$

for $t\in[0,1]$.

We characterize measurable by seminorm Pettis integrable functions. The analogous result in the case in which the range is a Banach space, has been proved in [**3**, Theorem 1].

Theorem 1. *Let $f:[0,1]\to X$ be a measurable by seminorm Pettis integrable function. Then, for each $p\in\mathcal{P}(X)$, there are two functions g and h such that $f=g+h$, with $h(t)=\sum_n x_n\chi_{E_n}(t)$, where the sets E_n are disjoint, the series $\sum_n|E_n|x_n$ is unconditionally convergent in X_p and the function g is bounded in X_p. If f is integrable by seminorm, then the series $\sum_n|E_n|x_n$ is absolutely convergent in X_p.*

Proof. Let $p\in\mathcal{P}(X)$ and $\varepsilon>0$. Since f is measurable by seminorm, we can assume that $f([0,1])$ is separable for p, so there exists a countable subset $\{x_1,x_2,\dots\}$ dense for p in $f([0,1])$. The collection

$$U_n^p=\{t\in[0,1]:p(f(t)-x_n)<\varepsilon\}\quad n=1,2,\dots$$

covers $f([0,1])$. Put

$$E_1=U_1$$

and

$$E_n = U_n \setminus \bigcup_{i=1}^{n-1} U_i, \quad n > 1$$

and define the function $h(t) = x_n$ for $t \in E_n$. The function h is a countably valued strongly measurable function, then $g(t) = f(t) - h(t)$ is measurable by seminorm and $p(g(t)) < \varepsilon$. Moreover the function

$$g_p = \pi_p \circ g : [0,1] \to X_p$$

is strongly measurable and has an essentially bounded range. Therefore it is Bochner integrable; in particular, it is Pettis integrable, see [**4**].

Being h is the difference of two Pettis integrable functions, it is scalarly integrable. Then for each $x^* \in X^*$, $x^*h \in L^1([0,1])$ and for every $E \in \mathcal{M}$

$$\int_E x^*h = \sum_n x^*(x_n)|E \cap E_n|$$

and

$$\int_E |x^*h| = \sum_n |x^*(x_n)|\,|E \cap E_n| < \infty. \tag{9}$$

Moreover, there is a $\nu_h(E)$ such that

$$x^*(\nu_h(E)) = \int_E x^*h.$$

To prove that the series

$$\sum_n x_n|E \cap E_n| \tag{10}$$

converges unconditionally in the Banach space X_p for every $E \in \mathcal{M}$, by the Pettis-Orlicz theorem it suffices to show that every subseries of it converges weakly to an element in X_p. If $(n_k)_k$ is a subsequence of natural numbers and $A = \cup_k E_{n_k}$, then

$$x^*(\nu_h(E \cap A)) = \int_{E\cap A} x^*h = \sum_n x^*(x_n)|E \cap E_n \cap A|$$
$$= \sum_k x^*(x_{n_k})|E \cap E_{n_k}|.$$

By (9) the last series converges for all $x^* \in X^*$, thus (10) converges unconditionally in X_p.

If the function f is integrable by seminorm, then the same is true for $h(t) = f(t) - g(t)$. Since the sets E_n are disjoint $\int_E p(h) = \sum_{n=1}^{\infty} p(x_n)|E \cap E_n|$ and the series $\sum_{n=1}^{\infty} x_n|E_n|$ is absolutely convergent in X_p. □

Corollary 1. *Let $f : [0,1] \to X$ be a measurable Pettis integrable function. Then there are two functions g and h such that $f = g + h$, with $h(t) = \sum_n x_n \chi_{E_n}(t)$, where the sets E_n are disjoint, the series $\sum_n |E_n| x_n$ is unconditionally convergent in X and the function g is bounded in X. If f is strongly integrable then the series $\sum_n |E_n| x_n$ is absolutely convergent in X.*

Proof. The function f is measurable, so we can assume that $f([0,1])$ is separable. If $\{x_1, x_2, \dots\}$ is a countable subset dense in $f([0,1])$ the proof follows as in Theorem 1 with slight changes. □

Theorem 2. *Let X be a locally convex space whose topology is generated by a sequence of seminorms, and let $f : [0,1] \to X$ be a measurable by seminorm Pettis integrable function. Then there are two functions g and h such that $f = g + h$, with $h(t) = \sum_n x_n \chi_{E_n}(t)$, where the sets E_n are disjoint, the series $\sum_n |E_n| x_n$ is unconditionally convergent in X and the function g is bounded in X.*

Proof. If X is a locally convex space with a countable family of seminorms, then a measurable by seminorm function is measurable, [**8**, p. 247]. Therefore the assertion follows from Corollary 1. □

4. Relations between McShane, Pettis and Kurzweil-Henstock integrals. We now proceed to prove that every measurable by seminorm, Pettis integrable function is McShane integrable.

Theorem 3. *Let $f : [0,1] \to X$ be a function which is Pettis integrable and measurable by seminorm, then it is McS-integrable (then KH-integrable) and the two integrals coincide.*

Proof. Let f be a function which is Pettis integrable and measurable by seminorm and let $p \in \mathcal{P}(X)$. Then

$$f_p = \pi_p \circ f : [0,1] \longrightarrow X_p$$

is a measurable Pettis integrable function, so by [**9**, Theorem 17], we get that $f_p(t)$ is McS-integrable with integral $(McS)\int_0^1 f_p = \pi_p(\nu_f([0,1]))$. Let $\varepsilon > 0$ be fixed; then there is a gauge δ_p such that if $P = \{(I_i, t_i) : i = 1, \dots, s\}$ is a δ_p-fine partition of $[0,1]$, we have

$$p\biggl(\sigma(f_p, P) - (McS)\int_0^1 f_p\biggr) < \varepsilon. \tag{11}$$

Since

$$p\biggl(\sigma(f_p, P) - (McS)\int_0^1 f_p\biggr) = p\bigl(\pi_p\left(\sigma(f,P) - \nu_f([0,1])\right)\bigr)$$

we obtain by (11)

$$p\bigl(\pi_p\left(\sigma(f,P) - \nu_f([0,1])\right)\bigr) = p\bigl(\sigma(f,P) - \nu_f([0,1])\bigr) < \varepsilon,$$

and the assertion holds true. □

Remark 1. By the Pettis measurability theorem [**2**, Theorem 2.2], it follows that in separable by seminorm spaces every Pettis integrable function is McShane integrable.

When the range is a Banach space, the following proposition has been proved in [**5**, Theorem 8].

Proposition 2. *Let $f : [0,1] \to X$. Then f is McS-integrable if and only if f is Pettis integrable and KH-integrable.*

Proof. If f is McS-integrable, then it is KH-integrable. The Pettis integrability follows by [**12**, Theorem 2]. To prove the converse, let f be a function which is Pettis integrable and KH-integrable and let $p \in \mathcal{P}(X)$ be fixed. Then the function

$$f_p = \pi_p \circ f : [0,1] \longrightarrow X_p$$

is Pettis integrable and KH-integrable. So, by [**5**, Theorem 8], we get that $f_p(t)$ is McS-integrable with integral $(McS)\int_0^1 f_p = \pi_p((KH)\int_0^1 f)$. With the same computation of Theorem 3 it follows that the function f is McS-integrable and the assertion holds true. □

Proposition 3. *Let $f : [0,1] \to X$ be a KH-integrable function. If, for every $p \in \mathcal{P}(X)$, the real valued function $p(f)$ is KH-integrable, then f is Pettis integrable.*

Proof. Since f is KH-integrable, for all $x^* \in X^*$ the real valued function x^*f is KH-integrable [**12**, Proposition 1], therefore it is measurable, see [**10**, Theorem 9.12). Moreover being $p(f)$ KH-integrable for all $p \in \mathcal{P}(X)$, it is also Lebesgue integrable, see [**10**, Theorem 9.13]. For each $p \in \mathcal{P}(X)$, for all sets $E \in \mathcal{M}$ and for every $x^* \in X_p^*$, it follows

$$\int_E |x^*f| \leq \int_E p(f) < \infty.$$

Thus f is equiscalarly integrable, see [**1**, Definition 2.5]. If $[a,b] \subset [0,1]$, the Kurzweil-Henstock integrability of f implies $(KH)\int_a^b f = \nu(a,b) \in X$. Fix $\varepsilon > 0$. The Lebesgue integrability of $p(f)$ implies the existence of a positive number η such that if $|E| < \eta$ then $\int_E p(f) < \varepsilon$. Thus, if $|E| < \eta$, we get

$$\sup_{x^* \in V_p^0} \left| \int_E x^* f \right| \leq \sup_{x^* \in V_p^0} \int_E |x^* f| \leq \int_E p(f) \leq \varepsilon. \tag{12}$$

Considering [**6**, Proposition 2B], by (12) we get that the function $f_p = \pi_p \circ f : [0,1] \to X_p$ is Pettis integrable. Since the space is complete, applying [**1**, Lemma 2.9], we get that the function f is Pettis integrable. □

Corollary 2. *Let $f : [0,1] \to X$ be a KH-integrable function. If, for every $p \in \mathcal{P}(X)$, the real valued function $p(f)$ is KH-integrable, then f is McS-integrable.*

Proof. By Proposition 3 f is Pettis integrable, then by Proposition 2 it is McS-integrable. □

5. A convergence theorem. We will prove a convergence theorem for the KH-integral.

Definition 5. Let $\{f_\alpha \in KH([0,1],X)\}_\alpha$ be a family of KH-integrable functions. The family $\{f_\alpha\}$ is said to be uniformly KH-integrable on $[0,1]$ if, for each $\varepsilon > 0$ and $p \in \mathcal{P}(X)$, there exists a gauge δ_p on $[0,1]$ such that, for each δ_p-fine partition $P = \{(I_i, t_i) : i = 1, \dots, s\}$ of $[0,1]$, we have

$$\sup_\alpha p\left(\sigma(f_\alpha, P) - (KH)\int_0^1 f_\alpha\right) < \varepsilon.$$

The following result is due to James [**11**].

Theorem 4 [**11**, Theorem 1]. *Let X be a complete locally convex space and let C be a bounded weakly closed subset of X. Then the following are equivalent*:

(i) *C is weakly compact.*

(ii) *There does not exist a positive number θ, a sequence $\{z_n\}$ in C, and an equicontinuous sequence $\{g_n\}$ of linear functionals such that*

$$g_n(z_k) > \theta \quad \text{if} \quad n \le k, \qquad g_n(z_k) = 0 \quad \text{if} \quad n > k.$$

Theorem 5. *Let $(f_n \in KH([0,1],X))_n$ be a sequence of KH-integrable functions. Suppose that $\{f_n\}_n$ converges to f weakly in $[0,1]$. If the family $\{f_n\}_n$ is uniformly KH-integrable on $[0,1]$, then f is KH-integrable on $[0,1]$ and*

$$(KH)\int_0^1 f = \lim_{n\to\infty}(KH)\int_0^1 f_n \quad \text{weakly.}$$

Proof. According to the uniform KH-integrability for each $\varepsilon > 0$ and $p \in \mathcal{P}(X)$, there exists a gauge δ_p on $[0,1]$ such that, for each δ_p-fine Perron partition $P = \{(I_i, t_i) : i = 1, \dots, s\}$ of $[0,1]$, we have

$$\sup_{n\in\mathbf{N}} p\left(\sigma(f_n, P) - (KH)\int_0^1 f_n\right) < \frac{\varepsilon}{3}.$$

Then if $x^* \in X_p^*$ we have

$$\sup_{n\in\mathbf{N}} \left| \left(\sigma(x^* f_n, P) - x^*(KH)\int_0^1 f_n \right) \right| < \frac{\varepsilon}{3} \tag{13}$$

for every δ_p-fine Perron partition P of $[0,1]$. Since, if $x^* \in X^*$, there exist a positive constant C and a seminorm p such that $|x^*(y)| \le Cp(y)$ for all $y \in X$, see [**7**, p. 158], it follows that $(x^* f_n)_n$ is a sequence of real valued uniformly KH-integrable functions. Thus by [**10**, Theorem 13.16], $x^* f$ is a real valued KH-integrable function and

$$x^*(KH)\int_0^1 f_n = (KH)\int_0^1 x^* f_n \to (KH)\int_0^1 x^* f.$$

Fix $t_0 \in [0,1]$ and denote by C the weak closure of the set $((KH)\int_0^{t_0} f_n)_n$. Since $((KH)\int_0^{t_0} f_n)_n$ is a weakly Cauchy sequence, it is bounded. Moreover $C \setminus \{(KH)\int_0^{t_0} f_n : n \in \mathbf{N}\}$ contains at most one point. We want to prove that C is weakly compact. Assume by contradiction that C is not weakly compact. Then applying Theorem 4, there are $\theta > 0$, $(x_m) \subset C$ and a sequence (y_m^*) of equicontinuous functionals of X^* such that $\langle y_k^*, x_m \rangle = 0$ if $k > m$ and $\langle y_k^*, x_m \rangle > \theta$ if $k \le m$. Thus we can find a subsequence (g_m) of (f_n) such that:

(i) $(KH)\int_0^{t_0} y_k^* g_m = 0$ if $k > m$;

(ii) $(KH)\int_0^{t_0} y_k^* g_m > \theta$ if $k \le m$;

(iii) $\lim_{m\to\infty}(KH)\int_0^{t_0} x^* g_m = (KH)\int_0^{t_0} x^* f$ for each $x^* \in X^*$.

Now we are going to prove that the sequence $(y_m^* f)_m$ is uniformly KH-integrable. Since the sequence $(y_m^*)_m$ is equicontinuous, it is also equibounded. So by (13), the family $\{y_m^* g_n : n, m \in \mathbf{N}\}$ is uniformly KH-integrable. Moreover, for each Perron partition $P = \{(A_i, t_i) : i = 1, \dots, p\}$ and for each $m \in \mathbf{N}$ we have

$$\left| \left(\sigma(y_m^* f, P) - (KH)\int_0^{t_0} y_m^* f \right) \right| = \lim_{n\to\infty} \left| \left(\sigma(y_m^* g_n, P) - (KH)\int_0^{t_0} y_m^* g_n \right) \right| < \frac{\varepsilon}{3}.$$

Then also the sequence $(y_m^* f)_m$ is uniformly KH-integrable.

Thus there exists a KH-integrable function $h : [0, t_0] \to \mathbf{R}$ such that

$$\lim_{m\to\infty}(KH)\int_0^{t_0} y_m^* f = (KH)\int_0^{t_0} h. \tag{14}$$

By (iii) and (ii), $(KH)\int_0^{t_0} y_m^* f = \lim_{n\to\infty}(KH)\int_0^{t_0} y_m^* g_n \geq \theta$ for all m; then

$$(KH)\int_0^{t_0} h \geq \theta. \tag{15}$$

Let z_0^* be a *weak**-cluster point of the sequence $(y_m^*)_m$ and let $(w_s^*)_s$ be a subnet *weakly** converging to z_0^*. Then, for each n and for each $t \in [0, t_0]$, we have

$$\lim_s w_s^* g_n(t) = z_0^* g_n(t). \tag{16}$$

Moreover by (13) the family $(w_s^* g_n)_s$ is uniformly KH-integrable in $[0, t_0]$, for each n. Thus, by [**10**, Theorem 13.16] and by (i) we get

$$\lim_s (KH)\int_0^{t_0} w_s^* g_n = (KH)\int_0^{t_0} z_0^* g_n = 0.$$

Therefore, by (iii), we infer

$$(KH)\int_0^{t_0} z_0^* f = 0. \tag{17}$$

As $(y_m^* f)_m$ is uniformly KH-integrable in $[0, t_0]$, the same holds for the family $(w_s^* f)_s$. Moreover for almost each $t \in [0, t_0]$, $\lim_s w_s^* f(t) = z_0^* f(t)$.

So, applying once again the convergence theorem for uniformly integrable real valued functions, we have

$$\lim_s \,(KH)\int_0^{t_0} w_s^* f = \,(KH)\int_0^{t_0} z_0^* f.$$

Then by (14) it follows that $(KH)\int_0^{t_0} z_0^* f = (KH)\int_0^{t_0} h$. Hence by (15) we get

$$(KH)\int_0^{t_0} z_0^* f \geq \theta,$$

in contradiction with (17). Thus the set C is weakly compact. Since t_0 is arbitrary, there is $F : [0,1] \to X$ such that $x^*(F(t)) = \lim_{n\to\infty}(KH)\int_0^t x^* f_n = (KH)\int_0^t x^* f$, for all $t \in [0,1]$ and for all $x^* \in X^*$. We want to prove that $F(1) = (KH)\int_0^1 f$.

Let $P = \{(I_i, t_i) : i = 1, \dots, s\}$ be a δ_p-fine Perron partition of $[0,1]$. Fix $x^* \in V_o^p$. Since $(x^* f_n)_n$ converges to $x^* f$ choose a natural number k such that

$$|\sigma(x^* f_k, P) - \sigma(x^* f, P)| < \frac{\varepsilon}{3} \tag{18}$$

and

$$\left|(KH)\int_0^1 x^* f_k - (KH)\int_0^1 x^* f\right| < \frac{\varepsilon}{3}. \tag{19}$$

By (18), (13) and (19) we get

$$\begin{aligned} |x^*(\sigma(f,P) - F(1))| &= |\sigma(x^* f, P) - x^* F(1)| \\ &= \left|\sigma(x^* f, P) - (KH)\int_0^1 x^* f\right| \\ &\le |\sigma(x^* f, P) - \sigma(x^* f_k, P)| \\ &\quad + \left|\sigma(x^* f_k, P) - (KH)\int_0^1 x^* f_k\right| \\ &\quad + \left|(KH)\int_0^1 x^* f_k - (KH)\int_0^1 x^* f\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

By the arbitrariness of $x^* \in V_o^p$, it follows that

$$p(\sigma(f,P) - F(1)) \le \varepsilon$$

and the assertion holds true. □

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