Probability Propagation in Selected Aristotelian Syllogisms

Niki Pfeifer[1]* and Giuseppe Sanfilippo[2]*

[1] Department of Philosophy, University of Regensburg, Germany
niki.pfeifer@ur.de
[2] Department of Mathematics and Computer Science, University of Palermo, Italy
giuseppe.sanfilippo@unipa.it

Abstract. This paper continues our work on a coherence-based probability semantics for Aristotelian syllogisms (Gilio, Pfeifer, and Sanfilippo, 2016; Pfeifer and Sanfilippo, 2018) by studying Figure III under coherence. We interpret the syllogistic sentence types by suitable conditional probability assessments. Since the probabilistic inference of $P|S$ from the premise set $\{P|M, S|M\}$ is not informative, we add $p(M|(S \vee M)) > 0$ as a probabilistic constraint (i.e., an "existential import assumption") to obtain probabilistic informativeness. We show how to propagate the assigned premise probabilities to the conclusion. Thereby, we give a probabilistic meaning to all syllogisms of Figure III. We discuss applications like generalised quantifiers (like Most S are P) and (negated) defaults.

Keywords: Aristotelian syllogisms · Coherence · Conditional events · Figure III · Imprecise probability · Default reasoning.

1 Motivation and Outline

Aristotelian syllogisms constitute one of the oldest logical reasoning systems. Given the over two millennia long history, not many authors proposed *probabilistic* semantics for Aristotelian syllogisms (see, e.g., [7,8,11,16,30]) to overcome formal restrictions inherited by deductive logic, like its *monotonicity* (i.e., the inability to retract conclusions in the light of new evidence) or its *bivalence* (i.e., the inability to express *degrees of belief*). This paper continues our work on categorical Aristotelian syllogisms within coherence-based probability logic (see, e.g., [5,10,12,16,39]; for other approach to probability logic see, e.g., [1,2,24,32]). We aim to manage nonmonotonicity and degrees of belief, which are necessary for the formalisation of commonsense reasoning. We have studied Figure I, which have transitive structures [16] and Figure II, where the middle term constitutes the consequents of both premises [41]. We extend this work by studying Figure III under coherence. The middle term constitutes the antecedents of the premises of Figure III syllogisms (see Table 1). After recalling some preliminary notions and results in Section 2, we show how to propagate the assigned probabilities to the sequence of conditional events $(P|M, S|M, M|(S \vee M))$ to the

* Both authors contributed equally to the article and are listed alphabetically.

AII	Datisi	*Every M is P, some M is S, therefore some S is P.*
AAI*	Darapti	*Every M is P, every M is S, therefore some S is P.*
EIO	Ferison	*No M is P, some M is S, therefore some S is not P.*
EAO*	Felapton	*No M is P, every M is S, therefore some S is not P.*
IAI	Disamis	*Some M is P, every M is S, therefore some S is P.*
OAO	Bocardo	*Some M is not P, every M is S, therefore some S is not P.*

Table 1: Traditional (logically valid) Aristotelian syllogisms of Figure III (term order: M—P, M—S, *therefore* S—P). * denotes syllogisms which require implicit existential import assumptions for logical validity (since universally quantifiers statements could be vacuously true, M must not be "empty", i.e., $\exists x Mx$).

conclusion $P|S$ in Section 3. This result is applied in Section 4, where we firstly give a probabilistic meaning to the traditionally valid syllogisms of Figure III (see Table 1). Secondly, we connect Aristotelian syllogistics with nonmonotonic reasoning by constructing syllogisms in terms of defaults and negated defaults. Section 5 concludes by remarks on further applications and future work.

2 Preliminary Notions and Results

In this section we recall selected key features of coherence (for more details see, e.g., [4,9,10,19,20,34,45]). Given two events E and H, with $H \neq \bot$, the *conditional event* $E|H$ is defined as a three-valued logical entity which is *true* if EH (i.e., $E \wedge H$) is true, *false* if $\bar{E}H$ is true, and *void* if H is false. In betting terms, assessing $p(E|H) = x$ means that, for every real number s, you are willing to pay an amount $s \cdot x$ and to receive s, or 0, or $s \cdot x$, according to whether EH is true, or $\bar{E}H$ is true, or $\bar{H}$ is true (i.e., the bet is called off), respectively. In these cases the random gain (that is, the difference between the (random) amount that you receive and the amount that you pay) is $\mathcal{G} = (sEH + 0\bar{E}H + sx\bar{H}) - sx = sEH + sx(1-H) - sx = sH(E-x)$. More generally speaking, consider a real-valued function $p: \mathcal{K} \to \mathbb{R}$, where $\mathcal{K}$ is an arbitrary (possibly not finite) family of conditional events. Let $\mathcal{F} = (E_1|H_1, \ldots, E_n|H_n)$ be a sequence of conditional events, where $E_i|H_i \in \mathcal{K}$, $i = 1, \ldots, n$, and let $\mathcal{P} = (p_1, \ldots, p_n)$ be the vector of values $p_i = p(E_i|H_i)$, where $i = 1, \ldots, n$. We denote by $\mathcal{H}_0$ the disjunction $H_1 \vee \cdots \vee H_n$. With the pair $(\mathcal{F}, \mathcal{P})$ we associate the random gain $\mathcal{G} = \sum_{i=1}^{n} s_i H_i (E_i - p_i)$, where $s_1, \ldots, s_n$ are n arbitrary real numbers. $\mathcal{G}$ represents the net gain of n transactions. Let $G_{\mathcal{H}_0}$ denote the set of possible values of $\mathcal{G}$ restricted to $\mathcal{H}_0$, that is, the values of $\mathcal{G}$ when at least one conditioning event is true.

Definition 1. *Function p defined on $\mathcal{K}$ is* coherent *if and only if, for every integer n, for every sequence $\mathcal{F}$ of n conditional events in $\mathcal{K}$ and for every $s_1, \ldots, s_n$, it holds that:* $\min G_{\mathcal{H}_0} \leqslant 0 \leqslant \max G_{\mathcal{H}_0}$.

Intuitively, Definition 1, means in betting terms that a probability assessment is coherent if and only if, in any finite combination of n bets, it cannot happen that the values in $G_{\mathcal{H}_0}$ are all positive, or all negative (*no Dutch Book*). Coherence can

be also characterized in terms of proper scoring rules ([6]), which can be related to the notion of entropy and extropy in information theory ([28,29]). Coherence can be checked, for example, by applying [13, Algorithm 1] or by the CkC-package [3]. We recall the fundamental theorem of de Finetti for conditional events, which states that a coherent assessment of premises can always be coherently extended to a conclusion [4,9,25,31,43,46]:

Theorem 1. *Let a coherent probability assessment $\mathcal{P} = (p_1, \ldots, p_n)$ on a sequence $\mathcal{F} = (E_1|H_1, \ldots, E_n|H_n)$ be given. Then, for a given further conditional event $E_{n+1}|H_{n+1}$, there exists a suitable closed interval $[z', z''] \subseteq [0, 1]$ such that the extension $(\mathcal{P}, z)$ of $\mathcal{P}$ to $(\mathcal{F}, E_{n+1}|H_{n+1})$ is coherent if and only if $z \in [z', z'']$.*

Definition 2. *An* imprecise, or set-valued, assessment *$\mathcal{I}$ on a finite sequence of n conditional events $\mathcal{F}$ is a (possibly empty) set of precise assessments $\mathcal{P}$ on $\mathcal{F}$.*

Definition 2, introduced in [13], states that an *imprecise (probability) assessment* $\mathcal{I}$ on a finite sequence $\mathcal{F}$ of n conditional events is just a (possibly empty) subset of $[0, 1]^n$. We recall the notions of g-coherence and total-coherence for imprecise (in the sense of set-valued) probability assessments [16].

Definition 3. *Let a sequence of n conditional events $\mathcal{F}$ be given. An imprecise assessment $\mathcal{I} \subseteq [0, 1]^n$ on $\mathcal{F}$ is* g-coherent *if and only if there exists a coherent precise assessment $\mathcal{P}$ on $\mathcal{F}$ such that $\mathcal{P} \in \mathcal{I}$.*

Definition 4. *An imprecise assessment $\mathcal{I}$ on $\mathcal{F}$ is* totally coherent *(t-coherent) if and only if the following two conditions are satisfied: (i) $\mathcal{I}$ is non-empty; (ii) if $\mathcal{P} \in \mathcal{I}$, then $\mathcal{P}$ is a coherent precise assessment on $\mathcal{F}$.*

We denote by Π the set of *all coherent precise* assessments on $\mathcal{F}$. We recall that if there are no logical relations among the events $E_1, H_1, \ldots, E_n, H_n$ involved in $\mathcal{F}$, that is $E_1, H_1, \ldots, E_n, H_n$ are logically independent, then the set Π associated with $\mathcal{F}$ is the whole unit hypercube $[0, 1]^n$. If there are logical relations, then the set Π *could be* a strict subset of $[0, 1]^n$. As it is well known $\Pi \neq \emptyset$; therefore, $\emptyset \neq \Pi \subseteq [0, 1]^n$.

Remark 1. Note that: $\mathcal{I}$ is g-coherent $\iff \Pi \cap \mathcal{I} \neq \emptyset$; $\mathcal{I}$ is t-coherent $\iff \emptyset \neq \Pi \cap \mathcal{I} = \mathcal{I}$. Then: $\mathcal{I}$ is t-coherent $\Rightarrow \mathcal{I}$ is g-coherent. Thus, g-coherence is weaker than t-coherence. For further details and relations to coherence see [16].

Given a g-coherent assessment $\mathcal{I}$ on a sequence of n conditional events $\mathcal{F}$, for each coherent precise assessment $\mathcal{P}$ on $\mathcal{F}$, with $\mathcal{P} \in \mathcal{I}$, we denote by $[z'_{\mathcal{P}}, z''_{\mathcal{P}}]$ the interval of coherent extensions of $\mathcal{P}$ to $E_{n+1}|H_{n+1}$; that is, the assessment $(\mathcal{P}, z)$ on $(\mathcal{F}, E_{n+1}|H_{n+1})$ is coherent if and only if $z \in [z'_{\mathcal{P}}, z''_{\mathcal{P}}]$. Then, defining the set $\Sigma = \bigcup_{\mathcal{P} \in \Pi \cap \mathcal{I}} [z'_{\mathcal{P}}, z''_{\mathcal{P}}]$, for every $z \in \Sigma$, the assessment $\mathcal{I} \times \{z\}$ is a g-coherent extension of $\mathcal{I}$ to $(\mathcal{F}, E_{n+1}|H_{n+1})$; moreover, for every $z \in [0, 1] \backslash \Sigma$, the extension $\mathcal{I} \times \{z\}$ of $\mathcal{I}$ to $(\mathcal{F}, E_{n+1}|H_{n+1})$ is not g-coherent. We say that Σ is the *set of coherent extensions* of the imprecise assessment $\mathcal{I}$ on $\mathcal{F}$ to the conditional event $E_{n+1}|H_{n+1}$.

3 Figure III: Propagation of Probability Bounds

We observe that the probabilistic inference of $C|A$ from the premise set $\{C|B, A|B\}$, which corresponds to the general form of syllogisms of Figure III, is probabilistically non-informative. Therefore, we add the probabilistic constraint $p(B|A \vee B) > 0$ to obtain probabilistic informativeness. This constraint serves as an "existential import assumption" (see also [16,41]). Contrary to first order monadic predicate logic, which requires existential import assumptions for Darapti and Felapton only (see Table 1), our probabilistic existential import assumption is required for all valid syllogisms of Figure III.

Remark 2. Let A, B, C be logically independent events. It can be proved that the assessment (x, y, z) on $(C|B, A|B, C|A)$ is coherent for every $(x, y, z) \in [0, 1]^3$, that is, the imprecise assessment $\mathcal{I} = [0, 1]^3$ on $(C|B, A|B, C|A)$ is totally coherent. Moreover, it can also be proved that the assessment (x, y, t) on $(C|B, A|B, B|A \vee B)$ is coherent for every $(x, y, t) \in [0, 1]^3$, that is, the imprecise assessment $\mathcal{I} = [0, 1]^3$ on $(C|B, A|B, B|A \vee B)$ is totally coherent. It is sufficient to check the coherence of each vertex of the unit cube [13].

Consider a coherent probability assessment (x, y, t) on the sequence of conditional events $(C|B, A|B, B|A \vee B)$. The next result allows for computing the lower and upper bounds, z' and z'' respectively, for the coherent extension $z = p(C|A)$.

Theorem 2. *Let A, B, C be three logically independent events and $(x, y, t) \in [0, 1]^3$ be a (coherent) assessment on the family $\big(C|B, A|B, B|A \vee B\big)$. Then, the extension $z = p(C|A)$ is coherent if and only if $z \in [z', z'']$, where*

$$z' = \begin{cases} 0, & \text{if } t(x+y-1) \leqslant 0, \\ \dfrac{t(x+y-1)}{1-t(1-y)}, & \text{if } t(x+y-1) > 0, \end{cases} \quad z'' = \begin{cases} 1, & \text{if } t(y-x) \leqslant 0, \\ 1 - \dfrac{t(y-x)}{1-t(1-y)}, & \text{if } t(y-x) > 0. \end{cases}$$

Proof. In order to compute the lower and upper probability bounds on the further event $C|A$ (i.e., the conclusion), we exploit Theorem 1 by applying [16, Algorithm 1] (which is originally based on [4, Algorithm 2]) in a symbolic way.

Computation of the lower probability bound z' on $C|A$.
Input. The assessment (x, y, t) on $\mathcal{F} = (C|B, A|B, B|A \vee B)$ and the event $C|A$.
Step 0. The constituents associated with $(C|B, A|B, B|A \vee B, C|A)$ are $C_0 = \bar{A}\bar{B}$, $C_1 = ABC$, $C_2 = A\bar{B}C$, $C_3 = AB\bar{C}$, $C_4 = A\bar{B}\bar{C}$, $C_5 = \bar{A}BC$, $C_6 = \bar{A}B\bar{C}$. We observe that $\mathcal{H}_0 = A \vee B$; then, the constituents contained in $\mathcal{H}_0$ are $C_1, \ldots, C_6$. We construct the starting system with the unknowns $\lambda_1, \ldots, \lambda_6, z$:

$$\begin{cases} \lambda_1 + \lambda_2 = z(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \\ \lambda_1 + \lambda_5 = x(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6), \\ \lambda_1 + \lambda_3 = y(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6), \\ \lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 = t(\sum_{i=1}^{6} \lambda_i), \\ \sum_{i=1}^{6} \lambda_i = 1,\ \lambda_i \geqslant 0,\ i = 1, \ldots, 6\,, \end{cases} \iff \begin{cases} \lambda_1 + \lambda_2 = z(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \\ \lambda_1 + \lambda_5 = xt, \\ \lambda_1 + \lambda_3 = yt, \\ \lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 = t, \\ \sum_{i=1}^{6} \lambda_i = 1,\ \lambda_i \geqslant 0,\ i = 1, \ldots, 6\,. \end{cases} \tag{1}$$

Step 1. By setting $z = 0$ in System (1), we obtain

$$\begin{cases} \lambda_1 + \lambda_2 = 0, \;\; \lambda_3 = yt, \;\; \lambda_5 = xt, \\ \lambda_3 + \lambda_5 + \lambda_6 = t, \\ \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1, \\ \lambda_i \geqslant 0, \; i = 1, \ldots, 6\,. \end{cases} \iff \begin{cases} \lambda_1 = \lambda_2 = 0, \\ \lambda_3 = yt, \;\; \lambda_4 = 1 - t, \;\; \lambda_5 = xt, \\ \lambda_6 = t(1 - x - y), \\ \lambda_i \geqslant 0, \; i = 1, \ldots, 6\,. \end{cases} \tag{2}$$

As $(x, y, t) \in [0, 1]^3$, the conditions $\lambda_h \geqslant 0$, $h = 1, \ldots, 5$, in System (2) are all satisfied. Then, System (2), i.e. System (1) with $z = 0$, is solvable if and only if $\lambda_6 = t(1 - x - y) \geqslant 0$. We distinguish two cases: (i) $t(1 - x - y) < 0$ (i.e. $t > 0$ and $x + y > 1$); (ii) $t(1 - x - y) \geqslant 0$, (i.e. $t = 0$ or $(t > 0) \wedge (x + y \leqslant 1)$). In Case (i), System (2) is not solvable and we go to Step 2 of the algorithm. In Case (ii), System (2) is solvable and we go to Step 3.

Case (i). By Step 2 we have the following linear programming problem: *Compute* $\gamma' = \min(\sum_{i:C_i \subseteq AC} \lambda_r) = \min(\lambda_1 + \lambda_2)$ *subject to:*

$$\begin{cases} \lambda_1 + \lambda_5 = x(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6), \; \lambda_1 + \lambda_3 = y(\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6), \\ \lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 = t(\sum_{i=1}^{6} \lambda_i), \; \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \\ \lambda_i \geqslant 0, \; i = 1, \ldots, 6. \end{cases} \tag{3}$$

We notice that y is positive since $x + y > 1$ (and $(x, y, t) \in [0, 1]^3$). Then, also $1 - t(1 - y)$ is positive and the constraints in (3) can be rewritten as

$$\begin{cases} \lambda_1 + \lambda_5 = xt(1 + \lambda_5 + \lambda_6), \\ \lambda_1 + \lambda_3 = yt(1 + \lambda_5 + \lambda_6), \\ \lambda_5 + \lambda_6 = (t - yt)(1 + \lambda_5 + \lambda_6) \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \\ \lambda_i \geqslant 0, \; i = 1, \ldots, 6, \end{cases} \iff \begin{cases} \lambda_5 + \lambda_6 = \frac{t(1-y)}{1-t(1-y)}, \\ \lambda_1 + \lambda_5 = xt(1 + \frac{t(1-y)}{1-t(1-y)}) = \frac{xt}{1-t(1-y)}, \\ \lambda_1 + \lambda_3 = yt(1 + \frac{t(1-y)}{1-t(1-y)}) = \frac{yt}{1-t(1-y)}, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \\ \lambda_i \geqslant 0, \; i = 1, \ldots, 6, \end{cases}$$

$$\iff \begin{cases} \max\{0, \frac{t(x+y-1)}{1-t(1-y)}\} \leqslant \lambda_1 \leqslant \min\{x, y\} \frac{t}{1-t(1-y)}, \\ 0 \leqslant \lambda_2 \leqslant \frac{1-t}{1-t(1-y)}, \quad \lambda_3 = \frac{yt}{1-t(1-y)} - \lambda_1, \quad \lambda_4 = \frac{1-t}{1-t(1-y)} - \lambda_2, \\ \lambda_5 = \frac{xt}{1-t(1-y)} - \lambda_1, \quad \lambda_6 = \frac{t(1-x-y)}{1-t(1-y)} + \lambda_1. \end{cases} \tag{4}$$

Thus, by recalling that $x + y - 1 > 0$, the minimum γ' of $\lambda_1 + \lambda_2$ subject to (3), or equivalently subject to (4), is obtained at $(\lambda'_1, \lambda'_2) = (\frac{t(x+y-1)}{1-t(1-y)}, 0)$. The *procedure stops* yielding as *output* $z' = \gamma' = \lambda'_1 + \lambda'_2 = \frac{t(x+y-1)}{1-t(1-y)}$.

Case (ii). We take Step 3 of the algorithm. We denote by Λ and $\mathcal{S}$ the vector of unknowns $(\lambda_1, \ldots, \lambda_6)$ and the set of solutions of System (2), respectively. We consider the following linear functions (associated with the conditioning events $H_1 = H_2 = B, H_3 = A \vee B, H_4 = A$) and their maxima in $\mathcal{S}$:

$$\begin{aligned} &\Phi_1(\Lambda) = \Phi_2(\Lambda) = \textstyle\sum_{r:C_r \subseteq B} \lambda_r = \lambda_1 + \lambda_3 + \lambda_5 + \lambda_6, \\ &\Phi_3(\Lambda) = \textstyle\sum_{r:C_r \subseteq A \vee B} \lambda_r = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6, \\ &\Phi_4(\Lambda) = \textstyle\sum_{r:C_r \subseteq A} \lambda_r = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \; M_i = \max_{\Lambda \in \mathcal{S}} \Phi_i(\Lambda), \; i = 1, 2, 3, 4\,. \end{aligned} \tag{5}$$

By (2) we obtain: $\Phi_1(\Lambda) = \Phi_2(\Lambda) = 0 + yt + xt + t - xt - yt = t$, $\Phi_3(\Lambda) = 1$, $\Phi_4(\Lambda) = yt + 1 - t = 1 - t(1-y)$, $\forall \Lambda \in \mathcal{S}$. Then, $M_1 = M_2 = t$, $M_3 = 1$, and $M_4 = 1 - (1-y)t$. We consider two subcases: $t < 1$; $t = 1$. If $t < 1$, then $M_4 = yt + 1 - t > yt \geqslant 0$; so that $M_4 > 0$ and we are in the first case of Step 3 (i.e., $M_{n+1} > 0$). Thus, the *procedure stops* and yields $z' = 0$ as *output*. If $t = 1$, then $M_1 = M_2 = M_3 = 1 > 0$ and $M_4 = y$. Hence, we are in the first case of Step 3 (when $y > 0$) or in the second case of Step 3 (when $y = 0$). Thus, the *procedure stops* and yields $z' = 0$ as *output*.

Computation of the upper probability bound z'' on $C|A$. Input and *Step 0* are the same as in the proof of z'. *Step 1.* By setting $z = 1$ in System (1), we obtain

$$\begin{cases} \lambda_1 + \lambda_2 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4,\ \lambda_1 + \lambda_5 = xt,\ \lambda_1 + \lambda_3 = yt, \\ \lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 = t,\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1,\ \lambda_i \geqslant 0,\ i = 1, \ldots, 6\,, \end{cases}$$

or equivalently

$$\begin{cases} \lambda_3 = \lambda_4 = 0,\ \lambda_1 + \lambda_5 = xt, \\ \lambda_1 = yt,\ \lambda_1 + \lambda_5 + \lambda_6 = t, \\ \lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 = 1, \\ \lambda_i \geqslant 0,\ i = 1, \ldots, 6\,; \end{cases} \iff \begin{cases} \lambda_1 = yt,\ \lambda_2 = 1 - t,\ \lambda_3 = \lambda_4 = 0, \\ \lambda_5 = (x-y)t,\ \lambda_6 = t(1-x), \\ \lambda_i \geqslant 0,\ i = 1, \ldots, 6\,. \end{cases} \tag{6}$$

As $(x, y, t) \in [0,1]^3$, the inequalities $\lambda_h \geqslant 0$, $h = 1, 2, 3, 4, 6$ are satisfied. Then, System (6), i.e. System (1) with $z = 1$, is solvable if and only if $\lambda_5 = (x-y)t \geqslant 0$. We distinguish two cases: (i) $(x-y)t < 0$, i.e. $x < y$ and $t > 0$; (ii) $(x-y)t \geqslant 0$, i.e. $x \geqslant y$ or $t = 0$. In Case (i), System (6) is not solvable and we go to Step 2 of the algorithm. In Case (ii), System (6) is solvable and we go to Step 3.
Case (i). By Step 2 we have the following linear programming problem:
Compute $\gamma'' = \max(\lambda_1 + \lambda_2)$ *subject to the constraints in (3).* As $(x, y, t) \in [0,1]^3$ and $x < y$, it follows that $\min\{x, y\} = x$ and $y > 0$. Then, in this case the quantity $1 - t(1-y)$ is positive and the constraints in (3) can be rewritten as in (4). Thus, the maximum γ'' of $\lambda_1 + \lambda_2$ subject to (4), is obtained at $(\lambda_1'', \lambda_2'') = (\frac{xt}{1-t(1-y)}, \frac{1-t}{1-t(1-y)})$. The *procedure stops* yielding as *output* $z'' = \gamma'' = \lambda_1'' + \lambda_2'' = \frac{xt}{1-t(1-y)} + \frac{1-t}{1-t(1-y)} = \frac{1-t+xt}{1-t+yt} = 1 - \frac{t(y-x)}{1-t+yt}$.

Case (ii). We take Step 3 of the algorithm. We denote by Λ and $\mathcal{S}$ the vector of unknowns $(\lambda_1, \ldots, \lambda_6)$ and the set of solutions of System (6), respectively. We consider the functions $\Phi_i(\Lambda)$ and the maxima M_i, $i = 1, 2, 3, 4$, given in (5). From System (6), we observe that the functions $\Phi_1, \ldots, \Phi_4$ are constant for every $\Lambda \in \mathcal{S}$, in particular it holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = t$, $\Phi_3(\Lambda) = 1$ and $\Phi_4(\Lambda) = yt + 1 - t + 0 + 0 = 1 - t(1-y)$ for every $\Lambda \in \mathcal{S}$. So that $M_1 = M_2 = t$, $M_3 = 1$, and $M_4 = 1 - t(1-y)$. We consider two subcases: $t < 1$; $t = 1$.
If $t < 1$, then $M_4 = yt + 1 - t > yt \geqslant 0$; so that $M_4 > 0$ and we are in the first case of Step 3 (i.e., $M_{n+1} > 0$). Thus, the *procedure stops* and yields $z'' = 1$ as *output*.
If $t = 1$, then $M_1 = M_2 = M_3 = 1 > 0$ and $M_4 = y$. Hence, we are in the first

case of Step 3 (when $y > 0$) or in the second case of Step 3 (when $y = 0$). Thus, the *procedure stops* and yields $z'' = 1$ as *output*. □

Remark 3. From Theorem 2, we obtain $z' > 0$ if and only if $t(x + y - 1) > 0$. Moreover, we obtain $z'' < 1$ if and only if $t(y - x) > 0$.

Based on Theorem 2, the next result presents the set of coherent extensions of a given interval-valued probability assessment $\mathcal{I} = ([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]) \subseteq [0, 1]^3$ on $\big(C|B, A|B, B|A \vee B\big)$ to the further conditional event $C|A$.

Theorem 3. *Let A, B, C be three logically independent events and $\mathcal{I} = ([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]) \subseteq [0, 1]^3$ be an imprecise assessment on $\big(C|B, A|B, B|A \vee B\big)$. Then, the set Σ of the coherent extensions of $\mathcal{I}$ on $C|A$ is the interval $[z^*, z^{**}]$, where*

$$z^* = \begin{cases} 0, & \text{if } t_1(x_1 + y_1 - 1) \leqslant 0, \\ \dfrac{t_1(x_1 + y_1 - 1)}{1 - t_1(1 - y_1)}, & \text{if } t_1(x_1 + y_1 - 1) > 0, \end{cases} \quad \text{and}$$

$$z^{**} = \begin{cases} 1, & \text{if } t_1(y_1 - x_2) \leqslant 0, \\ 1 - \dfrac{t_1(y_1 - x_2)}{1 - t_1(1 - y_1)}, & \text{if } t_1(y_1 - x_2) > 0. \end{cases}$$

Proof. Since the set $[0, 1]^3$ on $\big(C|B, A|B, B|A \vee B\big)$ is totally coherent (Remark 2), it follows that $\mathcal{I}$ is also totally coherent. For every precise assessment $\mathcal{P} = (x, y, t) \in \mathcal{I}$, we denote by $[z'_{\mathcal{P}}, z''_{\mathcal{P}}]$ the interval of the coherent extension of $\mathcal{P}$ on $C|A$, where $z'_{\mathcal{P}}$ and $z''_{\mathcal{P}}$ coincide with z' and z'', respectively, as defined in Theorem 2. Then, $\Sigma = \bigcup_{\mathcal{P} \in \mathcal{I}} [z'_{\mathcal{P}}, z''_{\mathcal{P}}] = [z^*, z^{**}]$, where $z^* = \inf_{\mathcal{P} \in \mathcal{I}} z'_{\mathcal{P}}$ and $z^{**} = \sup_{\mathcal{P} \in \mathcal{I}} z''_{\mathcal{P}}$.
Concerning the computation of z^* we distinguish the following alternative cases: (i) $t_1(x_1 + y_1 - 1) \leqslant 0$; (ii) $t_1(x_1 + y_1 > 1) > 0$. Case (i). By Theorem 2 it holds that $z'_{\mathcal{P}} = 0$ for $\mathcal{P} = (x_1, y_1, t_1)$. Thus, $\{z'_{\mathcal{P}} : \mathcal{P} \in \mathcal{I}\} \supseteq \{0\}$ and hence $z^* = 0$.
Case (ii). We note that the function $t(x + y - 1) : [0, 1]^3$ is nondecreasing in the arguments x, y, t. Then, $t(x + y - 1) \geqslant t_1(x_1 + y_1 - 1) > 0$ for every $(x, y, t) \in \mathcal{I}$. Hence by Theorem 2, $z'_{\mathcal{P}} = \frac{t(x+y-1)}{1-t(1-y)}$ for every $\mathcal{P} \in \mathcal{I}$. Moreover, the function $\frac{t(x+y-1)}{1-t(1-y)}$ is nondecreasing in the arguments x, y, t over the restricted domain $\mathcal{I}$; then, $\frac{t(x+y-1)}{1-t(1-y)} \geqslant \frac{t_1(x_1+y_1-1)}{1-t_1(1-y_1)}$. Thus, $z^* = \inf\{z'_{\mathcal{P}} : \mathcal{P} \in \mathcal{I}\} = \inf\left\{\frac{t(x+y-1)}{1-t(1-y)} : (x, y, z) \in \mathcal{I}\right\} = \frac{t_1(x_1+y_1-1)}{1-t_1(1-y_1)}$.
Concerning the computation of z^{**} we distinguish the following alternative cases: (i) $t_1(y_1 - x_2) \leqslant 0$; (ii) $t_1(y_1 - x_2) > 0$. Case (i). By Theorem 2 it holds that $z''_{\mathcal{P}} = 1$ for $\mathcal{P} = (x_2, y_1, t_1) \in \mathcal{I}$. Thus, $\{z''_{\mathcal{P}} : \mathcal{P} \in \mathcal{I}\} \supseteq \{1\}$ and hence $z^{**} = 1$.
Case (ii). We observe that $t(y - x) \geqslant t_1(y - x) \geqslant t_1(y_1 - x) \geqslant t_1(y_1 - x_2) > 0$ for every $(x, y, t) \in \mathcal{I}$. Then, the condition $t(y - x) > 0$ is satisfied for every $\mathcal{P} = (x, y, t) \in \mathcal{I}$ and hence by Theorem 2, $z''_{\mathcal{P}} = 1 - \frac{t(y-x)}{1-t(1-y)}$ for every $\mathcal{P} \in \mathcal{I}$. The function $1 - \frac{t(y-x)}{1-t(1-y)}$ is nondecreasing in the argument x and it is nonincreasing in the arguments y, t over the restricted domain $\mathcal{I}$. Thus,

$1 - \frac{t(y-x)}{1-t(1-y)} \leqslant 1 - \frac{t(y-x_2)}{1-t(1-y)} \leqslant 1 - \frac{t_1(y_1-x_2)}{1-t_1(1-y_1)}$ for every $(x, y, t) \in \mathcal{I}$. Then $z^{**} = \sup\{z''_{\mathcal{P}} : \mathcal{P} \in \mathcal{I}\} = \sup\left\{1 - \frac{t(y-x)}{1-t(1-y)} : (x, y, z) \in \mathcal{I}\right\} = 1 - \frac{t_1(y_1-x_2)}{1-t_1(1-y_1)}$. □

4 Selected Syllogisms of Figure III

In this section we consider examples of probabilistic categorical syllogisms of Figure III (Datisi, Darapti, Ferison, Felapton, Disamis, Bocardo) by suitable instantiations in Theorem 2. We consider three events P, M, S corresponding to the predicate, middle, and the subject term, respectively.

Datisi. The direct probabilistic interpretation of the categorical syllogism "*Every M is P, Some M is S, therefore Some S is P*" would correspond to infer $p(P|S) > 0$ from the premises $p(P|M) = 1$ and $p(S|M) > 0$; however, this inference is not justified. Indeed, by Remark 2, a probability assessment $(1, y, z)$ on $(P|M, S|M, P|S)$ is coherent for every $(y, z) \in [0, 1]^2$. In order to construct a probabilistically informative version of Datisi, a further constraint of the premise set is needed. Based on [16,41] we use $p(M|S \vee M) > 0$ as a further constraint (i.e., our existential import assumption). Then, by instantiating S, M, P in Theorem 2 for A, B, C with $x = 1, y > 0$ and $t > 0$, as $t(x + y - 1) = ty > 0$, it follows that $z' = \frac{t(x+y-1)}{1-t(1-y)} = \frac{ty}{1-t(1-y)} > 0$. Then,

$$p(P|M) = 1,\ p(S|M) > 0,\ \text{and}\ p(M|S \vee M) > 0 \Longrightarrow p(P|S) > 0\,. \tag{7}$$

Therefore, inference (7) is a probabilistically informative version of Datisi.

Darapti. From (7) it follows that

$$p(P|M) = 1,\ p(S|M) = 1,\ \text{and}\ p(M|S \vee M) > 0 \Longrightarrow p(P|S) > 0\,. \tag{8}$$

which is a probabilistically informative interpretation of Darapti (*Every M is P, Every M is S, therefore Some S is P*) under the existential import assumption ($p(M|S \vee M) > 0$).

Ferison. By instantiating S, M, P in Theorem 2 for A, B, C with $x = 0, y > 0$ and $t > 0$, as $t(y - x) = ty > 0$, it follows by Remark 3 that $z'' < 1$. Then, $p(P|M) = 0,\ p(S|M) > 0,$ and $p(M|S \vee M) > 0 \Longrightarrow p(P|S) < 1$, which can be rewritten as

$$p(\bar{P}|M) = 1,\ p(S|M) > 0,\ \text{and}\ p(M|S \vee M) > 0 \Longrightarrow p(\bar{P}|S) > 0\,. \tag{9}$$

Inference (9) is a probabilistically informative version of Ferison (*No M is P, Some M is S, therefore Some S is not P*) under the existential import.

Felapton. From (9) it follows that

$$p(\bar{P}|M) = 1,\ p(S|M) = 1,\ \text{and}\ p(M|S \vee M) > 0 \Longrightarrow p(\bar{P}|S) > 0\,, \tag{10}$$

which is a probabilistically informative interpretation of Felapton (*No M is P, Every M is S, therefore Some S is not P*) under the existential import.

Disamis. The direct probabilistic interpretation of the categorical syllogism "*Some M is P, Every M is S, therefore Some S is P*". By instantiating S, M, P for A, B, C in Theorem 2 with $x > 0$, $y = 1$, and $t > 0$, as $t(x + y - 1) > 0$, it follows that $z' > 0$ (see also Remark 3). Then,

$$p(P|M) > 0,\ p(S|M) = 1,\ \text{and}\ p(M|S \vee M) > 0 \Longrightarrow p(P|S) > 0\,. \tag{11}$$

Inference (11) is a probabilistically informative version of Disamis under the existential import assumption.

Bocardo. By instantiating S, M, P for A, B, C in Theorem 2 with $x < 1, y = 1$ and $t > 0$, as $t(y - x) > 0$, it follows that $z'' < 1$. Then, $p(P|M) < 1,\ p(S|M) = 1,$ and $p(M|S \vee M) > 0 \Longrightarrow p(P|S) < 1$, which can be rewritten as

$$p(\bar{P}|M) > 0,\ p(S|M) = 1,\ \text{and}\ p(M|S \vee M) > 0 \Longrightarrow p(\bar{P}|S) > 0\,. \tag{12}$$

Inference (12) is a probabilistically informative version of Bocardo (*Some M is not P, Every M is S, therefore Some S is not P*) under the existential import. Notice that Bocardo implies Felapton by strengthening the first premise (from $p(\bar{P}|M) > 0$ to $p(\bar{P}|M) = 1$).

Remark 4. We recall that $p(M) = p(M \wedge (S \vee M)) = p(M|S \vee M)p(S \vee M)$. Hence, if we assume that $p(M)$ is positive, then $p(M|S \vee M)$ must be positive too (the converse, however, does not hold). Therefore, the inferences (7)–(12) hold if $p(M|S \vee M) > 0$ is replaced by $p(M) > 0$. The constraint $p(M) > 0$ can be seen as a stronger version of an existential import assumption compared to the conditional event existential import.

Remark 5. We observe that, traditionally, the conclusions of logically valid categorical syllogisms of Figure III are neither in the form of sentence type A (*every*) nor of E (*no*). In terms of our probabilistic semantics, we study which assessments (x, y, t) on $(P|M, S|M, S|S \vee M)$ propagate to $z' = z'' = p(P|S) = 1$ in order to validate A in the conclusion. According to Theorem 2, the following conditions should be satisfied

$$\begin{cases} (x,y,t) \in [0,1]^3, t(x+y-1) > 0, \\ t(x+y-1) = 1 - t(1-y), \\ t(y-x) \leqslant 0, \end{cases} \Longleftrightarrow \begin{cases} (x,y,t) \in [0,1]^3, \\ 1 + yt - t > 0, \\ tx = 1, ty \leqslant 1, \end{cases} \Longleftrightarrow \begin{cases} x = 1, \\ 0 < y \leqslant 1, \\ t = 1. \end{cases}$$

Then, $z' = z'' = 1$ if and only if $(x, y, t) = (1, y, 1)$, with $y > 0$. However, for the syllogisms it would be too strong to require $t = 1$ as an existential import assumption, we only require that $t > 0$. Similarly, in order to validate E in the conclusion, it can be shown that assessments (x, y, t) on $(P|M, S|M, S|S \vee M)$ propagate to the conclusion $z' = z'' = p(P|S) = 0$ if and only if $(x, y, t) = (0, y, 1)$, with $y > 0$. Therefore, if t is just positive neither A nor E can be validate within in our probabilistic semantics of Figure III.

Application to default reasoning. We recall that the *default* $H \mid\!\sim E$ denotes the sentence "*E is a plausible consequence of H*" (see, e.g., [27]). Moreover, the *negated default* $H \not\mid\!\sim E$ denotes the sentence "*it is not the case, that: E is a plausible consequence of H*". Based on [16, Definition 8], we interpret the default $H \mid\!\sim E$ by the probability assessment $p(E|H) = 1$; while the negated default $H \not\mid\!\sim E$ is interpreted by the imprecise probability assessment $p(E|H) < 1$. Then, as the probability assessment $p(E|H) > 0$ is equivalent to $p(\bar{E}|H) < 1$, the negated default $H \not\mid\!\sim \bar{E}$ is also interpreted by $p(E|H) > 0$. Table 2 presents the syllogisms (7)–(12) of Figure III in terms of inference rules which involve defaults and negated defaults.

AII	Datisi	*from* $M \mid\!\sim P, M \not\mid\!\sim \bar{S}$, *and* $(S \vee M) \not\mid\!\sim \overline{M}$ *infer* $S \not\mid\!\sim \bar{P}$
AAI	Darapti	*from* $M \mid\!\sim P, M \mid\!\sim S$, *and* $(S \vee M) \not\mid\!\sim \overline{M}$ *infer* $S \not\mid\!\sim \bar{P}$
EIO	Ferison	*from* $M \mid\!\sim \bar{P}, M \not\mid\!\sim \bar{S}$, *and* $(S \vee M) \not\mid\!\sim \overline{M}$ *infer* $S \not\mid\!\sim P$
EAO	Felapton	*from* $M \mid\!\sim \bar{P}, M \mid\!\sim S$, *and* $(S \vee M) \not\mid\!\sim \overline{M}$ *infer* $S \not\mid\!\sim P$
IAI	Disamis	*from* $M \not\mid\!\sim \bar{P}, M \mid\!\sim S$, *and* $(S \vee M) \not\mid\!\sim \overline{M}$ *infer* $S \not\mid\!\sim \bar{P}$
OAO	Bocardo	*from* $M \not\mid\!\sim P, M \mid\!\sim S$, *and* $(S \vee M) \not\mid\!\sim \overline{M}$ *infer* $S \not\mid\!\sim P$

Table 2: Syllogisms of Figure III (see Table 1) in terms of defaults and negated defaults.

5 Concluding Remarks

In this paper we proved probability propagation rules for Aristotetlian syllogisms of Figure III by using an existential import assumption which we expressed in terms of a probability constraint. Although Aristotelian syllogistics is an ancient reasoning system, our probabilistic semantics allows for various applications including applications to (i) rational nonmonotonic reasoning (we showed how to express basic syllogistic sentence types in terms of defaults and negated defaults; see also [15,16] for connections between syllogisms and default reasoning), (ii) the psychology of reasoning as a new rationality framework (see, e.g., [26,35,36,37,38,42]), (iii) the square of opposition [39,40], and to (iv) formal semantics: by setting appropriate thresholds in Theorem 3, we can interpret generalised quantifiers (see, e.g., [33]) probabilistically (like interpreting *Almost all S are P* by $p(P|S) \geqslant t$, where t is a given—usually context dependent—threshold like $> .9$). Resulting probabilistic syllogisms are a much more plausible rationality framework for studying commonsense contexts compared to traditional Aristotelian syllogisms. We observe that our interpretation of syllogisms relies on conditionals. Thus, future work will be devoted to further generalise Aristotelian syllogisms by iterated conditionals where the S, M, or P terms are replaced by conditional events. We have shown in the context of conditional syllogisms [14,44,45], that the theory of conditional random quantities (see, e.g. [17,18,21,22,23]) is able manage nested conditionals without running into the notorious Lewis' triviality. Applying these results will yield further generalisations of Aristotelian syllogisms.

Acknowledgments. We thank three anonymous referees.

References

1. Adams, E.W.: A primer of probability logic. CSLI, Stanford (1998)
2. Amarger, S., Dubois, D., Prade, H.: Constraint Propagation with Imprecise Conditional Probabilities. In: Proc. of UAI-91. pp. 26–34. Morgan Kaufmann (1991)
3. Baioletti, M., Capotorti, A., Galli, L., Tognoloni, S., Rossi, F., Vantaggi, B.: CkC–Check Coherence package (version e6, 2016). `http://www.dmi.unipg.it/upkd/paid/software.html`
4. Biazzo, V., Gilio, A.: A generalization of the fundamental theorem of de Finetti for imprecise conditional probability assessments. Int. J. Approximate Reasoning 24(2-3), 251–272 (2000)
5. Biazzo, V., Gilio, A., Lukasiewicz, T., Sanfilippo, G.: Probabilistic logic under coherence: Complexity and algorithms. Ann. Math. Artif. Intell 45(1-2), 35–81 (2005)
6. Biazzo, V., Gilio, A., Sanfilippo, G.: Coherent Conditional Previsions and Proper Scoring Rules. In: Advances in Computational Intelligence, CCIS, vol. 300, pp. 146–156. Springer (2012)
7. Chater, N., Oaksford, M.: The probability heuristics model of syllogistic reasoning. Cognitive Psychology 38, 191–258 (1999)
8. Cohen, A.: Generics, frequency adverbs, and probability. Linguistics and Philosophy 22, 221–253 (1999)
9. Coletti, G., Scozzafava, R.: Characterization of coherent conditional probabilities as a tool for their assessment and extension. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 04(02), 103–127 (1996)
10. Coletti, G., Scozzafava, R., Vantaggi, B.: Possibilistic and probabilistic logic under coherence: Default reasoning and System P. Math. Slovaca 65(4), 863–890 (2015)
11. Dubois, D., Godo, L., López De Màntaras, R., Prade, H.: Qualitative reasoning with imprecise probabilities. J Intell Inf Syst 2(4), 319–363 (1993)
12. Gilio, A.: Probabilistic reasoning under coherence in System P. Annals of Mathematics and Artificial Intelligence 34, 5–34 (2002)
13. Gilio, A., Ingrassia, S.: Totally coherent set-valued probability assessments. Kybernetika 34(1), 3–15 (1998)
14. Gilio, A., Over, D., Pfeifer, N., Sanfilippo, G.: Centering and compound conditionals under coherence. In: Soft Methods for Data Science, AISC, vol. 456, pp. 253–260. Springer, Switzerland (2017)
15. Gilio, A., Pfeifer, N., Sanfilippo, G.: Transitive reasoning with imprecise probabilities. In: ECSQARU 2015, LNCS, vol. 9161, pp. 95–105. Springer, Berlin (2015)
16. Gilio, A., Pfeifer, N., Sanfilippo, G.: Transitivity in coherence-based probability logic. Journal of Applied Logic 14, 46–64 (2016)
17. Gilio, A., Sanfilippo, G.: Conditional random quantities and iterated conditioning in the setting of coherence. In: ECSQARU 2013, LNCS, vol. 7958, pp. 218–229. Springer, Berlin, Heidelberg (2013)
18. Gilio, A., Sanfilippo, G.: Conjunction, disjunction and iterated conditioning of conditional events. In: Synergies of Soft Computing and Statistics for Intelligent Data Analysis, AISC, vol. 190, pp. 399–407. Springer, Berlin (2013)
19. Gilio, A., Sanfilippo, G.: Probabilistic entailment in the setting of coherence: The role of quasi conjunction and inclusion relation. Int. J. Approximate Reasoning 54(4), 513–525 (2013)
20. Gilio, A., Sanfilippo, G.: Quasi conjunction, quasi disjunction, t-norms and t-conorms: probabilistic aspects. Information Sciences 245, 146–167 (2013)

21. Gilio, A., Sanfilippo, G.: Conditional random quantities and compounds of conditionals. Studia Logica 102(4), 709–729 (2014)
22. Gilio, A., Sanfilippo, G.: Conjunction of conditional events and t-norms. In: Kern-Isberner, G., Ognjanović, Z. (eds.) ECSQARU 2019, this volume. Springer (2019)
23. Gilio, A., Sanfilippo, G.: Generalized logical operations among conditional events. Applied Intelligence 49(1), 79–102 (2019)
24. Hailperin, T.: Sentential probability logic. Lehigh, Bethlehem (1996)
25. Holzer, S.: On coherence and conditional prevision. Bollettino dell'Unione Matematica Italiana 4(6), 441–460 (1985)
26. Kleiter, G.D., Fugard, A.J.B., Pfeifer, N.: A process model of the understanding of uncertain conditionals. Thinking & Reasoning 24(3), 386–422 (2018)
27. Kraus, S., Lehmann, D., Magidor, M.: Nonmonotonic reasoning, preferential models and cumulative logics. Artificial Intelligence 44, 167–207 (1990)
28. Lad, F., Sanfilippo, G., Agró, G.: Completing the logarithmic scoring rule for assessing probability distributions. AIP Conf. Proc. 1490(1), 13–30 (2012)
29. Lad, F., Sanfilippo, G., Agró, G.: Extropy: complementary dual of entropy. Statistical Science 30(1), 40–58 (2015)
30. Lambert, J.H.: Neues Organon. Wendler, Leipzig (1764)
31. Lehman, R.S.: On confirmation and rational betting. The Journal of Symbolic Logic pp. 251–261 (1955)
32. Ognjanović, Z., Rašković, M., Marković, Z.: Probability logics. Probability-based formalization of uncertain reasoning. Springer, Cham (2016)
33. Peters, S., Westerståhl, D.: Quantifiers in language and logic. Oxford University Press, Oxford (2006)
34. Petturiti, D., Vantaggi, B.: Envelopes of conditional probabilities extending a strategy and a prior probability. Int. J. Approximate Reasoning 81, 160 – 182 (2017)
35. Pfeifer, N.: The new psychology of reasoning: A mental probability logical perspective. Thinking & Reasoning 19(3–4), 329–345 (2013)
36. Pfeifer, N.: Reasoning about uncertain conditionals. Studia Logica 102(4), 849–866 (2014)
37. Pfeifer, N., Douven, I.: Formal epistemology and the new paradigm psychology of reasoning. The Review of Philosophy and Psychology 5(2), 199–221 (2014)
38. Pfeifer, N., Kleiter, G.D.: Towards a mental probability logic. Psychologica Belgica 45(1), 71–99 (2005)
39. Pfeifer, N., Sanfilippo, G.: Probabilistic squares and hexagons of opposition under coherence. Int. J. Approximate Reasoning 88, 282–294 (2017)
40. Pfeifer, N., Sanfilippo, G.: Square of opposition under coherence. In: Soft Methods for Data Science, pp. 407–414. AISC, vol. 456, Springer, Berlin (2017)
41. Pfeifer, N., Sanfilippo, G.: Probabilistic semantics for categorical syllogisms of Figure II. In: SUM 2018, LNCS, vol. 11142, pp. 196–211. Springer, Cham (2018)
42. Pfeifer, N., Tulkki, L.: Conditionals, counterfactuals, and rational reasoning. An experimental study on basic principles. Minds and Machines 27(1), 119–165 (2017)
43. Regazzini, E.: Finitely additive conditional probabilities. Rendiconti del Seminario Matematico e Fisico di Milano 55, 69–89 (1985)
44. Sanfilippo, G., Pfeifer, N., Gilio, A.: Generalized probabilistic modus ponens. In: ECSQARU 2017, LNCS, vol. 10369, pp. 480–490. Springer (2017)
45. Sanfilippo, G., Pfeifer, N., Over, D., Gilio, A.: Probabilistic inferences from conjoined to iterated conditionals. Int. J. Approximate Reasoning 93, 103 – 118 (2018)
46. Williams, P.M.: Notes on conditional previsions. Tech. rep., School of Mathematical and Physical Sciences, University of Sussex (1975), reprint in IJAR, 2007, 44(3)