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A Note on States and Traces from Biorthogonal Sets

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Abstract: In this paper, following Bagarello, Trapani, and myself, we generalize the Gibbs states and their related KMS-like conditions. We have assumed that H_0, H are closed and, at least, densely defined, without giving information on the domain of these operators. The problem we address in this paper is therefore to find a dense domain $\mathcal{D}$ that allows us to generalize the states of Gibbs and take them in their natural environment i.e., defined in $\mathcal{L}^\dagger(\mathcal{D})$.

Keywords: Gibbs states; non-Hermitian Hamiltonians; biorthogonal sets of vector

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1. Introduction and Notations

In this paper, we plan to further analyze the structure arising from two sets of orthonormal vectors in a more applied context. In fact, operatorial methods based on the dynamics of raising and lowering operators of quantum mechanics have been successfully used for the mathematical description of some macroscopic systems ([1–3]). These operators are as usually constructed by defining their action over the eigenstates $e_n, n \geq 0$, of the self-adjoint Hamiltonian operator (essentially the operator H_0 in this work), which is the energy-like operator including all the possible mechanisms between the actors of the system. We aim to further extend these approaches by using suitable non-self-adjoint Hamiltonians ($H \neq H^\dagger$), and to understand whether the dynamics produced by (12) can be used to determine a proper time evolution of the relevant observables of the system.

Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. As mentioned in the Introduction the problem of extending the Gibbs states ω defined on $\mathcal{D}$ may have, in some situations, easy solutions, namely when ω is closable.

This means that one of the two equivalent statements which follow is satisfied. Define

$$G_\omega = \{(a, \omega(a)) \in \mathcal{D} \times \mathbb{C};\, a \in \mathcal{D}\}$$

- If $a_\alpha \to 0$ with regard to τ and $\omega(a_\alpha) \to \ell$, then $\ell = 0$.
- $\overline{G_\omega}$, the closure of G_ω, does not contain couples $(0, \ell)$ with $\ell \neq 0$.

In this case, we define

$$D(\overline{\omega}) = \{a \in \mathcal{D} : \exists \{a_\alpha\} \subset \mathcal{D},\, a_\alpha \to a \text{ and } \omega(a_\alpha) \text{ is convergent}\},$$

and

$$\overline{\omega}(a) = \lim_\alpha \omega(a_\alpha), \quad a \in D(\overline{\omega}).$$

The closability of ω implies that $\overline{\omega}$ is well defined. The functional $\overline{\omega}$ is linear and is the minimal closed extension of ω (i.e., $G_{\overline{\omega}}$ is closed). However, in general w is not closable.

We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all closable, see [4], linear operators X such that $\mathcal{D}(X) = \mathcal{D}$, $\mathcal{D}(X^*) \supseteq \mathcal{D}$.

The set $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a partial *-algebra, see [4], with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^* \upharpoonright \mathcal{D}$ and the *(weak)* partial multiplication $X_1 \,\square\, X_2 = {X_1}^{\dagger *} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (we shall write $X_2 \in R^{\rm w}(X_1)$ or $X_1 \in L^{\rm w}(X_2)$), i.e., iff $X_2\mathcal{D} \subset \mathcal{D}({X_1}^{\dagger *})$ and $X_1^*\mathcal{D} \subset \mathcal{D}(X_2^*)$.

Let $\mathcal{L}^\dagger(\mathcal{D})$ be the subspace of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ consisting of all its elements which leave, together with their adjoints, the domain $\mathcal{D}$ invariant. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra with respect to the usual operations (see [4]).

In concrete applications in physics it may happen that (the clousure of) three different operators, $H_0 = H_0^\dagger, H \in \mathcal{L}^\dagger(\mathcal{D})$ and $H^\dagger$ have only point spectra, and that all the eigenvalues coincide. In particular, in [5] several such triples of operators have been discussed and the following eigenvalues equations are deduced:

$$H_0 e_n = n e_n, \qquad H\varphi_n = n\varphi_n, \qquad H^\dagger \psi_n = n\psi_n, \tag{1}$$

where $e_n, \varphi_n, \psi_n \in \mathcal{D},$ for all n. Here $\mathcal{F}_e = \{e_n,\, n \geq 0\}$ is an orthonormal basis for the Hilbert space $\mathcal{H}$, while $\mathcal{F}_\varphi = \{\varphi_n,\, n \geq 0\}$ and $\mathcal{F}_\psi = \{\psi_n,\, n \geq 0\}$ are two biorthogonal sets, $\langle \varphi_n, \psi_m \rangle = \delta_{n,m}$, not necessarily bases for $\mathcal{H}$. However, quite often, $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are complete in $\mathcal{H}$ and, see [5], they are also $\mathcal{D}$-quasi bases, i.e., they produce a weak resolution of the identity in a dense subspace $\mathcal{D}$ of $\mathcal{H}$:

$$\sum_n \langle f, \varphi_n \rangle \langle \psi_n, g \rangle = \sum_n \langle f, \psi_n \rangle \langle \varphi_n, g \rangle = \langle f, g \rangle\,,$$

for all $f, g \in \mathcal{D}$. The o.n. basis $\mathcal{F}_e$ are used to define a Gibbs state on $B(\mathcal{H})$ as follows:

$$\omega_0(X) = \frac{1}{Z_0} \sum_n \left\langle e_n, \mathrm{e}^{-\beta H_0} X e_n \right\rangle, \tag{2}$$

where $Z_0 := \sum_n \left\langle e_n, \mathrm{e}^{-\beta H_0} e_n \right\rangle = \sum_n \mathrm{e}^{-\beta n} = \frac{\mathrm{e}^\beta}{\mathrm{e}^\beta - 1}$. Here β is the inverse temperature, always larger than zero. Sometimes ω_0 is written as $\omega_0(X) = \mathrm{tr}(\rho X)$, where $\rho := \frac{1}{Z_0}\,\mathrm{e}^{-\beta H_0}$.

Hence, in view of (1), in [5] we have seen what happens if, in (2), we replace H_0 with H or with H^*, and the e_n's with the ψ_n's or with the φ_n's. To do this, in [5] we assumed that H_0, H and $H^\dagger$ are closed and, at least, densely defined, and $H_0, H \in \mathcal{L}^\dagger(\mathcal{D})$. A generalization of the above definition (2) could be useful therefore we assume $H_0, H \in \mathcal{L}^\dagger(\mathcal{D})$.

In this paper, particular attention is devoted to these operators. Moreover, they can be used to define a new ω_0 as a Gibbs state on $\mathcal{L}^\dagger(\mathcal{D})$. In [5] we have assumed that H_0, H are closed and, at least, densely defined, without giving information on the domain of these operators. The question then becomes: is there a dense domain $\mathcal{D}$ that allows us to generalize the states of Gibbs and taking them in their natural environment i.e., defined in $\mathcal{L}^\dagger(\mathcal{D})$?

To keep the paper sufficiently self-contained, we collect below some preliminary definitions and propositions that will be used in what follows. Throughout this paper we assume that $\mathcal{D}$ is a dense subspace of a Hilbert space $\mathcal{H}$ and $\mathcal{F}_e = \{e_n,\, n \geq 0\} \subseteq \mathcal{D}$ is an orthonormal basis for the Hilbert space $\mathcal{H}$. When $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are Riesz bases, we can find a bounded operator T, with bounded inverse, such that

$$\varphi_n = Te_n, \qquad \psi_n = (T^\dagger)^{-1} e_n, \tag{3}$$

for all n.

The paper is organized as follows. Section 2 is devoted to proving that is there a dense domain $\mathcal{D}$ that allows us to generalize the states of Gibbs and taking them in their natural environment i.e., defined in $\mathcal{L}^\dagger(\mathcal{D})$ therefore we enriched what was already discussed in [5]. In Section 3 we consider some special classes of trace possibilities ad further generalization of ω_0 enriching the information given in [5]. Our concluding remarks are given in Section 4.

Definition 1. *Let $(V, D(V))$, $(K, D(K))$ be two linear operators in the Hilbert spaces $\mathcal{H}$. Then, we say that V and K are similar, and write $V \sim K$ if there exists a bounded operator T with bounded inverse T^{-1} which intertwines K and V in the sense that $T : D(K) \to D(V)$ and $VT\xi = TK\xi$, for every $\xi \in D(K)$, and write $V \sim K$,*

A bounded operator T of the previous definition, see [4], is called a bounded intertwining operator for V and K.

Let $H, H_0, H^\dagger$ be as in (1) we denote by $\mathsf{H}, \mathsf{H}_0, \mathsf{H}^\dagger$ their closures and we assume that $\mathsf{H}_0 = \mathsf{H}_0{}^\dagger$.

The self-adjointness of H_0 implies, as we know, the existence and the self-adjointness of all its powers H_0^n in $D(\mathsf{H}_0^n)$ and it is also known that the domain: $D^\infty(\mathsf{H}_0) := \bigcap_{n\geq 0} D(\mathsf{H}_0^n)$ is dense.

In [5] we have shown that

Proposition 1. *Let $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ be Riesz bases as in (3). If $(\mathsf{H}_0, D(\mathsf{H}_0))$ is a self-adjoint operator such that*

$$T\mathsf{H}_0 e_n = \mathsf{H} T e_n, \qquad \mathsf{H}^\dagger (T^\dagger)^{-1} e_n = (T^\dagger)^{-1} \mathsf{H}_0 e_n,$$

and if the linear span of $\{Te_n\}$ is a core for H then $\mathsf{H} \sim \mathsf{H}_0$ and clearly $\mathsf{H}^\dagger \sim \mathsf{H}_0{}^\dagger = \mathsf{H}_0$, with intertwining operator T.

2. Stating the Problem and Results

Coming back to the Gibbs states, the o.n. basis $\mathcal{F}_e$ can be used to define a Gibbs state as follows:

$$\omega_0(X) = \frac{1}{Z_0} \sum_n \left\langle e_n, \mathrm{e}^{-\beta \mathsf{H}_0} X e_n \right\rangle, \tag{4}$$

on $B(\mathcal{H})$ the set of all bounded operators on $\mathcal{H}$, or on $\mathcal{L}^\dagger(\mathcal{D})$, where $Z_0 := \sum_n \left\langle e_n, \mathrm{e}^{-\beta \mathsf{H}_0} e_n \right\rangle = \sum_n \mathrm{e}^{-\beta n} = \frac{\mathrm{e}^\beta}{\mathrm{e}^\beta - 1}$.

The functional ω_0 is linear, normalized, continuous and positive: $\omega_0(X^\dagger X) \geq 0$, and faithful i.e., it is equal to zero only when $X = 0$. If we define the following standard Heisenberg time evolution on $B(\mathcal{H})$, by

$$\alpha_0^t(X) := e^{i\mathsf{H}_0 t} X e^{-i\mathsf{H}_0 t}, \tag{5}$$

$X \in B(\mathcal{H})$, ω_0 turns out to satisfy the following equation:

$$\omega_0\left(A\alpha_0^{i\beta}(B)\right) = \omega_0(BA), \tag{6}$$

for all $A, B \in B(\mathcal{H})$. The abstract version of this equation is known in the literature as the *KMS relation* [6], and it is used to analyze phase transitions.

Clearly, in general, if $\mathcal{D}$ is a dense generic subspace of $\mathcal{H}$ then $e^{-i\mathsf{H}_0 t} \notin \mathcal{L}^\dagger(\mathcal{D})$. This fact strongly depends on the set $\mathcal{D}$. Moreover,

$$e^{-i\mathsf{H}t} = \sum_{n=0}^{+\infty} \frac{(-i\mathsf{H}t)^n}{n!} = \sum_{n=0}^{+\infty} \frac{T(-i\mathsf{H}_0 t)^n T^{-1}}{n!} = Te^{-i\mathsf{H}_0 t} T^{-1} \notin \mathcal{L}^\dagger(\mathcal{D})$$

clearly $e^{-i\mathsf{H}t}, e^{-i\mathsf{H}_0 t} \in B(\mathcal{H})$. However, if we choose now $\mathcal{D} := D(\mathsf{H}_0)$ we have $e^{-i\mathsf{H}_0 t} \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$. Indeed, by the spectral representation, there exists a family of projection operators $\{E(\lambda)\}$ which commute with H_0 such that

$$D(\mathsf{H}_0) = \{f \in \mathcal{H}; \int_{-\infty}^{+\infty} \|\lambda\|^2 d(E(\lambda)f, f) < \infty\},$$

then if $f \in D(\mathsf{H}_0)$ we have

$$\int_{-\infty}^{+\infty} \|\lambda\|^2 d(E(\lambda)e^{-i\mathsf{H}_0 t}f, e^{-i\mathsf{H}_0 t}f) = \int_{-\infty}^{+\infty} \|\lambda\|^2 d(E(\lambda)f, f) < \infty$$

therefore $e^{-i\mathsf{H}_0 t}f \in D(\mathsf{H}_0)$.

Thus, if $X \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$ then

$$\alpha_0^t(X) := e^{i\mathsf{H}_0 t} X e^{-i\mathsf{H}_0 t} \in \mathcal{L}^\dagger(D(\mathsf{H}_0)). \tag{7}$$

What we are interested in here is the possibility of extending the state ω_0 to the situation where we know $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$, rather than $\mathcal{F}_e$. This is exactly what happens in pseudo-Hermitian quantum mechanics, and for this reason, we believe it is relevant in concrete situations. In this section, following [5], we will define the following functionals:

$$\begin{cases} \omega_{\varphi\varphi}(X) = \frac{1}{Z_{\varphi\varphi}} \sum_n \left\langle \mathrm{e}^{-\beta \mathsf{H}} \varphi_n, X\varphi_n \right\rangle = \frac{1}{Z_{\varphi\varphi}} \sum_n \mathrm{e}^{-\beta n} \left\langle \varphi_n, X\varphi_n \right\rangle, \\ \omega_{\psi\psi}(X) = \frac{1}{Z_{\psi\psi}} \sum_n \left\langle \mathrm{e}^{-\beta \mathsf{H}^\dagger} \psi_n, X\psi_n \right\rangle = \frac{1}{Z_{\psi\psi}} \sum_n \mathrm{e}^{-\beta n} \left\langle \psi_n, X\psi_n \right\rangle, \end{cases} \tag{8}$$

where X, for the time being, is just an operator on $\mathcal{H}$ such that the right-hand sides above both converge, and $Z_{\varphi\varphi} = \sum_n e^{-\beta n}\|\varphi_n\|^2$ and $Z_{\psi\psi} = \sum_n e^{-\beta n}\|\psi_n\|^2$ which is always satisfied in concrete examples.

Definition 2. *The biorthogonal sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are called* **well-behaved** *if $Z_{\varphi\varphi} < \infty$ and $Z_{\psi\psi} < \infty$.*

In view of its possible physical applications, it is also interesting to check what happens if we still have $\varphi_n = Te_n$ and $\psi_n = (T^\dagger)^{-1}e_n$, but at least one between T and T^{-1} is unbounded $T, T^{-1} \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$. In this case, the sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ might be $D(\mathsf{H}_0)$-quasi bases, i.e., if, for all $f, g \in D(\mathsf{H}_0)$, the following holds: for every $X, T, T^{-1} \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$ and $e_n \in D(\mathsf{H}_0)$

$$\langle f, g\rangle = \sum_{n\geq 0} \langle f, \varphi_n\rangle \langle \psi_n, g\rangle = \sum_{n\geq 0} \langle f, \psi_n\rangle \langle \varphi_n, g\rangle. \tag{9}$$

In this case,

$$\omega_{\varphi\varphi}(X) = \frac{Z_0}{Z_{\varphi\varphi}}\,\omega_0\left(T^\dagger X T\right), \qquad \omega_{\psi\psi}(X) = \frac{Z_0}{Z_{\psi\psi}}\,\omega_0\left(T^{-1}X(T^{-1})^\dagger\right), \tag{10}$$

which makes sense if $X \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$. This is relevant if we want to extend our results to unbounded operators.

Gibbs States

Equation (10) shows that $\omega_{\varphi\varphi}$ and $\omega_{\psi\psi}$ can be related to ω_0. Since we know that this vector satisfies the KMS-condition (6), we now investigate if some (generalized version of) KMS-relation is also satisfied by our states. In this section, we will always assume that $T, T^{-1} \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$. Going back the start point the starting point of our analysis are the following relations:

$$T\mathsf{H}_0 e_n = \mathsf{H}Te_n, \qquad \mathsf{H}^\dagger (T^\dagger)^{-1} e_n = (T^\dagger)^{-1}\mathsf{H}_0 e_n,$$

for all n, which implies also that for all complex γ,

$$T\mathrm{e}^{\gamma \mathsf{H}_0} e_n = \mathrm{e}^{\gamma\mathsf{H}} Te_n, \qquad \mathrm{e}^{\gamma\mathsf{H}^\dagger}(T^\dagger)^{-1}e_n = (T^\dagger)^{-1}\mathrm{e}^{\gamma\mathsf{H}_0}e_n, \tag{11}$$

for all n. Of course, these equalities can be extended to the linear span of the e_n's, which is dense in $\mathcal{H}$.

Recalling that the dynamics is one of the main ingredients of the KMS-condition, it is clear that we must face with this problem also here. Natural possibilities which extend that in (5) are the following

$$\alpha_\varphi^t(X) = \mathrm{e}^{it\mathsf{H}} X \mathrm{e}^{-it\mathsf{H}}, \qquad \alpha_\psi^t(X) = \mathrm{e}^{it\mathsf{H}^\dagger} X e^{-it\mathsf{H}^\dagger}, \tag{12}$$

for some X. These are two different, and both absolutely reasonable, definitions of *time evolution* of the operator X. However, it is evident that these definitions present some problems. First, being H and $\mathsf{H}^\dagger$ non self-adjoint and, quite often, unbounded, their exponentials should be properly defined (see Equation (11)). Moreover, in general, domain problems clearly occur: even if $f \in D(\mathrm{e}^{-it\mathsf{H}})$, and $X \in B(\mathcal{H})$ it is not guaranteed that $X\mathrm{e}^{-it\mathsf{H}} f \in D(\mathrm{e}^{it\mathsf{H}})$, in fact, but as we shall see if $X \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$ these problems are eliminated.

For this reason, it is convenient to define $\alpha_\varphi^t(X)$ as follows:

$$\alpha_\varphi^t(X) = T\alpha_0^t(T^{-1}XT)T^{-1}, \tag{13}$$

for all $X \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$. It is clear that the right-hand side of this equation is well defined. It is interesting to notice that α_φ^t has all the nice properties of a dynamics, [7,8].

Going back to (13), from (11) it follows that on a dense domain, $Te^{\pm it\mathsf{H}_0} = e^{\pm it\mathsf{H}}T$, so that for every $X \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$ we have

$$e^{\pm it\mathsf{H}} = Te^{\pm it\mathsf{H}_0}T^{-1} \in \mathcal{L}^\dagger(D(\mathsf{H}_0)),$$

$$\begin{aligned} T\alpha_0^t(T^{-1}XT)T^{-1} &= T\mathrm{e}^{it\mathsf{H}_0}T^{-1}XT\mathrm{e}^{-it\mathsf{H}_0}T^{-1} \\ &= \mathrm{e}^{it\mathsf{H}}TT^{-1}XTT^{-1}\mathrm{e}^{-it\mathsf{H}} = \mathrm{e}^{it\mathsf{H}}X\mathrm{e}^{-it\mathsf{H}} \in \mathcal{L}^\dagger(D(\mathsf{H}_0)), \end{aligned}$$

so that we go back to the natural definition of the dynamics proposed first. Now, following [5] but for all $A, B \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$, we deduce that

$$\omega_{\varphi\varphi}(BA) = \frac{1}{Z_{\varphi\varphi}} tr\left(\mathrm{e}^{-\beta\mathsf{H}}TT^\dagger BA\right), \tag{14}$$

while

$$\omega_{\varphi\varphi}(A\alpha_\varphi^{i\beta}(B)) = \frac{1}{Z_{\varphi\varphi}} tr\left(\mathrm{e}^{-\beta H}BTT^\dagger A\right). \tag{15}$$

Therefore, if B commutes with $TT^\dagger$, $[B, TT^\dagger] = 0$, then

$$\omega_{\varphi\varphi}(BA) = \omega_{\varphi\varphi}(A\alpha_\varphi^{i\beta}(B)). \tag{16}$$

It is interesting to notice that the role of A in the relevant assumption for (16) to hold is absolutely not relevant. Also, in case we have $T = 1$, everything collapses to the standard situation described at the beginning of Section 2.

Similar computations of [5] and similar considerations can be repeated for $\omega_{\psi\psi}$ for all $X, A, B \in \mathcal{L}^\dagger(D(\mathsf{H}_0))$.

3. A Possible Further Generalization of ω_0

How it is known sometimes ω_0 is written as $\omega_0(X) = \mathrm{tr}(\rho X)$, where $\rho := \frac{1}{Z_0}\mathrm{e}^{-\beta H_0}$ then further generalization of ω_0 is possible proposed defining. Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$. A locally convex topology t on $\mathcal{D}$ finer than the topology induced by the Hilbert norm defines, in standard fashion, a *rigged Hilbert space*

$$\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[t^\times], \tag{17}$$

where $\mathcal{D}^\times$ is the vector space of all continuous conjugate linear functionals on $\mathcal{D}[t]$, i.e., the conjugate dual of $\mathcal{D}[t]$, endowed with the *strong dual topology* $t^\times = \beta(\mathcal{D}^\times, \mathcal{D})$, which can be defined by the seminorms (see [4])

$$q_{\mathcal{M}}(F) = \sup_{g\in\mathcal{M}} |\langle F, g\rangle|, \quad F \in \mathcal{D}^\times, \tag{18}$$

where $\mathcal{M}$ is a bounded subset of $\mathcal{D}[t]$.

Since the Hilbert space $\mathcal{H}$ can be identified with a subspace of $\mathcal{D}^\times[t^\times]$, we will systematically read (17) as a chain of topological inclusions: $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$. These identifications imply that the sesquilinear form $B(\cdot,\cdot)$ that puts $\mathcal{D}$ and $\mathcal{D}^\times$ in duality is an extension of the inner product of $\mathcal{H}$; i.e., $B(\xi,\eta) = \langle \xi, \eta\rangle$, for every $\xi, \eta \in \mathcal{D}$ (to simplify notations we adopt the symbol $\langle\cdot,\cdot\rangle$ for both of them) and also that the embedding map $I_{\mathcal{D},\mathcal{D}^\times} : \mathcal{D} \to \mathcal{D}^\times$ can be taken to act on $\mathcal{D}$ as $I_{\mathcal{D},\mathcal{D}^\times} f = f$ for every $f \in \mathcal{D}$.

Let now $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a rigged Hilbert space, and let $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ denote the vector space of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}^\times[t^\times]$. If $\mathcal{D}[t]$ is barreled (e.g., reflexive), an involution $X \mapsto X^\dagger$ can be introduced in $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ by the equality

$$\langle X^\dagger \eta, \xi\rangle = \overline{\langle X\xi, \eta\rangle}, \quad \forall \xi, \eta \in \mathcal{D}.$$

Hence, in this case, $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ is a †-invariant vector space.

If $\mathcal{D}[t]$ is a smooth space (e.g., Fréchet and reflexive), then $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ is a quasi *-algebra over $\mathcal{L}^\dagger(\mathcal{D})$ ([4] Definition 2.1.9).

We also denote by $\mathcal{L}(\mathcal{D}^\times)$ the algebra of all continuous linear operators $Z : \mathcal{D}^\times[t^\times] \to \mathcal{D}^\times[t^\times]$. If $\mathcal{D}[t]$ is reflexive, for every $Y \in \mathcal{L}(\mathcal{D})$ there exists a unique operator $Y^\times \in \mathcal{L}(\mathcal{D}^\times)$, the adjoint of Y, such that

$$\langle \Phi, Yg\rangle = \langle Y^\times \Phi, g\rangle, \forall \Phi \in \mathcal{D}^\times, g \in \mathcal{D}.$$

In similar way an operator $Z \in \mathcal{L}(\mathcal{D}^\times)$ has an adjoint $Z^\times \in \mathcal{L}(\mathcal{D})$ such that $(Z^\times)^\times = Z$. The problem of extending ω_0 defined on $\mathcal{L}^\dagger(\mathcal{D})$ in the quasi *-algebra $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ can be approached with the methods of Slight extensions (see [9–17]. In this case, we define

$$D(\overline{\omega_0}) = \{A \in \mathcal{L}(\mathcal{D}, \mathcal{D}^\times) : \exists \{A_\alpha\} \subset \mathcal{L}^\dagger(\mathcal{D}),\ A_\alpha \to A \text{ and } \omega_0(A_\alpha) \text{ convergent}\}$$

Let $\mathcal{S}_\omega$ denote the collection of all subspaces H of $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times) \otimes \mathbb{C}$ such that

(g1) $G_\omega \subseteq H \subset \overline{G_\omega}$
(g2) $(0, \ell) \in H$ if, and only if, $\ell = 0$.

Then, to every $H \in \mathcal{S}_\omega$, it corresponds an extension ω_H, to be called a *slight* extension of ω_0, defined on

$$D(\omega_H) = \{A \in \mathcal{L}(\mathcal{D}, \mathcal{D}^\times) : (A, \ell) \in H\}$$

by

$$\omega_H(A) = \ell,$$

where ℓ is the unique complex number such that $(A, \ell) \in H$.

Moreover, by applying Zorn's lemma to the family $\mathcal{S}_\omega$ one has

Proposition 2. *ω_0 admits a maximal slight extension.*

4. Concluding Remarks

In this paper, we have discussed the possibility of setting a dense domain $\mathcal{D}$ that allows us to generalize the states of Gibbs, taking them in their natural environment, i.e., defined in $\mathcal{L}^\dagger(\mathcal{D})$ enriching the information given in [5]. As is widely known, ω_0 is sometimes written as $\omega_0(X) = \mathrm{tr}(\rho X)$, where $\rho := \frac{1}{Z_0} e^{-\beta H_0}$, then a possible further generalization of ω_0 is proposed. Finally, to conclude, it is possible to consider the operator T as Drazin invertible operator. We hope to discuss this aspect in a further paper.

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