ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 6, 2009

THE VARIATIONAL MCSHANE INTEGRAL IN LOCALLY CONVEX SPACES

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ABSTRACT. The variational McShane integral for functions taking values in a locally convex space is defined, and it is characterized by means of the p-variations of the indefinite Pettis integral.

1. Introduction. Riemann generalized integrals taking values in locally convex space have been studied in [**9, 10**]. In this paper we go a bit further in studying the variational McShane integral for functions defined in a σ-finite quasi Radon measure space and taking values in locally convex spaces. In [**9**] it is proved that if the domain is a compact subinterval of the real line, the family of McShane integrable functions coincide with that of variationally McShane integrable ones if and only if the space is nuclear. It is known that, for Banach valued functions, the family of variationally McShane integrable functions can be significantly larger than that of Bochner integrable ones [**2**]. We extend this result to the setting of locally convex spaces. We prove some properties of the variational McShane integral. The main result is the characterization of the family of variational McShane integrable functions by means of the Pettis integrability and of the fact that, for each semi-norm p, the p-variation of the indefinite Pettis integral is moderated, Theorem 4. The proof is based on differentiability of the primitive with respect to a suitable base which we introduce in a quasi Radon measure space, Theorem 3. As a corollary we get that in compact Radon measure spaces the family of variationally McShane integrable functions coincide with that of integrable by semi-norm ones, Theorem 6. Moreover, we give an example of a function which is measurable by semi-norm and Pettis integrable, but such that the p-variation of the indefinite Pettis integral is not moderated.

2000 AMS *Mathematics subject classification.* Primary 28B05, 46G10.
Keywords and phrases. Pettis integral, McShane integral, variational McShane integral, locally convex spaces.
Received by the editors on July 3, 2007.

DOI:10.1216/RMJ-2009-39-6-1993

2. Definitions and notations. Let $(\Omega, \mathcal{T}, \mathcal{F}, \mu)$ be a nonempty σ-finite outer regular quasi-Radon measure space, where $\mathcal{T}$ is the family of the open sets in Ω, and $\mathcal{F}$ is the family of all μ-measurable sets. Unless specified otherwise, the terms "measure," "measurable" and "almost everywhere" refer to the measure μ. If E is any set, then we denote by χ_E and ${}^cE$, respectively, the characteristic function and the complement of E.

From now on, X will be a Hausdorff locally convex topological vector space (briefly a locally convex space) and X^* the topological dual. $\mathcal{P}(X)$ denotes a family of continuous semi-norms on X so that the topology is generated by $\mathcal{P}(X)$.

We recall the following definitions.

Definition 1. A function $f : \Omega \to X$ is said to be measurable by semi-norm if, for each $p \in \mathcal{P}(X)$, there exist a sequence $(f_n^p)_n$ of simple functions and a subset $X_0^p \subset \Omega$, with $\mu(X_0^p) = 0$, such that $\lim_{n\to\infty} p(f_n^p(t) - f(t)) = 0$ for all $t \in \Omega \setminus X_0^p$.

Definition 2. A function $f : \Omega \to X$ is said to be integrable by semi-norm if, for any $p \in \mathcal{P}(X)$, there exist a sequence $(f_n^p)_n$ of simple functions and a subset $X_0^p \subset \Omega$, with $\mu(X_0^p) = 0$, such that

(i) $\lim_{n\to\infty} p(f_n^p(t) - f(t)) = 0$ for all $t \in \Omega \setminus X_0^p$;

(ii) $p(f(t) - f_n^p(t)) \in L^1(\Omega)$ for each $n \in \mathbf{N}$, and $\lim_{n\to\infty} \int_\Omega p(f(t) - f_n^p(t))dt = 0$;

(iii) for each measurable subset A of Ω there exists an element $y_A \in X$ such that $\lim_{n\to\infty} p(\int_A f_n^p(t) - y_A) = 0$.

Then we put $\int_A f = y_A$.

If, in the previous definitions, both the negligible set X_0 and the sequence of simple functions $(f_n)_n$ do not depend on the semi-norm p, the function f is said to be respectively *measurable* and *Bochner integrable*. Clearly a Bochner integrable function is integrable by semi-norm, and in a Banach space the two definitions coincide.

Definition 3. A function $f : \Omega \to X$ is said to be Pettis integrable if x^*f is Lebesgue integrable on Ω for each $x^* \in X^*$, and for every measurable set $E \subset \Omega$ there is a vector $\nu(E) = \int_E f \in X$ such that $x^*(\nu(E)) = \int_E x^* f \, d\mu$ for all $x^* \in X^*$.

The set function $\nu : \mathcal{F} \to X$ is called the indefinite Pettis integral of f. As it is known (see, for example [**15**, page 65]) ν is a countably additive vector measure, continuous with respect to μ (in the sense that for each $\varepsilon > 0$ there is an $\eta > 0$ such that if $\mu(E) < \eta$ then $\nu(E) < \varepsilon$). If p is a semi-norm on X, the p-variation ν_p of ν is the smallest nonnegative measure such that $p(\nu(E)) \leq \nu_p(E)$ for each $E \in \mathcal{F}$ (see [**8**, page 16]). If ν is the Pettis integral of f, then for each $p \in \mathcal{P}(X)$, ν_p is a measure of σ-finite variation.

A *generalized McShane partition* (or simply a *partition*), see [**5**, Definitions 1A], in Ω is a countable (eventually finite) set of pairs $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ where $(E_i)_i$ is a disjoint family of measurable sets of finite measure and $t_i \in \Omega$ for each $i = 1, 2 \dots$. If $\mu(\Omega \setminus \cup_i E_i) = 0$, we say that P is a *partition of* Ω. A *gauge* on Ω is a function $\Delta : \Omega \to \mathcal{T}$ such that $\omega \in \Delta(\omega)$ for each $\omega \in \Omega$. We say that a partition $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ is *subordinate to a gauge* Δ if $E_i \subset \Delta(t_i)$ for $i = 1, 2, \dots$.

Definition 4. A function $f : \Omega \to X$ is said to be McShane integrable, see [**11**, Definition 5], on Ω, if there exists a vector $w \in X$ satisfying the following property: given $\varepsilon > 0$ and $p \in \mathcal{P}(X)$, there exists a gauge Δ_p on Ω such that for each partition $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ of Ω subordinate to Δ_p, we have

$$\limsup_{n \to \infty} p\left(\sum_{i=1}^{n} \mu(E_i) f(t_i) - w \right) < \varepsilon.$$

If f is a McShane integrable function on Ω we set $z = (\mathrm{McS})\int_\Omega f$.

3. Main results. We extend the definition of variationally McShane integrable functions to the setting of locally convex spaces.

Definition 5. A function $f : \Omega \to X$ is said to be variationally McShane integrable on Ω, if there exists a countably additive set function $F : \mathcal{F} \to X$ such that, given $\varepsilon > 0$ and $p \in \mathcal{P}(X)$, there exists a gauge Δ_p on Ω such that, for each partition $P = \{(E_i, t_i) : i = 1, 2, \ldots\}$ of Ω subordinate to Δ_p, we have

$$\sum_{i=1}^{\infty} p\left(\mu(E_i) f(t_i) - F(E_i)\right) < \varepsilon. \tag{1}$$

We call F the McShane variational primitive of f.

Proposition 1. *If $f : \Omega \to X$ is variationally McShane integrable, then it is McShane integrable and also Pettis integrable.*

Proof. Let $\varepsilon > 0$ and $p \in \mathcal{P}(X)$. Then there is a gauge Δ_p on Ω such that (1) is satisfied for each partition $P = \{(E_i, t_i) : i = 1, 2, \ldots\}$ of Ω subordinate to Δ_p. Since $\Omega = \cup_{i=1}^{\infty} E_i$ and F is countably additive, there is an $N \in \mathbf{N}$ such that if $n > N$ then $p(F(\cup_{i=n}^{\infty} E_i)) < \varepsilon/2$. Therefore, for $n > N$, we have

$$\begin{aligned} p\bigg(\sum_{i=1}^{n} \mu(E_i) f(t_i) - F(\Omega)\bigg) & \\ &\leq p\bigg(\sum_{i=1}^{n} (\mu(E_i) f(t_i) - F(E_i))\bigg) + p\bigg(F\bigg(\bigcup_{n+1}^{\infty} E_i\bigg)\bigg) \\ &\leq \sum_{i=1}^{n} p\left(\mu(E_i) f(t_i) - F(E_i)\right) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which implies that f is McShane integrable and (McS) $\int_{\Omega} f = F(\Omega)$.

By [**11**, Theorem 2], the Pettis integrability follows from the McShane integrability of f. □

The following proposition can be proved in a standard way.

Proposition 2. *Let $f : \Omega \to X$ and $g : \Omega \to X$ be two variationally McShane integrable functions. Then*:

(i) *the function $f + g$ is variationally McShane integrable*;

(ii) *for each $\alpha \in \mathbf{R}$ the function αf is variationally McShane integrable*;

(iii) *if* $x^* \in X^*$, *the real valued function* $x^* f$ *is Lebesgue integrable*;

(iv) *if* $f = 0$ *almost everywhere, then* f *is variationally McShane integrable and* $F = 0$.

We recall that a function $f : \Omega \to X$ is called *simple* if there exist $x_1, x_2, \dots, x_n \in X$ and $A_1, A_2, \dots, A_n \in \mathcal{F}$ such that $f = \sum_{i=1}^n x_i \chi_{A_i}$. If $s = \sum_{i=1}^n x_i \chi_{A_i}$ and $A \in \mathcal{F}$, then $\int_A s = \sum_{i=1}^n \mu(A \cap A_i) x_i$.

Lemma 1. *If* $f : \Omega \to X$ *is a simple function, then* f *is variationally McShane integrable.*

Proof. Since the variational McShane integral is linear, it is sufficient to consider the case $f(t) = \chi_E(t) \cdot w$ where E is a measurable set in Ω and w is a non-null vector in X. For each $A \in \mathcal{F}$, put $F(A) = \mu(E \cap A) \cdot w$. Choose an open set G and a closed set H such that $H \subset E \subset G$. Define a gauge Δ_p on Ω in the following way:

$$\Delta_p(t) = \begin{cases} G & \text{if } t \in H \\ G \cap^c H & \text{if } t \in G \setminus H \\ {}^c H & \text{if } t \in \Omega \setminus G. \end{cases}$$

Let $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ be a partition of Ω subordinate to Δ_p; it follows that

$$\begin{aligned} \sum_{i=1}^{\infty} p\left(\mu(E_i) f(t_i) - F(E_i)\right) \\ &= \sum_{t_i \in E} p\left(\mu(E_i) f(t_i) - F(E_i)\right) + \sum_{t_i \notin E} p\left(F(E_i)\right) \\ &= \sum_{t_i \in E} p\left(\mu(E_i) \cdot w - \mu(E \cap E_i) \cdot w\right) \\ &\quad + \sum_{t_i \notin E} p\left(\mu(E \cap E_i) \cdot w\right) \\ &\le p(w) \sum_{t_i \in E} [\mu(E_i) - \mu(E \cap E_i)] \\ &\quad + p(w) \sum_{t_i \notin E} \mu(E \cap E_i) \le 2p(w) \cdot \mu(G \setminus H). \end{aligned}$$

If $p(w) = 0$, the assertion follows trivially; otherwise, we choose H and G such that $\mu(G \backslash H) < \varepsilon/2p(w)$. Therefore, f is variationally McShane integrable and, for each $A \in \mathcal{F}$, $F(A) = \mu(E \cap A) \cdot w$. $\square$

Lemma 2. *Let $f : \Omega \to X$ be a function. Given $p \in \mathcal{P}(X)$ and $\varepsilon > 0$, there is a gauge Δ_p such that*

$$\sum_{i=1}^{\infty} p(f(t_i))\mu(E_i) \leq \overline{\int_{\Omega}} p(f(t))d\mu + \epsilon$$

for every partition $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ of Ω subordinate to Δ_p, where the integral in the last inequality is the upper Lebesgue integral.

Proof. The proof follows as in [**11**, Lemma 3] with small changes.

Proposition 3. *If $f : \Omega \to X$ is an integrable by semi-norm function, then it is variationally McShane integrable and the two integrals coincide.*

Proof. Let $\epsilon > 0$ and $p \in \mathcal{P}(X)$. Let $\phi_p : \Omega \to X$ be a simple function such that

$$\int_{\Omega} p(f(t) - \phi_p(t))\, d\mu < \frac{\epsilon}{4}. \tag{2}$$

The function ϕ_p is variationally McShane integrable as we already proved; thus, there is a gauge Δ_1 such that

$$\sum_{i=1}^{\infty} p\Big(\phi_p(t_i)\mu(E_i) - \int_{E_i} \phi_p\, d\mu\Big) < \frac{\epsilon}{4} \tag{3}$$

for each partition $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ of Ω subordinate to Δ_1. By Lemma 2 there is a gauge Δ_2 such that

$$\sum_{i=1}^{\infty} p(f(t_i) - \phi_p(t_i))\mu(E_i) \leq \int_{\Omega} p(f(t) - \phi_p(t))\, d\mu + \frac{\epsilon}{4} \tag{4}$$

for every partition $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ of Ω subordinate to Δ_2. Let $\Delta = \Delta_1 \cap \Delta_2$, and take a partition $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ of Ω subordinate to Δ. By (2), (3) and (4) we get

$$\begin{aligned}\sum_{i=1}^{\infty} p\Big(f(t_i)\mu(E_i) - \int_{E_i} f d\mu\Big) &\leq \sum_{i=1}^{\infty} p\Big(f(t_i)\mu(E_i) - \phi_p(t_i)\mu(E_i)\Big) \\ &\quad + \sum_{i=1}^{\infty} p\Big(\phi_p(t_i)\mu(E_i) - \int_{E_i} \phi_p \, d\mu\Big) \\ &\quad + \sum_{i=1}^{\infty} p\Big(\int_{E_i} \phi_p d\mu - \int_{E_i} f \, d\mu\Big) \\ &< \int_{\Omega} p(f(t) - \phi_p(t)) d\mu + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &\quad + \sum_{i=1}^{\infty} \int_{E_i} p(f(t) - \phi_p(t)) \, d\mu \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \qquad \square\end{aligned}$$

Corollary 1. *If $f : \Omega \to X$ is a Bochner integrable function, then it is variationally McShane integrable and the two integrals coincide.*

For each $p \in \mathcal{P}(X)$, let X_p be the completion of the normed linear space $X/p^{-1}(0)$, and let i_p be the canonical mapping of X into X_p, see [**13**, 0.11.1]. Given a function $f : \Omega \to X$ and a semi-norm $p \in \mathcal{P}(X)$, define the function $f_p : \Omega \to X_p$ by

$$f_p(t) = (i_p \circ f)(t) = i_p(f(t)).$$

If $f : \Omega \to X$ is McShane integrable (variationally McShane integrable), then also $f_p : \Omega \to X_p$ is McShane integrable (variationally McShane integrable) and (McS) $\int_\Omega f_p =$ (McS) $\int_\Omega i_p \circ f = i_p \circ$(McS) $\int_\Omega f$, i.e., the McShane primitive (McShane variational primitive) F_p of f_p is equal to $i_p \circ F$.

Remark 1. We note that if $f : \Omega \to X$ is McShane integrable then, for each $A \subset \Omega$, $f \upharpoonright A : \Omega \to X$ is also McShane integrable. Indeed, since the function f is Pettis integrable the same is true also for $f \upharpoonright A$ and, for each $E \in \mathcal{F}$, $\nu_{f\upharpoonright A}(E) = \nu_f(E \cap A)$. Also, $f_p : \Omega \to X_p$ is McShane integrable and, by [**5**, Theorem 1N], the same is true for $f_p \upharpoonright A$. Choose $\varepsilon > 0$, $p \in \mathcal{P}(X)$ and find a gauge Δ_p such that for n big enough,

$$p\biggl(\sum_{i=1}^{n} \mu(E_i) f_p \upharpoonright A(t_i) - (\mathrm{McS}) \int_\Omega f_p \upharpoonright A\biggr) < \varepsilon, \tag{5}$$

for each partition $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ of Ω subordinate to Δ_p. Since

$$\begin{aligned} p\biggl(\sum_{i=1}^{n} \mu(E_i) f_p \upharpoonright A(t_i) - (\mathrm{McS}) \int_\Omega f_p \upharpoonright A\biggr) \\ = p\biggl(\sum_{i=1}^{n} \mu(E_i) f \upharpoonright A(t_i) - \nu_f(A)\biggr), \end{aligned}$$

we get from (5) the McShane integrability of $f \upharpoonright A$.

For a McShane integrable function, the following version of the Henstock lemma holds and can be proved as in the real case.

Lemma 3. *Let $f : \Omega \to X$ be a McShane integrable function. Then to each $\varepsilon > 0$ and each $p \in \mathcal{P}(X)$ there corresponds a gauge Δ_p such that*

$$p\biggl(\sum_{i=1}^{s} \Bigl(\mu(E_i) f(t_i) - (\mathrm{McS}) \int_{E_i} f\Bigr)\biggr) < \varepsilon$$

for each partition $P = \{(E_i, t_i) : i = 1, \dots, s\}$ in Ω subordinate to Δ_p.

We recall that a subset K of X is *totally bounded* if for each $p \in \mathcal{P}(X)$ and for each $\varepsilon > 0$ there exists a finite set B, $B \subset X$, such that $K \subset B + b_p(\varepsilon)$, where $b_p(\varepsilon)$ is the ball with center the null vector and radius ε.

Theorem 1. *Let $f : \Omega \to X$ be a McShane integrable function. Then $\nu_f(\mathcal{F})$ is totally bounded.*

Proof. Assume first that $\mu(\Omega) < \infty$. Let $\varepsilon > 0$ and $p \in \mathcal{P}(X)$ be fixed. Then, according to Lemma 3, there is a gauge Δ_p such that

$$p\bigg(\sum_{i=1}^{s}\Big(\mu(E_i)f(t_i) - (\mathrm{McS})\int_{E_i} f\Big)\bigg) < \frac{\varepsilon}{2} \tag{6}$$

for each partition $P = \{(E_i, t_i) : i = 1, \dots, s\}$ in Ω subordinate to Δ_p. Since the function f is Pettis integrable, ν_f is absolutely continuous. Then there is an $\eta > 0$ so that

$$p\,(\nu_f(A)) < \frac{\varepsilon}{2}, \tag{7}$$

whenever $\mu(A) < \eta$. Fix a partition $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ of Ω subordinate to Δ_p, and let N be such that $\mu(\Omega \setminus \cup_{i=1}^{N} E_i) < \eta$. Let $E \in \mathcal{F}$. Then $\{(E \cap E_i, t_i) : i = 1, \dots, N\}$ is a partition in Ω subordinate to Δ_p. By (6) we have

$$p\bigg(\sum_{i=1}^{N}(\mu(E \cap E_i)f(t_i) - \nu_f(E \cap E_i))\bigg) < \frac{\varepsilon}{2}. \tag{8}$$

Therefore, by (7) and (8), we get

$$\begin{aligned} p\bigg(\sum_{i=1}^{N}\mu(E \cap E_i)f(t_i) - \nu_f(E)\bigg) & \\ &\le p\bigg(\sum_{i=1}^{N}(\mu(E \cap E_i)f(t_i) - \nu_f(E \cap E_i))\bigg) \\ &\quad + p\left(\nu_f(E \setminus \cup_{i=1}^{N} E_i)\right) < \varepsilon. \end{aligned}$$

The set $\{\alpha f(t_i), 0 \le \alpha \le \mu(E_i), i = 1, \dots, N\}$ is compact; therefore, it is totally bounded [**7**, page 60]. Then it follows that $\nu_f(\mathcal{F})$ is totally bounded. The general case follows as in [**5**, Corollary 3E(b)]. $\square$

We will prove the existence of a strong derivative for the primitive of a variationally McShane integrable functions f.

We recall that a *derivation base* on Ω, see for example [**16**, Chapter 5], is a nonempty subset $\mathcal{B}$ of $\mathcal{F} \times \Omega$. For a set $E \subset \Omega$, we write

$$\mathcal{B}(E) = \{(A, \omega) \in \mathcal{B} : A \subset E\} \text{ and } \mathcal{B}[E] = \{(A, \omega) \in \mathcal{B} : \omega \in E\}.$$

If Δ is a gauge defined on Ω, we denote by

$$\mathcal{B}_{\Delta} = \{(A, \omega) \in \mathcal{B} : A \subset \Delta(\omega)\}.$$

We say that a base $\mathcal{B}$ is

- a *fine base* on a set $E \subset \Omega$ if for any $\omega \in E$ and for any gauge Δ the set $\mathcal{B}_{\Delta}[\{\omega\}]$ is nonempty;
- a *filtering base* if for each $\omega \in \Omega$, the set $\mathcal{B}[\{\omega\}]$ is a directed set.

We recall that a function $F : \Omega \to X$ is *$\mathcal{B}$-differentiable at* $\omega \in \Omega$ if there is an element α such that

$$\lim \frac{F(E)}{\mu(E)} = \alpha,$$

where the limit is taken over all E in the directed set $\mathcal{B}[\{\omega\}]$.

Definition 6. A derivation base $\mathcal{B}$ is said to satisfy the *strong Vitali property* if, for every $\mathcal{B}^* \subset \mathcal{B}$, fine on a set E, and every $\varepsilon > 0$, there exist finitely many couples $(A_1, \omega_1), (A_2, \omega_2), \dots, (A_n, \omega_n)$ in $\mathcal{B}^*$, such that the sets $A_1, A_2, \dots, A_n$ are pairwise disjoint and

$$\mu(E \nabla (\cup_{i=1}^{n} A_i)) < \varepsilon,$$

where the symbol ∇ denotes the symmetric difference.

The Vitali covering theorem is an important tool for classical derivation theorems of functions defined on subsets of $\mathbf{R}^n$. It is perhaps worth recalling at this point that any derivation base $\mathcal{B}$ with the strong Vitali property differentiates all L^1-primitives.

For the definition of decomposability see [**6**], and for the definition of lifting we refer to [**14**]. We recall the following theorem and corollary.

Theorem 2 [**6**, Theorem 72B]. *A quasi-Radon measure space* $(\Omega, \mathcal{T}, \mathcal{F}, \mu)$ *is decomposable.*

By the previous theorem and the lifting theorem ([**14**, page 1139]), we get

Corollary 2. *Each nontrivial quasi-Radon measure space* $(\Omega, \mathcal{T}, \mathcal{F}, \mu)$ *has a lifting.*

Let $\mathcal{F}_f \subseteq \mathcal{F}$ be the family of all sets of $\mathcal{F}$ of finite measure, and let ρ be a lifting. Set $R = \cup_{A \in \mathcal{F}_f} \rho(A)$, and define $g_\rho(\omega) = \{A \in \mathcal{F}_f : \omega \in A \subset \rho(A)\}$. For $\omega \in R$, let $a_\rho(\omega) = \{(g, \omega) : g$ is a cofinal subset of $g_\rho(\omega)\}$. Then $(a_\rho(\omega), \omega)_{\omega \in R}$ is a strong Vitali derivation basis [**14**, page 1146]. From now on let $\mathbf{B} = (a_\rho(\omega), \omega)_{\omega \in R}$. Since the set of partitions subordinate to any gauge Δ is not empty ([**5**, Remarks 1B]), the family $\mathbf{B}_\Delta$ is a fine base.

Theorem 3. *Let* $f : \Omega \to X$ *be a variationally McShane integrable function, and let* F *be its primitive. Then the function* F *is differentiable with respect to the derivation base* $\mathbf{B}$ *at almost all* $\omega \in \Omega$ *and* $F' = f$.

Proof. Let $p \in \mathcal{P}(X)$. Let N be the set of all $\omega \in \Omega$ for which $F(\omega)$ is not differentiable or $F'(\omega) \neq f(\omega)$. Given $\omega \in N$, there is an $\eta(\omega) > 0$ such that for each gauge Δ_p we can find a set $A \subset \Delta_p(\omega)$, with $\mu(A) < 1/\eta(\omega)$ and

$$p\,(f(\omega)\mu(A) - F(A)) \geq \eta(\omega)\mu(A).$$

Fix an integer $n \geq 1$, and set $N_n = \{\omega \in N : \eta(\omega) > 1/n\}$. If $\varepsilon > 0$, since f is variationally McShane integrable there is Δ_p^1 so that

$$\sum_{i=1}^{\infty} p\,(f(\omega_i)\mu(E_i) - F(E_i)) < \frac{\varepsilon}{n} \tag{9}$$

for each partition $\{(E_i, \omega_i) : i = 1, 2, \dots\}$ in Ω subordinate to Δ_p^1. Let $\mathcal{S}$ be the family of all sets A such that, for some $\omega_A \in N_n$, $A \subset \Delta_p(\omega_A)$ for some gauge Δ_p, with $\Delta_p(\omega_A) \subset \Delta_p^1(\omega_A)$ and

$$p\,(f(\omega_A)\mu(A) - F(A)) \geq \frac{1}{n}\mu(A). \tag{10}$$

Then the family $\mathcal{S}^* = \{(A, \omega_A)\}$ is a fine base of N_n. Indeed, for any $\omega \in N_n$ and for any gauge Δ_p with $\Delta_p(\omega) \subset \Delta_p^1(\omega)$, the set

$\mathcal{S}^*_\Delta[\{\omega\}]$ is not empty. By the strong Vitali property, there are couples $(A_1,\omega_1),(A_2,\omega_2),\ldots \in \mathcal{S}^*$ such that $A_1, A_2, \ldots$ are pairwise disjoint and $\mu(N_n \nabla (\cup_{i=1}^\infty A_i)) = 0$. Also $Q = \{(A_1,\omega_1),(A_2,\omega_2),\ldots\}$ is a partition in Ω subordinate to Δ_p^1. By (9) and (10), we get

$$\mu(N_n) \le \sum_{i=1}^\infty \mu(A_i) \le n \sum_{i=1}^\infty p\left(\mu(A_i)f(\omega_i) - F(A_i)\right) < \varepsilon.$$

By the arbitrariness of ε and of $p \in \mathcal{P}(X)$, it follows that $\mu(N_n) = 0$ and as $N = \cup_{n=1}^\infty N_n$, we get that $\mu(N) = 0$. Therefore, the assertion follows. □

In the following proposition, using a different technique from that used in [**2**], we will prove that each variationally McShane integrable function is measurable by semi-norm.

Proposition 4. *Let $f : \Omega \to X$ be a variationally McShane integrable function, and let F be its McShane variational primitive. Then the function f is measurable by semi-norm.*

Proof. It follows from Theorem 1 that $\nu_f(\mathcal{F})$ is totally bounded; therefore, the closed linear span of $\nu_f(\mathcal{F})$ is separable by semi-norm [**3**, page 75]. Moreover, by the previous theorem, the function F is differentiable almost everywhere and $F' = f$. For each $p \in \mathcal{P}(X)$, let Y_p be the closed linear span of $\{F_p(A) : A \in \Omega\}$. Then Y_p is separable in X_p and contains the set $\{f_p(t) : F_p'(t) = f_p(t)\}$. Hence, f_p is essentially separably valued. By the Pettis measurability theorem [**1**, Theorem 2.2], it follows that f is measurable by semi-norm. □

We will need the following lemma.

Lemma 4 [**4**, page 250]. *Let $f : \Omega \to X$ be an integrable by semi-norm function. Then, for each $p \in \mathcal{P}(X)$, the function $p(f(t))$ is Lebesgue integrable and*

$$p\left(\int_\Omega f\right) \le \int_\Omega p(f).$$

In the proof of the following proposition, we use a technique similar to that of [**2**, Lemma 4].

Proposition 5. *Let $f : \Omega \to X$ be a variationally McShane integrable function. Then the function f is measurable by semi-norm and Pettis integrable. Moreover, for each $p \in \mathcal{P}(X)$ the p-variation ν_p of ν is moderated.*

Proof. It remains to prove that, for each $p \in \mathcal{P}(X)$, ν_p is moderated. Let $p \in \mathcal{P}(X)$, and assume that ν_p is not moderated and that μ is positive on each nonempty open set. Since f is a measurable by the semi-norm Pettis integrable function, it follows from [**10**, Theorem 1] that there are two functions g and h such that $f = g + h$, with g bounded and measurable in X_p and $h(t) = \sum_n x_n \chi_{A_n}(t)$, where the sets A_n are disjoint, $\mu(A_n) < \infty$, $\Omega = \cup_{n=1}^{\infty} A_n$ and the series $\sum_n \mu(A_n) x_n$ is unconditionally convergent in X_p. Since $g : \Omega \to X_p$ is measurable and bounded, it is Bochner integrable; therefore, its p-variation is moderated since it is finite. Thus, we may assume $f = h = \sum_{n=1}^{\infty} x_n \chi_{A_n}$. Moreover, f is Pettis integrable but it is not integrable by semi-norm. Then there is an A_{n_0} such that if $U \supset A_{n_0}$ is open

$$\int_{U \setminus A_{n_0}} p(f(\omega))\, d\mu = +\infty. \tag{11}$$

Let $\varepsilon > 0$. For $n \in \mathbf{N}$, choose $G_n \supset A_n$ such that

$$\mu(G_n \setminus A_n) < \frac{\varepsilon}{2^{n+1}(p(x_n) + 1)}. \tag{12}$$

Let $\Delta_p : \Omega \to \mathcal{T}$ be any gauge such that $\Delta_p(\omega) \subset G_n$ if $\omega \in A_n$, and let $P = \{(E_i, \omega_i), i = 1, \dots s\}$ be a partition in Ω subordinate to Δ_p. For each $i = 1, \dots, s$, there is an n_i such that $\omega_i \in A_{n_i}$. Since P is subordinate to Δ_p, $E_{n_i} \subseteq G_{n_i}$. Set $C_i = E_i \cap A_{n_i}$ and $D_i = E_i \setminus A_{n_i}$. Observe that if $\omega \in A_{n_i}$ then $f(\omega) = x_{n_i}$; moreover, $\cup_{\omega_i \in A_n} D_i \subseteq G_n \setminus A_n$, thus

(13)
$$\sum_{i=1}^{s} p\Big(\mu(E_i) f(\omega_i) - \int_{E_i} f\Big) = \sum_{i=1}^{s} p\Big(\int_{E_i} (f(\omega_i) - f(\omega))\Big)$$

$$\begin{aligned}
&= \sum_{i=1}^{s} p\bigg(\int_{C_i} (f(\omega_i) - f(\omega)) + \int_{D_i} (f(\omega_i) - f(\omega))\bigg) \\
&\geq \sum_{i=1}^{s} p\bigg(\int_{D_i} (f(\omega_i) - f(\omega))\bigg) \\
&\geq \sum_{i=1}^{s} p\bigg(\int_{D_i} f(\omega)\bigg) - \sum_{i=1}^{s} p(f(\omega_i)\mu(D_i)) \\
&= \sum_{i=1}^{s} p\bigg(\int_{D_i} f(\omega)\bigg) - \sum_{n=1}^{\infty} \sum_{\omega_i \in A_n} p(f(\omega_i)\mu(D_i)) \\
&= \sum_{i=1}^{s} p\bigg(\int_{D_i} f(\omega)\bigg) - \sum_{n=1}^{\infty} p(x_n)\mu(\cup_{\omega_i \in A_n} D_i) \\
&> \sum_{i=1}^{s} p\bigg(\int_{D_i} f(\omega)\bigg) - \frac{\varepsilon}{2}.
\end{aligned}$$

Let Δ_p^1 be an arbitrary gauge satisfying $\Delta_p^1(\omega) \subseteq \Delta_p(\omega)$. We have that

$$A_{n_0} \subseteq_{\varsigma \in A_{n_0}} \Delta_p^1(\varsigma).$$

Set $U_{n_0} = G_{n_0} \cap (\cup_{\varsigma \in A_{n_0}} \Delta_p^1(\varsigma))$, and let ξ_i be a sequence of points from A_{n_0} satisfying

$$\mu(U_{n_0} \setminus \cup_i \Delta_p^1(\xi_i)) = 0.$$

Define $W_1 = \Delta_p^1(\xi_1) \cap U_{n_0}$ and, for $n \geq 2$, $W_i = \Delta_p^1(\xi_i) \cap U_{n_0} \setminus \cup_{j<i} W_j$ and assume $\mu(W_i) > 0$. Then, $\{(W_i, \xi_i)\}$ is a partition of U_{n_0} subordinate to Δ_p^1. Let $\{(F_l, \xi_l)\}$ be any partition of $\Omega \setminus U_{n_0}$ subordinate to Δ_p^1. If $\{(V_r, \xi_r)\} = \{(W_i, \xi_i) \cup (F_l, \xi_l)\}$, then $\{(V_r, \xi_r)\}$ is a partition of Ω subordinate to Δ_p^1. Since $p(f(\omega))$ is not Lebesgue integrable, there is a sequence of disjoint sets H_j such that $U_{n_0} \setminus A_{n_0} = \cup_j H_j$ and, according to Lemma 4,

$$\sum_j p\bigg(\int_{H_j} f\bigg) = +\infty. \tag{14}$$

Now change each (W_i, ξ_i) with the pair $\{(W_i \cap H_i, \xi_i) : \mu(W_i \cap H_i) > 0\} \cup \{(W_i \cap A_{n_0}, \xi_i) : \mu(W_i \cap A_{n_0}) > 0\}$. Again, this is a partition of Ω subordinate to Δ_p^1, and for each $\xi_i \in A_{n_0}$, $D_i \subseteq U_{n_0} \setminus A_{n_0}$; thus,

$\mu(D_i) = 0$ or $D_i \subseteq H_j$ for some j. Also, $\mu((U_{n_0} \setminus A_{n_0}) \setminus \cup_{\xi_i \in A_{n_0}} D_i) = 0$. By (13) and (14), we have

$$\begin{aligned} \sum_{i \in \mathbf{N}} p\bigg(\mu(E_i) f(\omega_i) - \int_{E_i} f\bigg) &\geq \sum_{\xi_i \in A_{n_0}} p\bigg(\int_{D_i} f\bigg) - \frac{\varepsilon}{2} \\ &\geq \sum_j p\bigg(\int_{H_j} f\bigg) - \frac{\varepsilon}{2} = \infty, \end{aligned} \tag{15}$$

which implies that f is not variationally McShane integrable. □

Proposition 6. *Let $f : \Omega \to X$ be a function which is Pettis integrable and measurable by semi-norm. If, for each $p \in \mathcal{P}(X)$, the p-variation ν_p of ν is moderated, then f is variationally McShane integrable.*

Proof. Observe that by [**11**, Proposition 6] the function f is McShane integrable and therefore also each $f_p : \Omega \to X_p$ is McShane integrable. Moreover, f_p is strongly measurable and the variation of its indefinite Pettis integral is moderated. Since X_p is a Banach space, by [**2**, Lemma 2] it follows that each f_p is variationally McShane integrable. Then, if $\varepsilon > 0$ is fixed, there is a gauge Δ_p such that if $P = \{(E_i, t_i) : i = 1, 2, \dots\}$ is a partition of Ω subordinate to Δ_p, we have

$$\begin{aligned} \sum_{i=1}^{\infty} p\left(\mu(E_i) f(t_i) - F(E_i)\right) &= \sum_{i=1}^{\infty} p\left(i_p\left(\mu(E_i) f(t_i) - F(E_i)\right)\right) \\ &= \sum_{i=1}^{\infty} p\left(\mu(E_i) f_p(t_i) - F_p(E_i)\right) < \varepsilon. \end{aligned} \tag{16}$$

Therefore, f is variationally McShane integrable. □

From Propositions 5 and 6 we have the following theorem which is the generalization to the locally convex space of the main result of [**2**].

Theorem 4. *Let $f : \Omega \to X$ be a function. Then f is variationally McShane integrable if and only if f is Pettis integrable, measurable by semi-norm and, for each $p \in \mathcal{P}(X)$, the p-variation ν_p of ν is moderated.*

We recall the following

Theorem 5 [**1**, Theorem 2.7]. *Let $f : \Omega \to X$ be a Pettis integrable function which is measurable by semi-norm. Then the induced vector measure ν has finite variation if and only if f is integrable by semi-norm. Moreover, for each $A \in \mathcal{F}$ $\nu_p(A) = \int_A p(f)\, d\mu$.*

As a corollary we get the following characterization.

Theorem 6. *Let $(\Omega, \mathcal{T}, \mathcal{F}, \mu)$ be a compact finite Radon measure space. Then a function $f : \Omega \to X$ is variationally McShane integrable if and only if f is integrable by semi-norm.*

Proof. The sufficient part follows from Proposition 3. To prove the necessity observe that, on a compact space, each moderated measure is finite. Therefore, the assertion follows from Theorems 4 and 5. □

The following is an example of a function which is McShane integrable (and then Pettis integrable), measurable by semi-norm, but for some $p \in \mathcal{P}(X)$, the p-variation ν_p is not moderated.

Example. Let $\Omega = [0, 1]$, C be the Cantor set in $[0, 1]$ of measure zero and X a locally convex space which is not nuclear. Let A_n be any sequence of open sets covering Ω. Then $C \subset \cup_n A_n$. There is an $n_0 \in \mathbf{N}$ such that $C \cap A_{n_0}$ is uncountable. In particular, there is an interval I_{n_0} such that $C \cap I_{n_0}$ is uncountable. Let $\sum_n x_n$ be a series in X which is unconditionally convergent but not absolutely convergent. Denote by (a_i^r, b_i^r), $r \geq 0$, $1 \leq i \leq 2^r$, the contiguous intervals of length $1/(3^{r+1})$ adjacent to C. Denote by d_i^r the center of (a_i^r, b_i^r).

Define

$$f(t) = \begin{cases} \phi & \text{if } t \in C \text{ or } t = d_i^r,\ i = 1, \dots, 2^r,\ r = 0, 1, \dots, \\ 3^r/2^r x_r & \text{if } t \in (a_i^r, d_i^r),\ i = 1, \dots, 2^r,\ r = 0, 1, \dots, \\ -3^r/2^r x_r & \text{if } t \in (d_i^r, b_i^r),\ i = 1, \dots, 2^r,\ r = 0, 1, \dots, \end{cases}$$

where ϕ is the null vector in X. We want to prove that f is McShane integrable to ϕ. Indeed, fix $\varepsilon > 0$, and let $p \in \mathcal{P}(X)$. By [**12**, Theorem

4], there is a natural number R such that

$$\sup_{\varepsilon_i=1 \text{ or } 0} p\biggl(\sum_{R+1}^{\infty} \varepsilon_i x_i\biggr) < \frac{\varepsilon}{4}.$$

For any sequence of real numbers $(\theta_i)_{R+1}^{\infty}$ satisfying the condition $|\theta_i| \le 1$ for $i = R+1, R+2, \dots$, we have

$$p\biggl(\sum_{R+1}^{\infty} \theta_i x_i\biggr) < \frac{\varepsilon}{2}.$$

In fact, let us expand θ_i into the dyadic form

$$\theta_i = \sum_{j=0}^{\infty} \frac{\varepsilon_{i,j}}{2^j}, \qquad \varepsilon_{i,j} = 0 \text{ or } 1.$$

Then

$$\begin{aligned} p\biggl(\sum_{R+1}^{\infty} \theta_i x_i\biggr) &= p\biggl(\sum_{R+1}^{\infty}\sum_{j=0}^{\infty} \frac{\varepsilon_{i,j} x_i}{2^j}\biggr) \\ &= p\biggl(\sum_{j=0}^{\infty} \frac{1}{2^j} \sum_{R+1}^{\infty} \varepsilon_{i,j} x_i\biggr) \\ &\le \sum_{j=0}^{\infty} \frac{1}{2^j} \sup_{\varepsilon_i=1 \text{ or } 0} p\biggl(\sum_{R+1}^{\infty} \varepsilon_i x_i\biggr) \\ &\le 2 \sup_{\varepsilon_i=1 \text{ or } 0} p\biggl(\sum_{R+1}^{\infty} \varepsilon_i x_i\biggr). \end{aligned} \tag{17}$$

For $i = 1, \dots, 2^r$, let $U_i^r = (a_i^r, b_i^r) \setminus \{d_i^r\}$, $U_R = \cup_{r=0}^{R} \cup_{i=1}^{2^r} U_i^r$, $U = \cup_{R=0}^{\infty} U_R$ and $V = C \cup (\cup_{r=0}^{\infty} \cup_{i=1}^{2^r} \{d_i^r\})$. Moreover, let $K = \max\{p(x_1), \dots, p(x_R)\}$, and choose a positive real number ρ such that $\rho K 3^{R+1} < \varepsilon/2$. For any $\xi \in [0,1]$, let $\delta = \min\{|\xi - a_i^r|, |\xi - b_i^r|, |\xi - d_i^r|\}$. Define $\Delta_p(\xi)$ as follows:

$$\Delta_p(\xi) = \begin{cases} (\xi - \delta,\ \xi + \delta) & \text{if } \xi \in U_i^r,\ i = 1, \dots, 2^r,\ r = 0, 1, \dots, \\ (\xi - \rho,\ \xi + \rho) & \text{if } \xi \in V. \end{cases}$$

Observe that, in order to prove the McShane integrability of a function f defined on a compact subinterval of the real line, it is enough to take finite partitions $P = \{(I_i, t_i) : i = 1, \dots, s\}$ where (I_i) is a collection of nonoverlapping subintervals of $[0,1]$, see [**5**, Proposition 1E]. Thus, let $P = \{(I_i, t_i) : i = 1, \dots, s\}$ be a partition of $[0,1]$ subordinate to Δ_p.

Now $p(\sum_{i=1}^{s} |I_i| f(t_i)) = p(\sum_{t_i \in U} |I_i| f(t_i))$. For $i = 1, \dots, s$, each interval I_i is either entirely in U_R or it is disjoint from U_R; thus,
(18)

$$p\left(\sum_{t_i \in U} |I_i| f(t_i)\right) \le p\left(\sum_{I_i \subset U_R} |I_i| f(t_i)\right) + p\left(\sum_{I_i \bigcap U_R = \Phi} |I_i| f(t_i)\right).$$

Let us estimate the two sums separately. From the definitions of f and of $\Delta_p(\xi)$ for $\xi \in U_i^r$, it follows that there are numbers θ_i^r, $0 \le r \le R$, $i = 1, \dots, 2^r$, such that $|\theta_i^r| < 2\rho$ and

$$\begin{aligned} p\left(\sum_{I_i \subset U_R} |I_i| f(t_i)\right) &\le \sum_{r=0}^{R} \sum_{i=1}^{2^r} \frac{3^r}{2^r} p(x_r) \theta_i^r \\ &\le \sum_{r=0}^{R} \frac{3^r}{2^r} p(x_r) \sum_{i=1}^{2^r} \theta_i^r \\ &\le \sum_{r=0}^{R} \frac{3^r}{2^r} p(x_r) \sum_{i=1}^{2^r} 2\rho \qquad (19) \\ &= 2\rho \sum_{r=0}^{R} \frac{3^r}{2^r} 2^r p(x_r) \le 2\rho K \frac{3^{R+1} - 1}{2} \\ &= K\rho(3^{R+1}) < \frac{\varepsilon}{2}. \end{aligned}$$

For $r \ge R$ we can find numbers θ_r for which $|\theta_r| < 1$ and

$$p\left(\sum_{I_i \bigcap U_R = \Phi} f(t_i)|I_i|\right) \le p\left(\sum_{r=R+1}^{\infty} x_r \theta_r\right) < \frac{\varepsilon}{2}. \qquad (20)$$

From (18), (19) and (20) we obtain

$$p\left(\sum_{t_i \in U} |I_i| f(t_i)\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, the function f is McShane integrable with the integral equal to ϕ. By [**9**, Theorem 2] it follows that f is Pettis integrable. Moreover, since f is a countably valued function, it is measurable. Now let

$$F(t) = \begin{cases} \phi & \text{if } t \in C \\ 3^r/2^r(t-a_i^r)x_r & \text{if } t \in (a_i^r, d_i^r],\ i=1,\dots,2^r,\ r=0,1,\dots \\ -3^r/2^r(t-b_i^r)x_r & \text{if } t \in (d_i^r, b_i^r),\ i=1,\dots,2^r,\ r=0,1,\dots \end{cases}$$

be the primitive of f. Since the series $\sum_{i=1}^{\infty} x_i$ is not absolutely convergent, there is a semi-norm $\overline{p} \in \mathcal{P}(X)$ such that, for all N,

$$\sum_{i=N+1}^{\infty} \overline{p}(x_i) = \infty. \tag{21}$$

The set $C \cap I_{n_0}$ is uncountable; let $r_0 \in \mathbf{N}$ be such that there is an i_0, $1 \le i_0 \le 2^{r_0}$, for which both the intervals $(a_{i_0}^{r_{i_0}}, b_{i_0}^{r_{i_0}})$ and $(a_{i_0+1}^{r_{i_0}}, b_{i_0+1}^{r_{i_0}})$ are contained in I_{n_0}. Let $\{(u_i^k, v_i^k), i \ge r_0,\ k=1,\dots,2^{i-r_0}\}$ be the contiguous intervals of the Cantor set which are contained in the interval $(b_{i_0}^{r_{i_0}}, a_{i_0+1}^{r_{i_0}})$, and denote by d_i^k their centers. Let $\{(u_i^k, d_i^k), i \ge r_0,\ k=1,\dots,2^{i-r_0}\}$ be a family of intervals. By (21) we have

$$\begin{aligned}\sum_{i=r_0+1}^{\infty}\sum_{k=1}^{2^{i-r_0}} \overline{p}(F([u_i^k, d_i^k])) &= \sum_{i=r_0+1}^{\infty}\sum_{k=1}^{2^{i-r_0}} \frac{3^i}{2^i}\frac{1}{2\cdot 3^{i+1}}\overline{p}(x_i) \\ &= \frac{1}{6}\sum_{i=r_0+1}^{\infty}\sum_{k=1}^{2^{i-r_0}} \frac{1}{2^i}\overline{p}(x_i) \\ &= \frac{1}{6}\sum_{i=r_0+1}^{\infty} \frac{1}{2^i}2^{i-r_0}\overline{p}(x_i) \\ &= \frac{1}{6}\sum_{i=r_0+1}^{\infty} \frac{1}{2^{r_0}}\overline{p}(x_i) \\ &= \frac{1}{3\cdot 2^{r_0+1}}\sum_{i=r_0+1}^{\infty} \overline{p}(x_i) \\ &= \infty.\end{aligned}$$

Since

$$\nu_{\overline{p}}(A_n) = \sum_{k=1}^{\infty} \nu_{\overline{p}}(I_k) \geq \sum_{i=N+1}^{\infty} \overline{p}(F([u_i^k, d_i^k])),$$

where $A_n = \cup_{k=1}^{\infty} I_k$, we get $\nu_{\overline{p}}(A_n) = \infty$. Therefore, the $\overline{p}$-variation $\nu_{\overline{p}}$ is not moderated. In particular, it follows that the function f is not variationally McShane integrable.

Acknowledgment. The author thanks Professors L. Di Piazza and K. Musial for helpful discussions and comments.

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