Varieties with at most cubic growth

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Abstract

Let $\mathcal{V}$ be a variety of non necessarily associative algebras over a field of characteristic zero. The growth of $\mathcal{V}$ is determined by the asymptotic behavior of the sequence of codimensions $c_n(\mathcal{V}), n = 1, 2, \ldots$, and here we study varieties of polynomial growth. We classify all possible growth of varieties $\mathcal{V}$ of algebras satisfying the identity $x(yz) \equiv 0$ such that $c_n(\mathcal{V}) < Cn^\alpha$, with $1 < \alpha < 3$, for some constant C. We prove that if $1 < \alpha < 2$ then $c_n(\mathcal{V}) \leq C_1 n$, and if $2 < \alpha < 3$, then $c_n(\mathcal{V}) \leq C_2 n^2$, for some constants C_1, C_2.

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1. Introduction

Let F be a field of characteristic zero and $F\{X\}$ the free non associative algebra on a countable set X over F. Let $\mathcal{V}$ be a variety of non necessarily associative algebras and $Id(\mathcal{V})$ be the T-ideal of identities of $\mathcal{V}$. In characteristic zero without loss of generality one can study the multilinear identities of $\mathcal{V}$ and a natural and well established way of measuring the identities of $\mathcal{V}$ is through the study of the asymptotic behavior of its sequence of codimensions $c_n(\mathcal{V})$, $n = 1, 2, \ldots$. More precisely, for every $n \geq 1$ let P_n be the space of multilinear polynomials in the variables $x_1, \ldots, x_n$. Since char $F = 0$, the T-ideal $Id(\mathcal{V})$ is determined by the multilinear polynomials it contains; hence the relatively free algebra $F\{X\}/Id(\mathcal{V})$ is determined by the sequence of subspaces $\{P_n/(P_n \cap Id(\mathcal{V}))\}_{n\geq 1}$. The integer $c_n(\mathcal{V}) = \dim P_n/(P_n \cap Id(\mathcal{V}))$ is called the n-th codimension of $\mathcal{V}$ and the growth function determined by the sequence of integers $\{c_n(\mathcal{V})\}_{n\geq 1}$ is the growth of the variety $\mathcal{V}$.

If $\mathcal{V} = var(A)$ is the variety generated by an algebra A, then we write $Id(\mathcal{V}) = Id(A)$ and $c_n(A) = c_n(\mathcal{V})$.

The first result on the asymptotic behavior of $c_n(\mathcal{V})$ is due to Regev ([16]). He proved that if $\mathcal{V}$ is a non-trivial variety of associative algebras, then the sequence of codimensions

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is exponentially bounded, i.e., there exist constants $\alpha, a > 0$ such that $c_n(\mathcal{V}) \leq \alpha a^n$, for all n. In case $\mathcal{V}$ is a variety of non associative algebras, such sequence has a much more involved behavior and can have overexponential growth ([15]). Nevertheless for varieties of associative and Lie algebras, no intermediate growth (between polynomial and exponential) and no exponential growth between 1 and 2 is allowed ([9],[10],[11]).

The exponential rate of growth of the sequence of codimensions of an associative algebra was determined in [5] and [6]. It was proved that for any associative PI-algebra A, the limit $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is a non-negative integer. In case of finite dimensional Lie algebra the same result was proved in [17]. This is not an expected behavior for Lie algebras, in fact in [18] was constructed an example of a Lie algebra whose sequence of codimensions grows exponentially but the rate of growth is not integer.

In this paper we consider varieties $\mathcal{V}$ of not necessarily associative algebras such that the sequence of codimensions is polynomially bounded, i.e., there exist constants $\alpha, t > 0$ such that $c_n(\mathcal{V}) \leq \alpha n^t$, for all n. The asymptotic behavior of the codimensions of a unitary algebra was described by Drensky ([3]). He proved that if $\mathcal{V}$ is a variety of associative or Lie algebras whose sequence of codimensions is polynomially bounded then the growth of the codimensions is exactly polynomial, i.e., there exist a positive integer k and a constant C such that $c_n(\mathcal{V}) = Cn^k + O(n^{k-1})$, where $O(n^{k-1})$ is a polynomial of degree $\leq k-1$.

In this paper we deal with the variety, $\mathcal{V} = {}_2\mathcal{N}$, of left nilpotent algebras of index two, that is the variety of algebras satisfying the identity

$$x(yz) \equiv 0.$$

For this class of algebras in [14] the authors constructed a variety $\mathcal{W} \subset {}_2\mathcal{N}$ such that for any $n \geq 25$

$$([\sqrt{n}\,] - 2)\frac{n(n-1)(n-5)}{6} \leq c_n(\mathcal{V}) \leq n^3\sqrt{n} + n^2(2n + 3\sqrt{n}) + n^2.$$

In other words, the variety $\mathcal{W}$ has fractional polynomial growth between 3 and 4, more precisely $\lim_{n\to\infty} \log_n c_n(\mathcal{V}) = \frac{7}{2}$.

Motivated by this results in ([12], [13]) we classified the growth of varieties of commutative and anticommutative algebras with at most quadratic growth. We proved that if $\mathcal{V}$ is a variety such that $c_n(\mathcal{V}) < Cn^\alpha$ with $0 < \alpha < 1$, then $c_n(\mathcal{V}) \leq 1$, for n large. Moreover if $1 < \alpha < 2$, then either $\lim_{n\to\infty} \log c_n(\mathcal{V}) = 1$ or $c_n(\mathcal{V}) \leq 1$, for n large.

The purpose of this paper is to prove that if $\mathcal{V}$ is the variety of algebras satisfying the identity $x(yz) \equiv 0$ and $c_n(\mathcal{V}) \leq Cn^\alpha$ with $1 < \alpha < 2$, then $c_n(\mathcal{V}) < C_1 n$, for some constant C_1. Moreover if $c_n(\mathcal{V}) \leq Cn^\alpha$, with $2 < \alpha < 3$, then $c_n(\mathcal{V}) < C_2 n^2$ for some constant C_2.

Preliminaries

Throughout F will be a field of characteristic zero, $X = \{x_1, x_2, \ldots\}$ a countable set and $F\{X\}$ the free non associative algebra on X over F. Let $\mathcal{V}$ be a variety and $Id(\mathcal{V}) = \{f \in F\{X\} | f \equiv 0 \text{ on } \mathcal{V}\}$ be the T-ideal of identities of $\mathcal{V}$. For every $n \geq 1$, let P_n be the space of multilinear polynomials of $F\{X\}$ in the first n variables $x_1, x_2, \ldots, x_n$. Since $\mathrm{char} F = 0$, it is well known that the sequence of spaces $P_n \cap Id(\mathcal{V})$, $n = 1, 2, \ldots$, carry all information about $Id(\mathcal{V})$. The symmetric group S_n acts on P_n by permuting variables: if $\sigma \in S_n$, $f(x_1, \ldots, x_n) \in P_n$,

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

The space $P_n \cap Id(\mathcal{V})$ is invariant under this action and one studies the structure of $P_n(\mathcal{V}) = P_n/(P_n \cap Id(\mathcal{V}))$ as an S_n-module. The S_n-character of $P_n(\mathcal{V})$, denoted $\chi_n(\mathcal{V})$, is called the nth cocharacter of A. Its degree $c_n(\mathcal{V}) = \chi_n(\mathcal{V})(1)$ is the nth codimension of $\mathcal{V}$. By complete reducibility one writes

$$\chi_n(\mathcal{V}) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{1}$$

where χ_λ is the irreducible S_n-character corresponding to the partition λ of n and $m_\lambda \geq 0$ is the multiplicity of χ_λ (see for instance [8] for the representation theory of the symmetric group).

Notice that in case $\mathcal{V}$ is a variety of associative algebras, for the multiplicities m_λ we have that $m_\lambda \leq d_\lambda$, where $d_\lambda = \deg \chi_\lambda$ is the degree of the character χ_λ. In the non associative case this inequality does not hold any more. For instance for the free non associative algebra $A = F\{X\}$ we have that, in $\chi_n(A)$, $m_\lambda = C_n d_\lambda$ where C_n is the nth Catalan number.

We next recall some basic properties of the representation theory of the symmetric group that we shall use in the sequel. Let $\lambda \vdash n$ and let T_λ be a Young tableau of shape $\lambda \vdash n$. We denote by e_{T_λ} the corresponding essential idempotent of the group algebra FS_n. Recall that $e_{T_\lambda} = \bar{R}_{T_\lambda} \bar{C}_{T_\lambda}$ where

$$\bar{R}_{T_\lambda} = \sum_{\sigma \in R_{T_\lambda}} \sigma,$$

$$\bar{C}_{T_\lambda} = \sum_{\tau \in C_{T_\lambda}} (\mathrm{sgn}\tau)\tau$$

and R_{T_λ}, C_{T_λ} are the row and column stabilizers of T_λ, respectively. Recall that if M_λ is an irreducible S_n-submodule of $P_n(\mathcal{V})$ corresponding to λ, there exists a polynomial $f(x_1, \ldots, x_n) \in P_n$ and a tableau T_λ such that $e_{T_\lambda} f(x_1, \ldots, x_n) \notin Id(\mathcal{V})$.

In what follows we shall use also the representation theory of the general linear group. Let $m \geq 1$ and $U = \mathrm{span}_F\{x_1, \ldots, x_m\}$. The group $GL(U) \cong GL_m$ acts naturally on the left on the space U and we can extend this action diagonally to get an action on $F_m\{X\} = F\{x_1, \ldots, x_m\}$, the free algebra of rank m.

The space $F_m\{X\} \cap Id(\mathcal{V})$ is invariant under this action, hence

$$F_m(\mathcal{V}) = \frac{F_m\{X\}}{F_m\{X\} \cap Id(\mathcal{V})}$$

inherits a structure of left GL_m-module. Let $F_{m,n}$ be the space of homogeneous polynomials of degree n in the variables $x_1, \ldots, x_m$, then

$$F_{m,n}(\mathcal{V}) = \frac{F_{m,n}}{F_{m,n} \cap Id(\mathcal{V})}$$

is a GL_m-submodule of $F_m(\mathcal{V})$ and we denote its character by $\psi_n(\mathcal{V})$. Write

$$\psi_n(\mathcal{V}) = \sum_{\lambda \vdash n} \bar{m}_\lambda \psi_\lambda$$

where ψ_λ is the irreducible GL_m-character associated to the partition λ and $\bar{m}_\lambda$ is the corresponding multiplicity. In [1] and [2] it was proved that if the character $\chi_n(\mathcal{V})$ has the decomposition given in (1) then $m_\lambda = \bar{m}_\lambda$, for all $\lambda \vdash n$ whose corresponding diagram has height at most m.

It is also known (see for instance [4, Theorem 12.4.12]) that any irreducible submodule of $F_{m,n}(\mathcal{V})$ corresponding to λ is generated by a non-zero polynomial f_λ, called highest weight vector, of the form

$$f_\lambda = \prod_{i=1}^{\lambda_1} St_{h_i(\lambda)}(x_1, \ldots, x_{h_i(\lambda)}) \sum_{\sigma \in S_n} \alpha_\sigma \sigma, \tag{2}$$

where $\alpha_\sigma \in F$, the right action of S_n on $F^T_{m,n}(A)$ is defined by place permutation, $h_i(\lambda)$ is the height of the ith column of the diagram of λ and

$$St_r(x_1, \ldots, x_r) = \sum_{\tau \in S_r} (\operatorname{sgn} \tau) x_{\tau(1)} \cdots x_{\tau(r)}$$

is the standard polynomial of degree r with a suitable arrangement of the parentheses. Recall that f_λ is unique up to a multiplicative constant.

For a Young tableau T_λ, denote by f_{T_λ} the highest weight vector obtained from (2) by considering the only permutation $\sigma \in S_n$ such that the integers $\sigma(1), \ldots, \sigma(h_1(\lambda))$, in this order, fill in from top to bottom the first column of T_λ, $\sigma(h_1(\lambda)+1), \ldots, \sigma(h_1(\lambda)+h_2(\lambda))$ the second column of T_λ, etc.

By [4, Proposition 12.4.14] we have that if

$$\psi_n(\mathcal{V}) = \sum_{\lambda \vdash n} \bar{m}_\lambda \psi_\lambda$$

is the GL_m-character of $F_{m,n}(\mathcal{V})$, then $\bar{m}_\lambda$ is equal to the maximal number of linearly independent highest weight vectors f_{T_λ} in $F_{m,n}(\mathcal{V})$.

2. Classifying varieties $\mathcal{V}$ such that $c_n(\mathcal{V}) \leq Cn^\alpha, 1 < \alpha < 2$

Throughout this section we shall assume that $\mathcal{V}$ is the variety of left nilpotent algebras of index two, that is the variety of algebras satisfying the identity

$$x(yz) \equiv 0$$

such that $c_n(\mathcal{V}) \leq Cn^\alpha$, for some $1 < \alpha < 2$, and for some constant C.

Our aim is to prove that for such variety $c_n(\mathcal{V}) < C_1 n$, for some constant C_1.
Notice that modulo the identity $x(yz) \equiv 0$ all non-zero monomials of the free algebra are left normed, i.e., are of the type $(((x_1x_2)x_3)\ldots)$. Since we shall be working modulo such identity throughout we shall omit the parenthesis in left normed monomials, hence we shall write $(((x_1x_2)x_3)\ldots x_n) = x_1x_2\ldots x_n$, and xy^2 for xyy.

In what follows we shall make use of the following lemma which was proved in [12].

Lemma 1. *If $\lambda \vdash n$ is such that $\lambda \notin \{(n), (1^n), (n-1, 1), (2, 1^{n-2})\}$ then $d_\lambda \geq \frac{1}{8}n^2$.*

From the above lemma it follows that if $\lambda \vdash n$ is distinct from $(n), (1^n), (n-1,1), (2, 1^{n-2})$, then there exists $N > 0$ such that for all $n \geq N$ we have that $d_\lambda > Cn^\alpha$ and hence $m_\lambda = 0$. We fix the integer N from now on.

The following remark is obvious.

Remark 1. *Let $\lambda \vdash n$. If either $\lambda = (n)$ or $\lambda = (1^n)$ then $m_\lambda \leq 1$.*

Let $\lambda \vdash n$ be a partition of n and f_i, $i = 1, 2, \ldots, \deg \chi_\lambda$, be polynomials corresponding to the standard Young tableaux of shape λ in P_n. For every i, let T_i be the corresponding standard tableaux and denote by g_i the polynomial obtained from f_i by identifying with x_1 all variables corresponding to the first row of T_i, with x_2 all variables corresponding to the second row of T_i and so on. Then, by [4, Proposition 12.4.14], m_λ equals the dimension of the space spanned by all g_i, $1 \leq i \leq \deg \chi_\lambda$, mod $Id(\mathcal{V})$. In what follows we shall use this fact without mention it.

We shall adopt the convention of marking a set of alternating variables with the same symbol $\bar{}$, $\tilde{}$. For instance, in $\bar{x}_1 y_1 \bar{x}_2 y_2 \bar{x}_3$ stands for $\sum_{\sigma \in S_3} (\mathrm{sgn}\sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 x_{\sigma(3)}$.

The following result concern the partition $\lambda = (n-1, 1)$.

Proposition 1. *If $\chi_n(\mathcal{V}) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ and $n \geq N$ we have that*

$$m_{(n-1,1)} \leq 2.$$

PROOF. Let $\lambda = (n-2, 1, 1) \vdash n$. For every $i = 0, \ldots, n-3$, let

$$f_i = \bar{x}_1 x_1^i \bar{x}_2 \bar{x}_3 x_1^{n-i-3}$$

be the left normed polynomials corresponding to the following standard tableaux

$$T_\lambda^{(i)} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & \ldots & i+1 & i+4 & \ldots & n \\ \hline i+2 \\ \cline{1-1} i+3 \\ \cline{1-1} \end{array}$$

Since if $n \geq N$, by Lemma 1, $d_\lambda > Cn^\alpha$, $1 < \alpha < 2$, then it follows that, for every $i = 0, \ldots, n-3$,

$$f_i \equiv 0 \pmod{Id(\mathcal{V})}.$$

Let consider the following substitution $x_1 = zx_1 + x_1$ then we obtain

$$zx_1^{i+1} \bar{x}_2 \bar{x}_3 x_1^{n-i-3} \equiv 0 \pmod{Id(\mathcal{V})}$$

and by putting $x_3 = x_1$ we have

$$zx_1^{i+1} x_2 x_1^{n-i-2} \equiv zx_1^{i+2} x_2 x_1^{n-i-3} \pmod{Id(\mathcal{V})} \tag{3}$$

for every $i = 0, \ldots, n-3$.

Let $n \geq N$ and consider, for every $j = 0, \ldots, n-2$, the polynomials

$$g_j = \bar{x}_1 x_1^j \bar{x}_2 x_1^{n-j-2}$$

corresponding to the standard Young tableaux of shape $\lambda = (n-1, 1)$.

Notice that, by the identity (3), we obtain

$$g_j \equiv g_1 \pmod{Id(\mathcal{V})}$$

for $j = 2, \ldots, n-2$. It follows that the subspace $\mathrm{span}\{g_0, \ldots, g_{n-2}\}$, modulo $Id(\mathcal{V})$, has dimension bounded by 2. Hence $m_{(n-1,1)} \leq 2$.

Our next objective is to find an upper bound for the multiplicity m_λ for the partition $\lambda = (2, 1^{n-2})$.

Proposition 2. *Let $\chi_n(\mathcal{V}) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$. If $n \geq N$ then $m_{(2,1^{n-2})} \leq 2$.*

PROOF. Let $\lambda = (3, 1^{N-3})$. For every $r = 1, \ldots, N-2$ let

$$f_r = \bar{x}_1 \bar{x}_2 \cdots \bar{x}_r x_1 \bar{x}_{r+1} \bar{x}_{N-2} x_1$$

be polynomials corresponding to the following standard tableaux

$$T_\lambda^{(r)} = \begin{array}{|c|c|c|} \hline 1 & r+1 & N \\ \hline . \\ \cline{1-1} . \\ \cline{1-1} . \\ \cline{1-1} r \\ \cline{1-1} r+2 \\ \cline{1-1} . \\ \cline{1-1} . \\ \cline{1-1} . \\ \cline{1-1} \end{array} .$$

By Lemma 1, it follows that for every $r = 1, \cdots, N-2$

$$\bar{x}_1 \bar{x}_2 \cdots \bar{x}_r x_1 \bar{x}_{r+1} \cdots \bar{x}_{N-2} x_1 \equiv 0 \pmod{Id(\mathcal{V})}.$$

Let consider the substitution $x_1 = z_1 z_2 + x_1$ the we obtain

$$z_1 z_2 \bar{x}_2 \cdots \bar{x}_r x_1 \bar{x}_{r+1} \cdots \bar{x}_{N-2} x_1 \equiv 0 \pmod{Id(\mathcal{V})}$$

for $r = 2, \ldots, N-2$. After multilinearization we have that

$$z_1 z_2 \bar{x}_2 \cdots \bar{x}_r x_1 \bar{x}_{r+1} \cdots \bar{x}_{N-2} z \equiv -z_1 z_2 \bar{x}_2 \cdots \bar{x}_r z \bar{x}_{r+1} \cdots \bar{x}_{N-2} x_1 \pmod{Id(\mathcal{V})}.$$

If we alternate on $x_1, x_2, \ldots, x_{N-2}$ it follows that

$$z_1 z_2 \bar{x}_1 \cdots \bar{x}_r \bar{x}_{r+1} \cdots \bar{x}_{N-2} z \equiv \alpha_{r,N} z_1 z_2 \bar{x}_1 \bar{x}_2 \cdots \bar{x}_{r-1} z \bar{x}_r \cdots \bar{x}_{N-2} \pmod{Id(\mathcal{V})} \qquad (4)$$

where $r \geq 3$ and $\alpha_{r,N} = \pm 1$ according to the parity of r and N.

Let now $n \geq N$, and $f_1, \ldots, f_{n-1}$ be polynomials corresponding to the standard Young tableaux of shape $(2, 1^{n-2})$ in P_n. Then, if $g_1, \ldots, g_{n-1}$ are the polynomials obtained from the f_i's by identifying with x_1 the two variables of the first row of the corresponding tableaux, for $i = 1, \ldots, n-1$, we have that

$$g_i = \bar{x}_1 \bar{x}_2 \cdots \bar{x}_i x_1 \bar{x}_{i+1} \cdots \bar{x}_{n-1}.$$

Let look at the dimension of the $span\{g_1, \ldots, g_{n-1}\}$.

Notice that, by the identity (4) we obtain

$$g_j \equiv \pm g_{n-1} \pmod{Id(\mathcal{V})}$$

for $j = 2, \ldots, n-2$.

It follows that the subspace $\mathrm{span}\{g_1, \ldots, g_{n-1}\}$, modulo $Id(\mathcal{V})$, has dimension bounded by 2. Hence $m_{(2,1^{n-2})} \leq 2$ and we are done.

Now we are able to prove the following

Theorem 1. *Let $\mathcal{V}$ be a variety of algebras satisfying the identity*

$$x(yz) = 0.$$

If $c_n(\mathcal{V}) \leq Cn^\alpha$ for some constant $C > 0$ and $1 < \alpha < 2$, then $c_n(\mathcal{V}) \leq 4n + C_1$ for some constant $C_1 > 0$.

Proof. Fix N so that, for all $n \geq N$, $d_\lambda > Cn^\alpha$ for $\lambda \notin \{(n), (1^n), (n-1,1), (2,1^{n-2})\}$. Then by Lemma 1, $m_\lambda = 0$ for every $\lambda \neq (n)(n-1,1), (1^n), (2, 1^{n-2})$.

Thus for $n \geq N$,

$$\chi_n(\mathcal{V}) = m_{(n)}\chi_{(n)} + m_{(n-1,1)}\chi_{(n-1,1)} + m_{(1^n)}\chi_{(1^n)} + m_{(2,1^{n-2})}\chi_{(2,1^{n-2})}.$$

Since $\deg \chi_{(n-1,1)} = \deg \chi_{(2,1^{n-2})} = n-1$ and $\deg \chi_{(1^n)} = \deg \chi_{(1^n)} = 1$, by recalling Remark 1, Proposition 1 and Proposition 2, we get

$$c_n(\mathcal{V}) \leq 1 + 2(n-1) + 1 + 2(n-1) \leq 4n - 2.$$

For $1 \leq n < N$, let C_1 be such that $c_n(\mathcal{V}) \leq 4n + C_1$ and we are done.

3. Varieties $\mathcal{V}$ such that $c_n(\mathcal{V}) \leq Cn^\alpha, 2 < \alpha < 3$

Let $\mathcal{V}$ be the variety of algebras satisfying the identity

$$x(yz) \equiv 0.$$

Throughout this section we shall assume that $c_n(\mathcal{V}) \leq Cn^\alpha$, for some constants C and α, $2 < \alpha < 3$.

Our aim is to prove that for such variety $c_n(\mathcal{V}) < C_1 n^2$, for some constant C_1.

Let observe that if $\lambda \in \{(n), (1^n), (n-1,1), (2,1^{n-2})\}$, then $m_\lambda d_\lambda < n^2$, so from now on we shall consider partitions $\lambda \notin \{(n), (1^n), (n-1,1), (2,1^{n-2})\}$. The strategy of the proof will be the following: we shall first prove that, for n large enough, $m_\lambda = 0$ for every $\lambda \notin \{(n-2,1,1), (3,1^{n-3}), (n-2,2), (2,2,1^{n-4})\}$ then for the above case we shall find an upper bound for the multiplicities m_λ.

Let start with the following

Lemma 2. *If $\lambda \vdash n$, $n \neq 6$, is such that $\lambda \notin \{(n-2,1,1), (3,1^{n-3}), (n-2,2), (2,2,1^{n-4})\}$ then $d_\lambda \geq \frac{n^3}{44}$.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ and let denote by $\lambda' = (\lambda'_1, \ldots, \lambda'_s) \vdash n$ the conjugate partition of λ. It easy to check that for any $n \neq 6$

$$d_{(n-3,2,1)} = d_{(3,2,1^{n-5})} > d_{(n-3,1,1,1)} = d_{(3,2,1^{n-5})} > d_{(n-3,3)} = d_{(2,2,2,1^{n-6})} > \frac{n^3}{44}.$$

It follows that if $\lambda_1 = n-3$, or $\lambda'_1 = n-3$ or $n \leq 9$ then the conclusion of the lemma follows by direct computation from the hook formula (see [8]).

Hence we may assume that $n \geq 10$, $\lambda_1 \leq n-4$ and $\lambda'_1 < n-4$. The proof will be by induction on n. If the shape of the diagram of λ is not a rectangle then there exist two subdiagrams corresponding to partitions μ and ν each containing $n-1$ boxes and satisfying the hypotheses of the lemma. But then, by using induction and the branching rule, we have that

$$d_\lambda \geq d_\mu + d_\nu \geq 2 \cdot \frac{(n-1)^3}{44} > \frac{n^3}{44}.$$

If the shape of the diagram of λ is a rectangle, there exist two different subdiagrams corresponding to partitions μ and ν, each with $n-2$ boxes, and they both satisfy the hypotheses of the lemma. In this case we have

$$d_\lambda \geq d_\mu + d_\nu \geq 2 \cdot \frac{(n-2)^3}{44} > \frac{n^3}{44}.$$

In fact let consider the sequence

$$a_n = 2 \cdot \frac{(n-2)^3}{44} - \frac{n^3}{44} = \frac{\left(\sqrt[3]{2}(n-2) - n\right)\left(\left(\sqrt[3]{2}(n-2)\right)^2 + \sqrt[3]{2}(n-2)n + n^2\right)}{44}.$$

As $n \geq 10 > \frac{2\sqrt[3]{2}}{\sqrt[3]{2}-1}$ then $\sqrt[3]{2}(n-2) - n > 0$. So, $a_n > 0$ for any $n \geq 10$ and we are done.

From the above lemma it follows that if $\lambda \vdash n$ is distinct from $(n-2,1,1), (3,1^{n-3}), (n-2,2), (2,2,1^{n-4})$, then there exists $m > 0$ such that for all $n \geq m$ we have that $d_\lambda > \frac{n^3}{44} > Cn^\alpha$ and so $m_\lambda = 0$. We fix the integer $m \geq 4$ from now and we shall also assume that the integer m has the further property that $c_n(\mathcal{V}) < \frac{n^3}{44}$ for all $n \geq m$.

Let start with the following

Lemma 3. *There exists r, with $0 \leq r \leq m-3$, such that*

$$z_1 z_2 x_1^r z x_1^{m-r-2} \equiv \sum_{i>r} \gamma_i z_1 z_2 x_1^i z x_1^{m-i-2} \quad (mod.\ Id(\mathcal{V})) \tag{5}$$

where, for some i, $\gamma_i \neq 0$.

Proof. Let $\lambda = (m-2,1,1) \vdash m$. For $p = 0, \ldots, m-3$, we define the tableaux

$$T_\lambda^{(p)} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & \ldots & p+1 & p+4 & \ldots \\ \hline p+2 \\ \cline{1-1} p+3 \\ \cline{1-1} \end{array}$$

and we associate to $T_\lambda^{(p)}$ the left-normed polynomials

$$g_p = \bar{x}_1 x_1^p \bar{x}_2 \bar{x}_3 x_1^{m-p-3}.$$

Notice that, for every $p = 0, \ldots, m-3$, the polynomials g_p are obtained from the essential idempotents corresponding to the tableaux $T_\lambda^{(p)}$ by identifying all the elements in each row of λ.

If the polynomials g_p are linearly independent then $m_\lambda \geq m-2$. In this case, by Lemma 1, $d_\lambda > \frac{m^2}{8}$ and we have

$$c_m(\mathcal{V}) \geq (m-2)\frac{m^2}{8} > \frac{m^3}{44} > Cm^\alpha$$

a contradiction. So, m_λ will be less than $m-2$ then it follows that the polynomials g_p are linearly dependent and this implies

$$\sum_{p=0}^{m-3} \alpha_p \bar{x}_1 x_1^p \bar{x}_2 \bar{x}_3 x_1^{m-p-3} \equiv 0 \pmod{Id(\mathcal{V})}.$$

Let us replace x_3 with $z_1 z_2$ and x_2 with z then we obtain

$$\sum_{p=0}^{m-3} \alpha_p z_1 z_2 x_1^p (z x_1 - x_1 z) x_1^{m-p-3} \equiv 0 \pmod{Id(\mathcal{V})}.$$

Let now r be the minimum p such that $\alpha_p \neq 0$ and t be the maximum p such that $\alpha_p \neq 0$.

If $t = r$ then

$$z_1 z_2 x_1^r z x_1^{m-r-2} \equiv z_1 z_2 x_1^{r+1} z x_1^{m-r-1} \pmod{Id(\mathcal{V})}.$$

If $t \neq r$ then

$$\sum_{i=r}^{t+1} \beta_i z_1 z_2 x_1^i z x_1^{m-i-2} \equiv 0 \pmod{Id(\mathcal{V})}$$

where $\beta_r = \alpha_r \neq 0$, $\beta_{t+1} = -\alpha_t \neq 0$ and $\beta_{r+j} = \alpha_{r+j} - \alpha_{r+j-1}$ for all $1 \leq j < t - r + 1$. It follows that

$$z_1 z_2 x_1^r z x_1^{m-r-2} \equiv \sum_{i=r+1}^{t+1} \frac{\beta_i}{\beta_r} z_1 z_2 x_1^i z x_1^{m-i-2} = \sum_{i=r+1}^{t+1} \gamma_i z_1 z_2 x_1^i z x_1^{m-i-2} \pmod{Id(\mathcal{V})}$$

where $\gamma_{t+1} = -\frac{\beta_{t+1}}{\beta_r} = \frac{\alpha_t}{\alpha_r} \neq 0$.

From this lemma it easily follows the following

Remark 2. *If $s \geq m$ and $t \geq m$ then*

$$z_1 x_1^s z x_1^t \equiv \sum_i \gamma_i z_1 x_1^{s_i} z x_1^{t_i} \pmod{Id(\mathcal{V})}$$

where $t_i < m$.

Lemma 4. *Let $k = 2m$. For a fixed q, with $0 \leq q < m$, there exists r_q, $0 \leq r_q \leq k - q - 3$, such that*

$$z_1 z_2 x_1^{r_q} \bar{x}_2 x_1^q \bar{x}_3 z x_1^{k-r_q-q-3} \equiv \sum_{0 \leq i < r_q} \beta_i z_1 z_2 x_1^i \bar{x}_2 x_1^q \bar{x}_3 z x_1^{k-i-q-3} \pmod{Id(\mathcal{V})}. \tag{6}$$

Proof. Let $k = 2m$. For a fixed q, $0 \leq q < m$, and for $p = 0, \ldots, k - q - 3$ let

$$h_{p,q} = \bar{x}_1 x_1^p \bar{x}_2 x_1^q \bar{x}_3 x_1^{k-p-q-3}$$

be the left normed polynomials associated to the tableaux

$$T_\lambda^{(p,q)} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & \ldots & p+1 & p+3 & \ldots & p+q+2 & \ldots \\ \hline p+2 \\ \cline{1-1} p+q+3 \\ \cline{1-1} \end{array}$$

where $\lambda = (k-2,1,1) \vdash k$.

As in the previous lemma, since $m \geq 4$, if $m_\lambda \geq k-q-2$ then,

$$(k-q-2)\frac{(2m)^2}{8} > (k-m-2)\frac{(2m)^2}{8} = (m-2)\frac{m^2}{2} > \frac{(2m)^3}{44} > Ck^\alpha \geq c_k(\mathcal{V})$$

a contradiction.

Then the polynomials $h_{p,q}$, for any q, must be linearly dependent and so

$$\sum_{p=0}^{k-q-3} \alpha_p h_{p,q} \equiv 0 \pmod{Id(\mathcal{V})}.$$

Let now consider the following substitution $x_1 = z_1z_2 + x_1$ and let r_q be the maximum p such that $\alpha_p \neq 0$. It follows that

$$z_1z_2x_1^{r_q}\bar{x}_2x_1^q\bar{x}_3x_1^{k-r_q-q-3} \equiv \sum_{0\leq i<r_q} \beta_i z_1z_2x_1^i\bar{x}_2x_1^q\bar{x}_3x_1^{k-i-q-3} \pmod{Id(\mathcal{V})},$$

and we are done.

From now on we shall assume that $k = 2m$.

We have the following

Remark 3. *For a fixed q, $0 \leq q < m$, and $s, t \geq k$ then either*

$$z_1x_1^s\bar{x}_2x_1^q\bar{x}_3x_1^t \equiv \sum_{i<k} \beta_i z_1x_1^i\bar{x}_2x_1^q\bar{x}_3x_1^{s+t-i} \quad (mod.\ Id(\mathcal{V}))$$

where $\beta_i \neq 0$ for some i, or

$$z_1x_1^s\bar{x}_2x_1^q\bar{x}_3x_1^t \equiv 0 \quad (mod.\ Id(\mathcal{V})).$$

Proof. Let consider the polynomial $z_1x_1^s\bar{x}_2x_1^q\bar{x}_3x_1^t$ with $q < m$, $s \geq k$ and $t \geq k$.

If in (6) there exists i such that $\beta_i \neq 0$, then

$$z_1x_1^s\bar{x}_2x_1^q\bar{x}_3x_1^t \equiv \sum_{i<k} \beta_i z_1x_1^i\bar{x}_2x_1^q\bar{x}_3x_1^{s+t-i} \pmod{Id(\mathcal{V})}$$

and we are done.

Otherwise if in (6), $\beta_i = 0$ for all i, then we have

$$z_1z_2x_1^{r_q}\bar{x}_2x_1^q\bar{x}_3x_1^{k-r_q-q-3} \equiv 0 \pmod{Id(\mathcal{V})}.$$

So, for any $s, t \geq k$, and $0 \leq q < m$ it follows that

$$z_1x_1^s\bar{x}_2x_1^q\bar{x}_3x_1^t \equiv 0 \pmod{Id(\mathcal{V})}.$$

Proposition 3. *Let $\chi_n(\mathcal{V}) = \sum_{\lambda\vdash n} m_\lambda\chi_\lambda$. If $\lambda = (n-2,1,1)$ then $m_{(n-2,1,1)} \leq 5m^2$.*

PROOF. If $n < k$ then $m_\lambda \leq d_\lambda < n^2 < 5m^2$.

Let $n \geq k$. The highest weight vectors corresponding to the standard Young tableaux of shape $\lambda = (n-2, 1, 1)$ are of the following type

$$g_{\alpha,\beta,\gamma} = \bar{x}_1 x_i^{\alpha} \bar{x}_2 x_1^{\beta} \bar{x}_3 x_1^{\gamma}$$

where $\alpha + \beta + \gamma = n - 3$. We want to find an upper bound for the dimension of the span $\{g_{\alpha,\beta,\gamma}\}$.

Let first assume $0 \leq \beta < m$.

If $\alpha \geq k$ and $\gamma \geq k$, by Remark 3, either $g_{\alpha,\beta,\gamma} \equiv 0$ or we can write $g_{\alpha,\beta,\gamma}$ as a linear combination of polynomials $\bar{x}_1 x_i^{\alpha'} \bar{x}_2 x_1^{\beta} \bar{x}_3 x_1^{\gamma'}$ where $\alpha' < k$.

Hence, if $\beta < m$ we have to consider polynomials $g_{\alpha,\beta,\gamma}$ such that either $\alpha < k$ or $\gamma < k$ and so we obtain at most $2km$ polynomials.

Let now $\beta \geq m$.

If $\gamma \geq m$ then, by Remark 2, we can write $g_{\alpha,\beta,\gamma}$ as a linear combination of polynomials $g_{\alpha',\beta',\gamma'}$ where $\gamma' < m$.

So let consider polynomials $g_{\alpha,\beta,\gamma}$ with $\beta \geq m$ and $\gamma < m$.

If $\alpha \geq m$ then, by Remark 2, $g_{\alpha,\beta,\gamma}$ is equivalent to a linear combination of polynomials $g_{\alpha',\beta',\gamma'}$ where $\beta' < m$.

So we have to consider polynomials $g_{\alpha,\beta,\gamma}$ with $\alpha < m$, $\beta \geq m$ and $\gamma < m$ and this are at most m^2.

It follows that the dim span$\{g_{\alpha,\beta,\gamma}\} \leq 5m^2$.

Lemma 5.

1) There exists r, $0 \leq r \leq m-2$, such that

$$z_1 z_2 \bar{x}_1 \cdots \bar{x}_r z \bar{x}_{r+1} \cdots \bar{x}_{m-2} \equiv \sum_{i>r} \gamma_i z_1 z_2 \bar{x}_1 \cdots \bar{x}_i z \bar{x}_{i+1} \cdots \bar{x}_{m-2} \quad (mod.\ Id(\mathcal{V})), \tag{7}$$

where $\gamma_{r+1} = -1$.

2) Let $k = 2m$, then for any fixed q, $0 \leq q < m$, there exists r_q, $0 < r_q < k - q - 2$, such that

$$z_1 z_2 \bar{x}_2 \cdots \bar{x}_{r_q} z \bar{x}_{r_q+1} \cdots \bar{x}_{r_q+q} z \bar{x}_{r_q+q+1} \cdots \bar{x}_{k-2} \equiv \tag{8}$$
$$\sum_{i<r_q} \beta_i z_1 z_2 \bar{x}_2 \cdots \bar{x}_i z \bar{x}_{i+1} \cdots \bar{x}_{i+q} z \bar{x}_{i+q+1} \cdots \bar{x}_{k-2} \quad (mod.\ Id(\mathcal{V}))$$

PROOF. 1) Let $\lambda = (3, 1^{m-3}) \vdash m$. For every $p \in P = \{1, 3, 5, \cdots\}$, we define the standard tableaux

$$T_\lambda^{(p)} = \begin{array}{|c|c|c|} \hline 1 & p+1 & p+2 \\ \hline . \\ \cline{1-1} . \\ \cline{1-1} . \\ \cline{1-1} p \\ \cline{1-1} p+3 \\ \cline{1-1} . \\ \cline{1-1} . \\ \cline{1-1} . \\ \cline{1-1} \end{array}$$

and we associate to $T_\lambda^{(p)}$ the left-normed polynomials

$$g_p = \bar{x}_1 \cdots \bar{x}_p x_1 x_1 \bar{x}_{p+1} \cdots \bar{x}_{m-2}.$$

Notice that for every $p \in P = \{1, 3, 5, \cdots\}$, the $[(m+1)/2]$ polynomials g_p are obtained from the essential idempotents corresponding to the tableaux $T_\lambda^{(p)}$ by identifying all the elements in each row of λ. Since $c_m(\mathcal{V}) < \frac{m^3}{44} < [(m+1)/2]\frac{m^2}{8}$ it follows that the polynomials g_p are linearly dependent then

$$\sum_{p \in P} \alpha_p g_p \equiv 0 \pmod{Id(\mathcal{V})}.$$

By making the substitution $x_1 = z_1 z_2 + x_1$ we obtain

$$\sum_{p \in P} \alpha_p z_1 z_2 \bar{x}_2 \cdots \bar{x}_p x_1 x_1 \bar{x}_{p+1} \cdots \bar{x}_{m-2} \equiv 0 \pmod{Id(\mathcal{V})}.$$

Let r be the minimum p such that $\alpha_p \neq 0$, then

$$z_1 z_2 \bar{x}_2 \cdots \bar{x}_r x_1 x_1 \bar{x}_{r+1} \cdots \bar{x}_{m-2} \equiv \sum_{i>r} \beta_i z_1 z_2 \bar{x}_2 \cdots \bar{x}_i x_1 x_1 \bar{x}_{i+1} \cdots \bar{x}_{m-2} \pmod{Id(\mathcal{V})}.$$

By substituting x_1 with $x_1 + z_1$ and by alternating on $x_1, x_2, \ldots, x_{m-2}$ we obtain

$$z_1 z_2 \bar{x}_1 \cdots \bar{x}_r z \bar{x}_{r+1} \cdots \bar{x}_{m-2} \equiv \sum_{i>r} \gamma_i z_1 z_2 \bar{x}_1 \cdots \bar{x}_i z \bar{x}_{i+1} \cdots \bar{x}_{N-2} \pmod{Id(\mathcal{V})}.$$

Let observe that $\gamma_{r+1} = -1$ and we are done.

2) Let now $k = 2m$, for a fixed $q = 0, \ldots, m-1$ and for $p = 1, 2, \ldots, k-q-2$ let

$$T_\lambda^{(p,q)} = \begin{array}{|c|c|c|}
\hline
1 & p+1 & p+q+2 \\
\hline
2 \\
\cline{1-1}
. \\
\cline{1-1}
. \\
\cline{1-1}
. \\
\cline{1-1}
p \\
\cline{1-1}
p+2 \\
\cline{1-1}
. \\
\cline{1-1}
. \\
\cline{1-1}
. \\
\cline{1-1}
p+q+1 \\
\cline{1-1}
p+q+3 \\
\cline{1-1}
. \\
\cline{1-1}
. \\
\cline{1-1}
. \\
\cline{1-1}
\end{array}$$

be standard tableaux of shape $\lambda = (3, 1^{k-3}) \vdash k$. We associate to any tableaux $T_\lambda^{(p,q)}$, $p = 1, 2, \ldots, k-q-2$, the left-normed polynomials

$$h_{p,q} = \bar{x}_1 \cdots \bar{x}_p x_1 \bar{x}_{p+1} \cdots \bar{x}_{p+q} x_1 \bar{x}_{p+q+1} \cdots \bar{x}_{k-2}.$$

As before, by the hypothesis on $c_k(\mathcal{V})$ we have that the polynomials $h_{p,q}$ are linearly dependent then

$$\sum \alpha_{p,q} h_{p,q} \equiv 0 \pmod{Id(\mathcal{V})}.$$

Let $r_q > 0$ be the maximum p such that $\alpha_{p,q} \neq 0$, then after the substitution $x_1 = z_1 z_2 + z$ we obtain

$$z_1 z_2 \bar{x}_2 \cdots \bar{x}_{r_q} z \bar{x}_{r_q+1} \cdots \bar{x}_{r_q+q} z \bar{x}_{r_q+q+1} \cdots \bar{x}_{k-2} \equiv$$
$$\sum_{i<r_q} \beta_i z_1 z_2 \bar{x}_2 \cdots \bar{x}_i z \bar{x}_{i+1} \cdots \bar{x}_{i+q} z \bar{x}_{i+q+1} \cdots \bar{x}_{k-2} \pmod{Id(\mathcal{V})},$$

and we are done.

Remark 4.

1) If $s \geq m$ and $t \geq m$ then

$$z_1 \bar{x}_1 \cdots \bar{x}_s z \bar{x}_{s+1} \cdots \bar{x}_{s+t} \equiv \sum_i \gamma_i z_1 \bar{x}_1 \cdots \bar{x}_i z \bar{x}_{i+1} \cdots \bar{x}_{i+t'} \quad (mod.\ Id(\mathcal{V})),$$

where $t' < m$.

2) If $0 \leq q < m$, $s \geq k$ and $t \geq k$ then either

$$\bar{x}_1 \cdots \bar{x}_s z \bar{x}_{s+1} \cdots \bar{x}_{s+q} z \bar{x}_{s+q+1} \cdots \bar{x}_{s+q+t} \equiv$$
$$\sum_{s'<k} \gamma_{s'} \bar{x}_1 \cdots \bar{x}_{s'} z \bar{x}_{s'+1} \cdots \bar{x}_{s'+q} z \bar{x}_{s'+q+1} \cdots \bar{x}_{s'+q+t'} \quad (mod.\ Id(\mathcal{V})),$$

where $\gamma_{s'} \neq 0$, for some s' or

$$z_1 \bar{x}_1 \cdots \bar{x}_s z \bar{x}_{s+1} \cdots \bar{x}_{s+t} \equiv 0 \quad (mod.\ Id(\mathcal{V})).$$

PROOF. 1) Let consider the polynomial $z_1 \bar{x}_1 \cdots \bar{x}_s z \bar{x}_{s+1} \cdots \bar{x}_{s+t}$. If $s \geq m$ and $t \geq m$, by (7) it follows that

$$z_1 \bar{x}_1 \cdots \bar{x}_s z \bar{x}_{s+1} \cdots \bar{x}_{s+t} \equiv \sum_i \gamma_i z_1 \bar{x}_1 \cdots \bar{x}_i z \bar{x}_{i+1} \cdots \bar{x}_{i+t'} \pmod{Id(\mathcal{V})},$$

where $t' < m$.

2) Let now consider the polynomial $\bar{x}_1 \cdots \bar{x}_s z \bar{x}_{s+1} \cdots \bar{x}_{s+q} z \bar{x}_{s+q+1} \cdots \bar{x}_{s+q+t}$ with $0 \leq q < m$, $s \geq k$ and $t \geq k$. If in (8), $\beta_i \neq 0$ for some i, then

$$\bar{x}_1 \cdots \bar{x}_s z \bar{x}_{s+1} \cdots \bar{x}_{s+q} z \bar{x}_{s+q+1} \cdots \bar{x}_{s+q+t} \equiv$$
$$\sum_{s'<k} \gamma_{s'} \bar{x}_1 \cdots \bar{x}_{s'} z \bar{x}_{s'+1} \cdots \bar{x}_{s'+q} z \bar{x}_{s'+q+1} \cdots \bar{x}_{s'+q+t'} \pmod{Id(\mathcal{V})},$$

where $\gamma_{s'} \neq 0$, for some s'.

If in (8), $\beta_i = 0$ for any i, then we have

$$z_1 z_2 \bar{x}_2 \cdots \bar{x}_{r_q} z \bar{x}_{r_q+1} \cdots \bar{x}_{r_q+q} z \bar{x}_{r_q+q+1} \cdots \bar{x}_{k-2} \equiv 0 \pmod{Id(\mathcal{V})}$$

and this implies that

$$\bar{x}_1 \cdots \bar{x}_s z \bar{x}_{s+1} \cdots \bar{x}_{s+q} z \bar{x}_{s+q+1} \cdots \bar{x}_{s+q+t} \equiv 0 \pmod{Id(\mathcal{V})},$$

for $0 \leq q < m$ and $s, t \geq k$.

Proposition 4. *Let $\lambda = (3, 1^{n-3}) \vdash n$, then $m_\lambda \leq 5m^2$.*

PROOF. If $n < k$ then $m_\lambda \leq d_\lambda < n^2 < 5m^2$.

So, let $n \geq k$. For $p = 1, \cdots n-2$ and $p+q+r = n-2$ let

$$f_{p,q,r} = \underbrace{\bar{x}_1 \bar{x}_2 \cdots \bar{x}_p}_{p} x_1 \underbrace{\bar{x}_{p+1} \cdots \bar{x}_{p+q}}_{q} x_1 \underbrace{\bar{x}_{p+q+1} \cdots \bar{x}_{n-2}}_{r}$$

be the highest weight vectors corresponding to the standard tableaux of the partition $(3, 1^{n-3})$. The dimension of the space spanned by all $f_{p,q,r}$, equals m_λ. We want to find an upper bound of this dimension.

Let first assume that $q < m$.

If $p > k$ and $r > k$, by Remark 4, $f_{p,q,r}$ is equivalent to a linear combination of polynomials of the type $f_{p',q,r'}$ where $p' < k$, and these polynomials are at most km.

If $p > k$ and $r < k$ we obtain again at most km polynomials.

So let assume that $q \geq m$.

If $p > m$, by the first part of Remark 4, we obtain that $f_{p,q,s}$ is a linear combination of polynomials $f_{p',q',r'}$ with $q' < m$.

So let suppose $p < m$, $q \geq m$, and $r > m$. By Remark 4, we can write $f_{p,q,r}$ as a linear combination of polynomials $f_{p',q',r'}$ with $r' < m$. It follows that, if $q \geq m$, the linearly independent polynomials $f_{p,q,r}$ are at most m^2.

As consequence we obtain that $m_\lambda \leq 5m^2$.

Next we shall prove that for $\lambda = (n-2, 2) \vdash n$ the multiplicity m_λ is bounded by some constant.

Lemma 6. *For any $\alpha, \beta, \gamma, \delta, \eta$ with $\alpha + \beta + \gamma + \delta + \eta \geq m - 5$ we have that*

$$z_1 z_2 x_1^\alpha \bar{y}_1 x_1^\beta \bar{y}_2 x_1^\gamma \tilde{y}_1 x_1^\delta \tilde{y}_2 x_1^\eta \equiv 0 \pmod{Id(\mathcal{V})}, \tag{9}$$

$$z_1 z_2 x_1^\alpha \bar{y}_1 x_1^\beta \tilde{y}_1 x_1^\gamma \tilde{y}_2 x_1^\delta \bar{y}_2 x_1^\eta \equiv 0 \pmod{Id(\mathcal{V})}. \tag{10}$$

PROOF. Let $\lambda = (2, 2, 1) \vdash 5$ and $\mu = (n-5) \vdash n-5$. Let $M_\lambda \hat{\otimes} M_\mu$ be the the S_n- module outer tensor product of the irreducible modules M_λ and M_μ, (see [7]). If we consider the polynomials

$$\bar{y}_3 x_1^\alpha \bar{y}_1 x_1^\beta \bar{y}_2 x_1^\gamma \tilde{y}_1 x_1^\delta \tilde{y}_2 x_1^\eta$$
$$\bar{y}_3 x_1^\alpha \bar{y}_1 x_1^\beta \tilde{y}_1 x_1^\gamma \tilde{y}_2 x_1^\delta \bar{y}_2 x_1^\eta$$

obtained from the two modules M_λ and M_μ, from remark after Lemma 2, it follows that these polynomials are identities of the variety $\mathcal{V}$. Let substitute y_3 with $z_1 z_2$ and we are done.

Proposition 5. *Let $\lambda = (n-2, 2) \vdash n$ then $m_\lambda \leq \bar{C}$ where $\bar{C} = max\{4m+3, m^2\}$.*

PROOF. We shall construct polynomials corresponding to essential idempotents of the group algebra of S_n. Let $e_{T_\lambda} \in FS_n$ be the essential idempotent corresponding to the tableau T_λ, we shall identify e_{T_λ} with the polynomial $e_{T_\lambda}(x_n, \ldots, x_1) = e_{T_\lambda} x_n \cdots x_1$ obtained by acting with e_{T_λ} on the left normed monomial $x_n \cdots x_1$. We shall then identify all variables corresponding to each row of the tableau. Let consider the standard Young tableaux corresponding to $\lambda = (n-2, 2)$. This tableaux are of different types.

Type 1. Let consider first tableaux of the following types

$$T_\lambda^i = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & n-1 & n \\ \hline i & j \\ \cline{1-2}\end{array}$$

where $4 \leq i < j \leq n-2$ or

$$T_\lambda^i = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & n-1 & n \\ \hline 3 & i \\ \cline{1-2}\end{array}$$

where $4 \leq i \leq n-2$ or

$$T_\lambda^i = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & \cdots & n-1 & n \\ \hline 2 & i \\ \cline{1-2}\end{array}$$

where $4 \leq i \leq n-2$.

For this tableaux we obtain, by Lemma 6, that $e_{T_\lambda^i} \equiv 0$.

Type 2. Let now consider tableaux of the following type

$$T_\lambda^i = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & \cdots & n-2 & n-1 \\ \hline 2 & n \\ \cline{1-2}\end{array}$$

or

$$T_\lambda^i = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & \cdots & n-2 & n \\ \hline 2 & n-1 \\ \cline{1-2}\end{array}$$

In the first case we obtain

$$e_{T_\lambda^i} = \tilde{x}_2 x_1^{n-4} \tilde{x}_1 \bar{x}_2 \bar{x}_1$$

and in the second case

$$e_{T_\lambda^i} = x_1 \tilde{x}_2 x_1^{n-3} \tilde{x}_1 \bar{x}_2 \bar{x}_1.$$

Type 3. If

$$T_\lambda^i = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n-2 \\ \hline n-1 & n \\ \cline{1-2}\end{array}$$

then we obtain the polynomial

$$e_{T_\lambda^i} = \tilde{x}_2 \bar{x}_2 x_1^{n-4} \tilde{x}_1 \bar{x}_1$$

Type 4. Let now

$$T_\lambda^i = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline i & n-1 \\ \cline{1-2}\end{array}$$

where $2 < i < n-1$ then $e_{T_\lambda^i}$ are polynomials of the following type

$$x_1 \tilde{x}_2 x_1^{n-i-2} \bar{x}_2 x_1^{i-3} \tilde{x}_1 \bar{x}_1.$$

By Lemma 3 and Remark 2, it follows that they are linear combination of polynomials of the same type with $n-i-2 < m$ or $i-3 < m$. So, in this case the linearly independent polynomials are less than $2m$.

Type 5. Let now consider the tableaux

$$T^i_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n-1 \\ \hline i & n & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$$

where $2 < i < n-1$ then we obtain the polynomials

$$e_{T^i_\lambda} = \tilde{x}_2 x_1^{n-i-1} \bar{x}_2 x_1^{i-3} \tilde{x}_1 \bar{x}_1$$

By Remark 2, these polynomials are linear combinations of polynomials of the same type with $n-i-2 < m$ or $i-3 < m$. So, also in this case the polynomials that are linearly independent are less than $2m$.

It follows that, for $n \geq m$, $m_\lambda \leq 4m+3$.

If $n < m$ then $m_\lambda \leq d_\lambda < m^2$. Hence for any n, $m_\lambda \leq \bar{C}$ where $\bar{C} = max\{4m+3, m^2\}$.

Now we shall prove that the multiplicity for the partition $\lambda = (2,2,1^{n-4}) \vdash n$ is bounded by some constant.

Lemma 7. *For any $\alpha, \beta, \gamma, \delta, \eta$ with $\alpha+\beta+\gamma+\delta+\eta \geq m-5$ we have*

$$z_1 z_2 \bar{x}_1 \bar{x}_2 \ldots \bar{x}_\alpha \bar{\bar{y}}_1 x_{\alpha+1} x_{\alpha+2} \ldots x_{\alpha+\beta} \bar{\bar{y}}_2 x_{\alpha+\beta+1} \ldots x_{\alpha+\beta+\gamma} \tilde{y}_1 x_{\alpha+\beta+\gamma+1} x_{\alpha+\beta+\gamma+2} \cdots \tag{11}$$

$$\ldots x_{\alpha+\beta+\gamma+\delta} \tilde{y}_2 x_{\alpha+\beta+\gamma+\delta+1} \ldots x_{\alpha+\beta+\gamma+\delta+\eta} \equiv 0 \quad (\textit{mod. } Id(\mathcal{V})).$$

Proof. Let $\lambda = (3,2) \vdash 5$ and $\mu = (1^{n-5}) \vdash n-5$ and let $M_\lambda \widehat{\otimes} M_\mu$ be the S_n module outer tensor product of the S_5-module M_λ and of the S_{n-5}-module M_μ.

Consider the polynomials

$$y_1 \bar{x}_1 \bar{x}_2 \ldots \bar{x}_\alpha \bar{\bar{y}}_1 x_{\alpha+1} x_{\alpha+2} \ldots x_{\alpha+\beta} \bar{\bar{y}}_2 x_{\alpha+\beta+1} \ldots x_{\alpha+\beta+\gamma} \tilde{y}_1 x_{\alpha+\beta+\gamma+1} x_{\alpha+\beta+\gamma+2} \cdots$$

$$\ldots x_{\alpha+\beta+\gamma+\delta} \tilde{y}_2 x_{\alpha+\beta+\gamma+\delta+1} \ldots x_{\alpha+\beta+\gamma+\delta+\eta}$$

obtained from the two modules M_λ and M_μ. From remark after Lemma 2 these polynomials are identities of the variety $\mathcal{V}$. Let substitute y_1 with $y_1 + z_1 z_2$ and we are done.

Let's now examine some consequences of the previous Lemma.

In (11) let substitute y_1 with $y_1 + z_3$, y_2 with $y_2 + z_4$, and multilinearize. For simplicity we will not write the $x_i' s$, then, modulo $Id(\mathcal{V})$, we obtain

$$z_1 z_2 \ldots \bar{\bar{y}}_1 \ldots \bar{\bar{y}}_2 \ldots \tilde{z}_3 \ldots \tilde{z}_4 \ldots + z_1 z_2 \ldots \bar{\bar{z}}_3 \ldots \bar{\bar{z}}_4 \ldots \tilde{y}_1 \ldots \tilde{y}_2 \ldots +$$

$$z_1 z_2 \ldots \bar{\bar{z}}_3 \ldots \bar{\bar{y}}_2 \ldots \tilde{y}_1 \ldots \tilde{z}_4 \ldots + z_1 z_2 \ldots \bar{\bar{y}}_1 \ldots \bar{\bar{z}}_4 \ldots \tilde{z}_3 \ldots \tilde{y}_2 \ldots \equiv 0.$$

In particular we have that

$$z_1 z_2 \ldots \bar{\bar{y}}_1 \ldots \bar{\bar{y}}_2 \ldots \tilde{z}_3 \ldots \tilde{z}_4 \ldots + z_1 z_2 \ldots \bar{\bar{z}}_3 \ldots \bar{\bar{z}}_4 \ldots \tilde{y}_1 \ldots \tilde{y}_2 \ldots +$$

$$z_1 z_2 \ldots z_3 \ldots y_2 \ldots y_1 \ldots z_4 \ldots - z_1 z_2 \ldots y_2 \ldots z_3 \ldots y_1 \ldots z_4 \ldots -$$

$$z_1 z_2 \ldots z_3 \ldots y_2 \ldots z_4 \ldots y_1 \ldots + z_1 z_2 \ldots y_2 \ldots z_3 \ldots z_4 \ldots y_1 \ldots +$$

$$z_1 z_2 \ldots y_1 \ldots z_4 \ldots z_3 \ldots y_2 \ldots - z_1 z_2 \ldots z_4 \ldots y_1 \ldots z_3 \ldots y_2 \ldots -$$

$$z_1 z_2 \ldots y_1 \ldots z_4 \ldots y_2 \ldots z_3 \ldots + z_1 z_2 \ldots z_4 \ldots y_1 \ldots y_2 \ldots z_3 \ldots \equiv 0.$$

If we alternate on y_1, y_2 and on z_3, z_4, we obtain that, modulo $Id(\mathcal{V})$,

$$2z_1z_2\dots\bar{\bar{y}}_1\dots\bar{\bar{y}}_2\dots\tilde{z}_3\dots\tilde{z}_4\dots \equiv -2z_1z_2\dots\bar{\bar{z}}_3\dots\bar{\bar{z}}_4\dots\tilde{y}_1\dots\tilde{y}_2\dots +$$
$$2z_1z_2\dots\tilde{z}_3\dots\bar{\bar{y}}_1\dots\bar{\bar{y}}_2\dots\tilde{z}_4\dots - 2z_1z_2\dots\bar{\bar{y}}_1\dots\tilde{z}_3\dots\bar{\bar{y}}_2\dots\tilde{z}_4\dots -$$
$$2z_1z_2\dots\tilde{z}_3\dots\bar{\bar{y}}_1\dots\tilde{z}_4\dots\bar{\bar{y}}_2\dots + 2z_1z_2\dots\bar{\bar{y}}_1\dots\tilde{z}_3\dots\tilde{z}_4\dots\bar{\bar{y}}_2\dots$$

Let now consider the following substitutions $z_i = x_{\alpha+\beta+\gamma+\delta+\eta+i}$, for $i = 1,2,3,4$, and let alternate on the variables $x_1, x_2, \dots, x_{\alpha+\beta+\gamma+\delta+\eta+4}$, then it follows that, modulo $Id(\mathcal{V})$,

$$\bar{x}_1\bar{x}_2\dots\bar{x}_{\alpha+2}\tilde{y}_1\bar{x}_{\alpha+3}\dots\bar{x}_{\alpha+\beta+2}\tilde{y}_2\bar{x}_{\alpha+\beta+3}\dots\bar{x}_{\alpha+\beta+\gamma+\delta+\eta+4} \equiv$$
$$a\cdot\bar{x}_1\bar{x}_2\dots\bar{x}_{\alpha+2}\tilde{y}_1\bar{x}_{\alpha+3}\dots\bar{x}_{\alpha+\beta+\gamma+3}\tilde{y}_2\bar{x}_{\alpha+\beta+\gamma+4}\dots\bar{x}_{\alpha+\beta+\gamma+\delta+\eta+4}+$$
$$b\cdot\bar{x}_1\bar{x}_2\dots\bar{x}_{\alpha+2}\tilde{y}_1\bar{x}_{\alpha+3}\dots\bar{x}_{\alpha+\beta+\gamma+\delta+4}\tilde{y}_2\bar{x}_{\alpha+\beta+\gamma+\delta+5}\dots\bar{x}_{\alpha+\beta+\gamma+\delta+\eta+4}+$$
$$c\cdot\bar{x}_1\bar{x}_2\dots\bar{x}_{\alpha+\beta+3}\tilde{y}_1\bar{x}_{\alpha+\beta+4}\dots\bar{x}_{\alpha+\beta+\gamma+3}\tilde{y}_2\bar{x}_{\alpha+\beta+\gamma+4}\dots\bar{x}_{\alpha+\beta+\gamma+\delta+\eta+4}+$$
$$d\cdot\bar{x}_1\bar{x}_2\dots\bar{x}_{\alpha+\beta+3}\tilde{y}_1\bar{x}_{\alpha+\beta+4}\dots\bar{x}_{\alpha+\beta+\gamma+\delta+4}\tilde{y}_2\bar{x}_{\alpha+\beta+\gamma+\delta+5}\dots\bar{x}_{\alpha+\beta+\gamma+\delta+\eta+4}+$$
$$e\cdot\bar{x}_1\bar{x}_2\dots\bar{x}_{\alpha+\beta+\gamma+4}\tilde{y}_1\bar{x}_{\alpha+\beta+\gamma+5}\dots\bar{x}_{\alpha+\beta+\gamma+\delta+4}\tilde{y}_2\bar{x}_{\alpha+\beta+\gamma+\delta+5}\dots\bar{x}_{\alpha+\beta+\gamma+\delta+\eta+4}$$

where a, b, c, d, e are equals to 1 or -1 according to the parity of the numbers $\alpha, \beta, \gamma, \delta, \eta$.

Let remark that from these identities it follows that it is possible to change the number of alternating variables between different alternating pair and we shall use this observation in the next proposition.

Proposition 6. *If* $\lambda = (2, 2, 1^{n-4})$ *then* $m_\lambda \leq 3m^2$.

Proof. Let $e_{T_\lambda} \in FS_n$ be the essential idempotent corresponding to the tableau T_λ, we shall identify e_{T_λ} with the polynomial $e_{T_\lambda}(x_1, \dots, x_n) = e_{T_\lambda}x_1\cdots x_n$ obtained by acting with e_{T_λ} on the left normed monomial $x_1\cdots x_n$.

The polynomials corresponding to standard tableaux are of the following type

$$\bar{x}_1\bar{x}_2\dots\bar{x}_\alpha\tilde{x}_1x_{\alpha+1}x_{\alpha+2}\dots x_{\alpha+\beta}\tilde{x}_2x_{\alpha+\beta+1}\dots x_{\alpha+\beta+\gamma}.$$

If $\alpha \geq m$, $\beta \geq m$ and $\gamma \geq m$ or $\alpha \geq m$, $\beta \geq m$ but $\gamma < m$ then, by Remark 4, we rewrite such polynomials as a linear combination of polynomials of the same type with $\beta < m$ and $\gamma < m$, then we have less than m^2 linearly independent polynomials.

If $\beta < m$ but $\gamma \geq m$ by the consequences of Lemma 7, we rewrite such polynomials as a linear combinations of polynomials of the same type with $\beta \geq m$ or $\beta < m$, $\gamma < m$.
So, any polynomial is a linear combination of polynomials with $\alpha < m$ and $\beta < m$ or $\beta < m$ and $\gamma < m$ or $\alpha < m$ and $\gamma < m$.

It follows that, for $n \geq m$, $m_\lambda \leq 3m^2$.

If $n < m$ then $m_\lambda \leq d_\lambda < m^2$. Then, for any n, $m_\lambda \leq 3m^2$ and we are done.

Now we are able to prove the following

Theorem 2. *Let* $\mathcal{V}$ *be a variety of algebras satisfying the identity*

$$x(yz) = 0.$$

If $c_n(\mathcal{V}) \leq Cn^\alpha$ *for some constant* $C > 0$ *and* $2 < \alpha < 3$, *then* $c_n(\mathcal{V}) \leq C_1n^2$ *for some constant* $C_1 > 0$.

PROOF. By Lemma 2, it follows that

$$\chi_n(\mathcal{V}) = m_{(n)}\chi_{(n)} + m_{(1^n)}\chi_{(1^n)} + m_{(n-1,1)}\chi_{(n-1,1)} + m_{(2,1^{n-2})}\chi_{(2,1^{n-2})} +$$
$$m_{(n-2,1,1)}\chi_{(n-2,1,1)} + m_{(3,1^{n-3})}\chi_{(3,1^{n-3})} + m_{(n-2,2)}\chi_{(n-2,2)} + m_{(2,2,1^{n-4}}\chi_{(2,2,1^{n-4})}.$$

Since $\deg\chi_{(n)} = \deg\chi_{(1^n)} = 1, \deg\chi_{(n-1,1)} = \deg\chi_{(2,1^{n-2})} = n-1, \deg\chi_{(3,1^{n-3})} = \frac{(n-1)(n-2)}{2}$ and $\deg\chi_{(n-2,2)} = \deg\chi_{(2,2,1^{n-4})} = \frac{n(n-3)}{2}$, by recalling Proposition 3, Proposition 4, Proposition 5 and Proposition 6 we get

$$c_n(\mathcal{V}) \leq 2 + 2(n-1)^2 + \frac{(n-1)(n-2)}{2}(5m^2) + \frac{(n-1)(n-2)}{2}(5m^2) +$$
$$\frac{n(n-3)}{2}\bar{C} + \frac{n(n-3)}{2}3m^2 \leq n^2(2 + 6.5m^2 + \bar{C}).$$

So there exists a costant C_1 such that $c_n(\mathcal{V}) \leq C_1 n^2$.

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