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ON A ROBIN (p,q)-EQUATION WITH A LOGISTIC REACTION

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Abstract. We consider a nonlinear nonhomogeneous Robin equation driven by the sum of a p-Laplacian and of a q-Laplacian ((p,q)-equation) plus an indefinite potential term and a parametric reaction of logistic type (superdiffusive case). We prove a bifurcation-type result describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies. Also, we show that for every admissible parameter $\lambda > 0$, the problem admits a smallest positive solution.

Keywords: positive solutions, superdiffusive reaction, local minimizers, maximum principle, minimal positive solutions, Robin boundary condition, indefinite potential.

Mathematics Subject Classification: 35J20, 35J60.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric (p,q)-equation with Robin boundary condition:

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{\theta-1} - f(z,u(z)) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)u^{p-1} = 0 \quad \text{on } \partial\Omega, \quad u > 0,\, 1 < q < p < \theta < p^*,\, \lambda > 0. \end{cases} \tag{P_λ}$$

Here for any $r \in (1, +\infty)$ by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u) \text{ for all } u \in W^{1,r}(\Omega).$$

So, in problem (P_λ) in the left hand side we have the sum of two differential operators of different nature. Such situations arise in the mathematical models of many physical processes. We mention the works of Cherfils–Il'yasov [5] (reaction-diffusion systems) and of Zhikov [32] (homogenization of composites consisting of two different materials with distinct hardening exponents, double phase problems). The differential operator of problem (P_λ) is nonhomogeneous and this is a source of difficulties in the analysis of problem (P_λ). Many of the arguments in the study of superdiffusive logistic equations driven by the Laplacian or p-Laplacian depend heavily on the homogeneity of the operator (see, for example [8, 18, 19, 22, 24]). So, in the present setting they have to be modified. Another, new feature in the problem (P_λ) is the presence of the potential term $\xi(z)u^{p-1}$. The potential function $\xi(\cdot)$ is sign-changing. This adds to the difficulties of problem (P_λ) since the left hand side of (P_λ) is not coercive and so various estimations and bounds are more difficult to produce. The reaction (right hand side) is a generalization of the classical superdiffusive reaction

$$x \to \lambda x^{\theta-1} - cx^{r-1}$$

with $c > 0$,

$$p < \theta < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p \end{cases}$$

(the critical Sobolev exponent corresponding to p). Here cx^{r-1} is replaced by a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous) which is $(\theta-1)$-superlinear in the $x \in \mathbb{R}$ variable. However, we do not impose any positivity requirement on $f(z,\cdot)$ which may be sign-changing. Logistic equations are important in mathematical biology in the description of the steady state of the dynamics of a biological population, whose mobility is state-dependent (see Gurtin–MacCamy [12]). Other physical phenomena also lead to logistic type equations (see Dong [6]).

In the boundary condition $\dfrac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative of u defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \to (|\nabla u|^{p-2} + |\nabla u|^{q-2})\frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$, we recover the Neumann problem.

In the past almost all the works on logistic type equations, examined Dirichlet problems driven by the Laplacian or p-Laplacian. To the best of our knowledge this is the first work dealing with Robin (p,q)-logistic equations. Our aim is to describe the changes in the set of positive solutions of problem (P_λ) as the parameter $\lambda > 0$ varies. In this direction, we prove a bifurcation-type theorem, which produces a critical parameter value $\lambda_* > 0$ such that

- for all $\lambda > \lambda_*$ problem (P_λ) has at least two positive solutions;
- for $\lambda = \lambda_*$ problem (P_λ) has at least one positive solution;
- for all $\lambda \in (0, \lambda_*)$ problem (P_λ) has no positive solutions.

Moreover, we show that for all $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$ problem (P_λ) admits a smallest positive solution $\overline{u}_\lambda$.

If we have a p-Laplace equation with a reaction of the form

$$x \to \lambda x^{q-1} - x^{r-1},\ x \geq 0,\ \text{ with } 1 < q \leq p < r,$$

then we have subdiffusive ($q < p$) and equidiffusive ($q = p$) logistic equations, which have a different behaviour than the superdiffusive ones. Subdiffusive and equidiffusive equations were examined in Ambrosetti–Lupo [2], Ambrosetti–Mancini [3], Marano–Papageorgiou [16], Rădulescu–Repovš [27], Struwe [28, 29] (all dealing with semilinear Dirichlet equations driven by the Laplacian) and also Kamin–Veron [14], Marano–Papageorgiou [16], Papageorgiou–Papalini [18, 19], Papageorgiou–Winkert [22] (nonlinear equations driven by the p-Laplacian). The superdiffusive case was investigated by Cardinali–Papageorgiou–Rubbioni [4], Dong–Chen [7], Filippakis–O'Regan–Papageorgiou [8], Papageorgiou–Rădulescu-Repovš [24], Takeuchi [30, 31] (nonlinear equations driven by the Laplacian and p-Laplacian).

2. MATHEMATICAL BACKGROUND – HYPOTHESES

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence".

This compactness-type condition on φ leads to minimax theorems for the critical values of φ. We formulate one of them, the so-called "mountain pass theorem" which we will use in the sequel.

Theorem 2.1. *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$,*

$$\|u_1 - u_0\|_X > r > 0, \quad \max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\|_X = r\} = m_r$$

and $c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \geq m_r$ and c is a critical value of φ (that is, there exists $\widehat{u} \in X$ such that $\varphi'(\widehat{u}) = 0$, $\varphi(\widehat{u}) = c$).

The analysis of problem (P_λ) involves the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue space $L^p(\partial\Omega)$. In what follows by $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left[\|u\|_p^p + \|\nabla u\|_p^p\right]^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ \, : \, u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

In fact D_+ is also the interior of C_+, when $C^1(\overline{\Omega})$ is endowed with the relative $C(\overline{\Omega})$-norm topology.

Also we will use the order cone

$$\widehat{C}_+ = \left\{ u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ on } \overline{\Omega}, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega \cap u^{-1}(0)} \leq 0 \right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the "boundary" Lebesgue spaces $L^q(\partial\Omega)$, $1 \leq q \leq +\infty$.

The theory of Sobolev spaces says that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u\big|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map gives meaning to the notion of "boundary values" for all Sobolev functions. The trace map is not surjective and $\ker \gamma_0 = W_0^{1,p}(\Omega)$. Moreover it is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{p(N-1)}{N-p})$ when $p < N$ and into $L^q(\partial\Omega)$ for all $q \in [1, +\infty)$ when $N \leq p$. In the sequel, to simplify our notation, we drop the use of the trace map γ_0. All restrictions of the Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Our hypotheses on the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$ are the following:

$H(\xi)$ $\xi \in L^\infty(\Omega)$,

$H(\beta)$ $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark 2.2. If $\beta \equiv 0$, then we recover the Neumann problem.

In what follows by $\gamma : W^{1,p}(\Omega) \to \mathbb{R}$ we denote the C^1-functional defined by

$$\gamma(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q + \frac{1}{p}\int_\Omega \xi(z)|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma$$

for all $u \in W^{1,p}(\Omega)$.

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z,x)| \leq a_0(z)[1 + |x|^{r-1}] \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)$, $1 < r \leq p^*$. We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \gamma(u) - \int_\Omega F_0(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

From Papageorgiou–Rădulescu [21, Proposition 8], we have the following result.

Proposition 2.3. *If hypotheses $H(\xi)$, $H(\beta)$ hold and $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that*

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \text{ for all } h \in C^1(\overline{\Omega}),\ \|h\|_{C^1(\overline{\Omega})} \le \rho_1,$$

then $u_0 \in C^{1,\tau}(\overline{\Omega})$ for some $\tau \in (0,1)$ and u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_2 > 0$ such that

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \text{ for all } h \in W^{1,p}(\Omega),\ \|h\| \le \rho_2.$$

The next result is a strong comparison principle for (p,q)-equations which will be helpful in producing multiple positive solutions for problem (P_λ). The result is a special case of a more general result of Papageorgiou–Rădulescu [25] (see also Gasiński–Papageorgiou [11]).

Proposition 2.4. *If $\eta \in L^\infty(\Omega)$, $\eta(z) \ge 0$ for a.a. $z \in \Omega$, $h_1, h_2 \in L^\infty(\Omega)$ are such that $0 < c \le h_2(z) - h_1(z)$ for a.a $z \in \Omega$, $u \in C^1(\overline{\Omega})$, $u \ne 0$, $v \in D_+$, $u \le v$ and satisfy*

$$\begin{aligned} &-\Delta_p u - \Delta_q u + \eta(z)|u|^{p-2}u = h_1 \text{ for a.a } z \in \Omega,\\ &-\Delta_p v - \Delta_q v + \eta(z)v^{p-1} = h_2 \text{ for a.a } z \in \Omega, \end{aligned}$$

then $v - u \in \operatorname{int} \widehat{C}_+$.

For $r \in (1, +\infty)$ let $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ be the nonlinear map defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\, dz \text{ for all } u, h \in W^{1,r}(\Omega).$$

The next proposition summarizes the main properties of this map (see Gasiński–Papageorgiou [10, Problem 2.192, p. 279]).

Proposition 2.5. *The map $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is, "$u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A_r(u_n), u_n - u\rangle \le 0 \Rightarrow u_n \to u$ in $W^{1,p}(\Omega)$".*

Suppose that $\varphi \in C^1(X, \mathbb{R})$ is coercive (that is, $\varphi(u) \to +\infty$ as $\|u\|_X \to +\infty$) and $\varphi' = A + K$ with $A : X \to X^*$ of type $(S)_+$ and $K : X \to X^*$ completely continuous (that is, $u_n \xrightarrow{w} u$ in $X \Rightarrow K(u_n) \to K(u)$ in X^*), then φ satisfies the C-condition (see Marano–Papageorgiou [17, Proposition 2.2]).

Let $x \in \mathbb{R}$. We set $x^\pm = \max\{\pm x, 0\}$ and for $u \in W^{1,r}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Given $\varphi \in C^1(X, \mathbb{R})$ by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $\widehat{\sigma}(p)$ denote the spectrum of this eigenvalue problem. From Fragnelli–Mugnai–Papageorgiou [9] and Papageorgiou–Rădulescu [20], we know that $\widehat{\sigma}(p) \subseteq \mathbb{R}$ is closed and there exists a smallest eigenvalue $\widehat{\lambda}_1(p) \in \mathbb{R}$ such that

(a) $\widehat{\lambda}_1(p)$ is isolated (that is, there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1(p), \widehat{\lambda}_1(p)+\varepsilon) \cap \widehat{\sigma}(p) = \emptyset$).
(b) $\widehat{\lambda}_1(p)$ is simple (that is, if $\widehat{u}$, $\widehat{v} \in W^{1,p}(\Omega)$ are eigenfunctions corresponding to $\widehat{\lambda}_1(p)$, then $\widehat{u} = \vartheta\widehat{v}$ for some $\vartheta \in \mathbb{R} \setminus \{0\}$).
(c) $\widehat{\lambda}_1(p)$ admits the following variational characterization

$$\widehat{\lambda}_1(p) = \inf \left[\frac{\gamma_p(u)}{\|u\|_p^p} \,:\, u \in W^{1,p}(\Omega),\, u \neq 0 \right], \tag{2.1}$$

where $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ is the C^1-functional defined by

$$\gamma_p(u) = \|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \text{ for all } u \in W^{1,p}(\Omega).$$

The infimum in (2.1) is realized on the corresponding one dimensional eigenspace. Note that if $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a $z \in \Omega$ and $\xi \not\equiv 0$, then $\widehat{\lambda}_1(p) > 0$.

Now we introduce the hypotheses on the perturbation function $f(z,x)$:
$H(f)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$ $f(z,0) = 0$ and

(i) $|f(z,x)| \leq a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geq 0$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;
(ii) $\lim_{x\to+\infty} \dfrac{f(z,x)}{x^{\theta-1}} = +\infty$ uniformly for a.a. $z \in \Omega$;
(iii) there exist $\delta \in (0,1)$ and $\eta_0, \widetilde{\eta}_0 > 0$ such that

$$\eta_0 x^{q-1} \leq f(z,x) \quad \text{for a.a } z \in \Omega, \text{ all } 0 \leq x \leq \delta,$$
$$\limsup_{x\to 0^+} \frac{f(z,x)}{x^{q-1}} \leq \widetilde{\eta}_0 \text{ uniformly for a.a. } z \in \Omega;$$

(iv) for every $\rho > 0$, there exist $\widehat{\xi}_\rho > 0$ and $\widehat{d}_\rho > -\widehat{\lambda}_1(p)$ such that

$$x \to \widehat{\xi}_\rho x^{p-1} - f(z,x) \text{ is nondecreasing on } [0,\rho],$$
$$\widehat{d}_\rho x^{p-1} \leq f(z,x) \text{ for all } 0 \leq x \leq \rho.$$

Remark 2.6. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality we assume that

$$f(z,x) = 0 \text{ for a.a. } z \in \Omega, \text{ all } x \leq 0. \tag{2.2}$$

If $\widehat{\lambda}_1(p) \leq 0$, hypotheses $H(f)$(iv) implies that $f(z,x) \geq 0$ for a.a $z \in \Omega$, all $x \in \mathbb{R}$.

Example 2.7. The following functions satisfy hypotheses $H(f)$. For the sake of simplicity we drop the z-dependence:

$$f_1(x) = c(x^{r-1} + x^{q-1}) \text{ for all } x \geq 0,\ c > 0 \text{ and } c > -\widehat{\lambda}_1(p) \text{ if } \widehat{\lambda}_1(p) < 0,\quad q < p < \theta < r < p^*;$$

$$f_2(x) = c(x^{\theta-1} \ln(1+x) + x^{q-1}) \text{ for all } x \geq 0,\ c > 0 \text{ and } c > -\widehat{\lambda}_1(p) \text{ if } \widehat{\lambda}_1(p) < 0,\quad q < p < \theta < p^*.$$

3. POSITIVE SOLUTIONS

We introduce the following two sets

$$\begin{aligned} \mathcal{L} &= \{\lambda > 0 :\ \text{problem } (P_\lambda) \text{ admits a positive solution}\}, \\ S_\lambda &= \ \text{the set of positive solutions of problem } (P_\lambda). \end{aligned}$$

If $u \in S_\lambda$, then from Papageorgiou-Rădulescu [21, Proposition 7], we have $u \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [15, p. 320] implies that $u \in C_+ \setminus \{0\}$.

Let $\rho = \|u\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)$(iv). We have

$$\begin{aligned} & -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{\theta-1} - f(z,u(z)) \text{ for a.a } z \in \Omega \\ & \qquad \text{(see Papageorgiou–Rădulescu [20])} \\ \Rightarrow\quad & -\Delta_p u(z) - \Delta_q u(z) + [\xi(z) + \widehat{\xi}_\rho]\, u(z)^{p-1} \geq 0 \text{ for a.a } z \in \Omega \\ & \qquad \text{(see hypothesis } H(f) \text{ (iv))} \\ \Rightarrow\quad & \Delta_p u(z) + \Delta_q u(z) \leq [\|\xi\|_\infty + \widehat{\xi}_\rho]\, u(z)^{p-1} \text{ for a.a } z \in \Omega \\ & \qquad \text{(see hypothesis } H(\xi)) \\ \Rightarrow\quad & u \in D_+ \text{ (see Pucci–Serrin [26, pp. 111 and 120]).} \end{aligned}$$

Therefore we can say that

$$S_\lambda \subseteq D_+. \tag{3.1}$$

Proposition 3.1. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $\mathcal{L} \neq \emptyset$.*

Proof. Let $\lambda > 0$, $\mu > \|\xi\|_\infty$ (see hypothesis $H(\xi)$) and $F(z,x) = \int_0^x f(z,s)ds$. We consider the C^1-functional $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \gamma(u) + \frac{\mu}{p}\|u^-\|_p^p - \frac{\lambda}{\theta}\|u^+\|_\theta^\theta + \int_\Omega F(z,u^+)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Hypotheses $H(f)$ (i), (ii) imply that given any $\eta > 0$, we can find $c_\eta > 0$ such that

$$F(z,x) \geq \frac{\eta}{\theta}x^\theta - c_\eta \text{ for a.a } z \in \Omega, \text{ all } x \geq 0. \tag{3.2}$$

Using (3.2) in (3.1), we obtain

$$\begin{aligned}\varphi_\lambda(u) &\geq \gamma(u) + \frac{\mu}{p}\|u^-\|_p^p + \frac{1}{\theta}[\eta - \lambda]\|u^+\|_\theta^\theta - c_1 \text{ for some } c_1 > 0 \\ &\geq c_2\|u^-\|_p^p + \frac{1}{p}\|\nabla u^+\|_p^p + \frac{1}{\theta}[\eta - \lambda]\|u^+\|_\theta^\theta - c_3\|u^+\|_p^p - c_1\end{aligned}$$

for some $c_2, c_3 > 0$ (see hypotheses $H(\xi)$, $H(\beta)$ and recall that $\mu > \|\xi\|_\infty$).

We choose $\eta > \lambda$. Since $\theta > p$, we can find $c_4 > 0$ such that

$$\begin{aligned}\varphi_\lambda(u) &\geq c_2\|u^-\|^p + \frac{1}{p}\|\nabla u^+\|_p^p + c_4\|u^+\|_p^\theta - c_3\|u^+\|_p^p \\ &= c_2\|u^-\|^p + \frac{1}{p}\|\nabla u^+\|_p^p + [c_4\|u^+\|_p^{\theta-p} - c_3]\|u^+\|_p^p \\ &\geq c_5\|u\|^p \text{ for some } c_5 > 0, \text{ all } u \in W^{1,p}(\Omega) \text{ with } \|u^+\|_p > \left(\tfrac{c_3}{c_4}\right)^{\frac{1}{\theta-p}}.\end{aligned}$$

Therefore $\varphi_\lambda(\cdot)$ is coercive.

Also, from the Sobolev embedding theorem and the compactness of the trace map, we have that $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous.

Then by the Weierstrass–Tonelli theorem we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \inf[\varphi_\lambda(u) : u \in W^{1,p}(\Omega)]. \tag{3.3}$$

Evidently choosing $\lambda > 0$ big we can guarantee that $\varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0)$, so $u_\lambda \neq 0$ for all $\lambda > 0$ big.

From (3.3) we have $\varphi'_\lambda(u_\lambda) = 0$, so

$$\begin{aligned}&\langle A_p(u_\lambda), h\rangle + \langle A_q(u_\lambda), h\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma \\ &- \mu\int_\Omega (u_\lambda^-)^{p-1} h dz = \int_\Omega [\lambda(u_\lambda^+)^{\theta-1} - f(z, u_\lambda)] h dz \text{ for all } h \in W^{1,p}(\Omega).\end{aligned} \tag{3.4}$$

In (3.4) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\|\nabla u_\lambda^-\|_p^p + \|\nabla u_\lambda^-\|_q^q + \int_\Omega [\xi(z) + \mu](u_\lambda^-)^p dz \leq 0 \text{ (see hypothesis } H(\beta) \text{ and (2.2)).}$$

Hence $c_6\|u_\lambda^-\|^p \leq 0$ for some $c_6 > 0$ (recall that $\mu > \|\xi\|_\infty$). Consequently, $u_\lambda \geq 0$, $u_\lambda \neq 0$.

Then from (3.4) we have

$$\begin{cases} -\Delta_p u_\lambda(z) - \Delta_q u_\lambda(z) + \xi(z)u_\lambda(z)^{p-1} = \lambda u_\lambda(z)^{\theta-1} - f(z, u_\lambda(z)) \text{ for a.a } z \in \Omega, \\ \dfrac{\partial u_\lambda}{\partial n_{pq}} + \beta(z)u_\lambda^{p-1} = 0 \quad \text{on } \partial\Omega \end{cases}$$

(see Papageorgiou–Rădulescu [11]). Finally, $u_\lambda \in S_\lambda \subseteq D_+$ for $\lambda > 0$ big and so $\mathcal{L} \neq \emptyset$. $\square$

Next we show that the admissible set $\mathcal{L}$ is an unbounded interval.

Proposition 3.2. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, $\lambda \in \mathcal{L}$ and $\eta > \lambda$, then $\eta \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S_\lambda \subseteq D_+$ (see (3.1)). As before, let $\mu > \|\xi\|_\infty$.

We introduce the Carathéodory function $k_\eta(z,x)$ defined by

$$k_\eta(z,x) = \begin{cases} \eta u_\lambda(z)^{\theta-1} - f(z,u_\lambda(z)) + \mu u_\lambda(z)^{p-1} & \text{if } x \le u_\lambda(z), \\ \eta x^{\theta-1} - f(z,x) + \mu x^{p-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.5}$$

We set $K_\eta(z,x) = \int_0^x k_\eta(z,s)ds$ and consider the C^1-functional $\psi_\eta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\eta(u) = \gamma(u) + \frac{\mu}{p}\|u\|_p^p - \int_\Omega K_\eta(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Note that

$$\lim_{x\to+\infty} \frac{k_\eta(z,x)}{x^{\theta-1}} = -\infty \text{ uniformly for a.a } z \in \Omega$$

(see (3.5) and hypothesis $H(f)$ (iii)). Therefore, given any $\tau > 0$, we can find $c_\tau > 0$ such that

$$-K_\eta(z,x) \ge \tau(x^+)^\theta - c_\tau \text{ for a.a } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

This implies that ψ_η is coercive (recall $\mu > \|\xi\|_\infty$).

Also $\psi_\eta(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $u_\eta \in W^{1,p}(\Omega)$ such that

$$\psi_\eta(u_\eta) = \inf[\psi_\eta(u) : u \in W^{1,p}(\Omega)].$$

Hence $\psi_\eta'(u_\eta) = 0$, and consequently

$$\begin{aligned} &\langle A_p(u_\eta), h\rangle + \langle A_q(u_\eta), h\rangle + \int_\Omega [\xi(z)+\mu]|u_\eta|^{p-2}u_\eta h dz + \int_{\partial\Omega} \beta(z)|u_\eta|^{p-2}u_\eta h d\sigma \\ &= \int_\Omega k_\eta(z,u_\eta)hdz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.6}$$

In (3.6) we choose $h=(u_\lambda-u_\eta)^+\in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A_p(u_\eta),(u_\lambda-u_\eta)^+\rangle+\langle A_q(u_\eta),(u_\lambda-u_\eta)^+\rangle\\
&\quad+\int_\Omega[\xi(z)+\mu]|u_\eta|^{p-2}u_\eta(u_\lambda-u_\eta)^+dz\\
&\quad+\int_{\partial\Omega}\beta(z)|u_\eta|^{p-2}u_\eta(u_\lambda-u_\eta)^+d\sigma\\
&=\int_\Omega[\eta u_\lambda^{\theta-1}-f(z,u_\lambda)+\mu u_\lambda^{p-1}](u_\lambda-u_\eta)^+dz\quad\text{(see (3.5))}\\
&\geq\int_\Omega[\lambda u_\lambda^{\theta-1}-f(z,u_\lambda)+\mu u_\lambda^{p-1}](u_\lambda-u_\eta)^+dz\quad\text{(since }\lambda<\eta)\\
&=\langle A_p(u_\lambda),(u_\lambda-u_\eta)^+\rangle+\langle A_q(u_\lambda),(u_\lambda-u_\eta)^+\rangle\\
&\quad+\int_\Omega[\xi(z)+\mu]u_\lambda^{p-1}(u_\lambda-u_\eta)^+dz\\
&\quad+\int_{\partial\Omega}\beta(z)u_\lambda^{p-1}(u_\lambda-u_\eta)^+d\sigma\quad\text{(since }u_\lambda\in S_\lambda).
\end{aligned}$$

Therefore, $u_\lambda\leq u_\eta$ (see Proposition 2.5 and recall that $\mu>\|\xi\|_\infty$). Then from (3.5) and (3.6) it follows that $u_\eta\in S_\eta\subseteq D_+$ and $\eta\in\mathcal{L}$. □

Let $\lambda_*=\inf\mathcal{L}$.

Proposition 3.3. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $\lambda_*>0$.*

Proof. If $\widehat{\lambda}_1(p)\leq 0$, then from hypothesis $H(f)$(iv) it follows that $f(z,x)\geq 0$ for a.a $z\in\Omega$, all $x\in\mathbb{R}$ (see (2.2)). Then using hypothesis $H(f)$(ii), we see that we can find $\overline{\lambda}_0^*>0$ small such that

$$\overline{\lambda}_0^*x^{\theta-1}-f(z,x)\leq\widehat{\lambda}_1(p)x^{p-1}\text{ for a.a }z\in\Omega,\text{ all }x\geq 0.$$

If $\widehat{\lambda}_1(p)>0$, then on account of hypothesis $H(f)$(iii), we have

$$\lambda x^{\theta-1}-f(z,x)\leq\lambda x^{\theta-1}-\eta_0x^{q-1}\text{ for a.a }z\in\Omega,\text{ all }0\leq x\leq\delta.$$

Since $\delta\in(0,1)$ and $q<p<\theta$, we can find $\overline{\lambda}_0\in(0,\overline{\lambda}_0^*]$ such that

$$\overline{\lambda}_0x^{\theta-1}-\eta_0x^{q-1}\leq\widehat{\lambda}_1(p)x^{p-1}\text{ for all }0\leq x\leq\delta.$$

Therefore,

$$\overline{\lambda}_0x^{\theta-1}-f(z,x)\leq\widehat{\lambda}_1(p)x^{p-1}\text{ for a.a }z\in\Omega,\text{ all }0\leq x\leq\delta.\tag{3.7}$$

Hypothesis $H(f)$ (ii) implies that we can find $\overline{M} > 0$ such that

$$\overline{\lambda}_0 x^{\theta-1} - f(z,x) \leq \widehat{\lambda}_1(p)x^{p-1} \text{ for a.a } z \in \Omega, \text{ all } x \geq \overline{M}. \tag{3.8}$$

Finally, for the interval $[\delta, \overline{M}]$, we choose $\overline{\lambda} \in (0, \overline{\lambda}_0]$ small such that

$$\overline{\lambda}\,\overline{M}^{\theta-p} \leq \widehat{d}_{\overline{M}} + \widehat{\lambda}_1(p) \quad \text{(see hypothesis } H(f)\text{(iv))}.$$

Then on account of hypothesis $H(f)$ (iv) we have

$$\overline{\lambda} x^{\theta-1} - f(z,x) \leq \widehat{\lambda}_1(p)x^{p-1} \text{ for a.a } z, \Omega, \text{ all } \delta \leq x \leq \overline{M}. \tag{3.9}$$

From (3.7), (3.8), (3.9) we see that for some $0 < \overline{\lambda} \leq \overline{\lambda}_0 \leq \overline{\lambda}_0^*$ small, we have

$$\overline{\lambda} x^{\theta-1} - f(z,x) \leq \widehat{\lambda}_1(p)x^{p-1} \text{ for a.a } z \in \Omega, \text{ all } x \geq 0. \tag{3.10}$$

Now let $\lambda \in (0, \overline{\lambda})$ and assume that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq D_+$ (see (2.2)) and we have

$$\begin{aligned} &\langle A_p(u_\lambda), h\rangle + \langle A_q(u_\lambda), h\rangle + \int_\Omega \xi(z)u_\lambda^{p-1}h dz + \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}h d\sigma \\ &= \int_\Omega [\lambda u_\lambda^{\theta-1} - f(z,u_\lambda)]h dz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.11}$$

In (3.11) we choose $h = u_\lambda \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} \gamma_p(u_\lambda) &\leq \int_\Omega [\lambda u_\lambda^{\theta-1} - f(z,u_\lambda)]u_\lambda dz \\ &< \int_\Omega [\overline{\lambda} u_\lambda^{\theta-1} - f(z,u_\lambda)]u_\lambda dz \quad (\text{since } \lambda < \overline{\lambda}). \end{aligned}$$

It follows that

$$\gamma_p(u_\lambda) < \widehat{\lambda}_1(p)\|u_\lambda\|_p^p \quad \text{(see (3.10))}.$$

This contradicts (2.1). Therefore, $\lambda \notin \mathcal{L}$ and so $0 < \overline{\lambda} \leq \lambda_*$. □

Proposition 3.4. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold and $\lambda > \lambda_*$, then problem (P_λ) admits at least two positive solutions $u_0, \widehat{u} \in D_+$, $u_0 \neq \widehat{u}$.*

Proof. Let $\eta \in (\lambda_*, \lambda) \cap \mathcal{L}$. Then we can find $u_\eta \in S_\eta \subseteq D_+$ (see (2.2)). Let $\mu > \|\xi\|_\infty$ and consider the Carathéodory function $e_\lambda(z,x)$ defined by

$$e_\lambda(z,x) = \begin{cases} \lambda u_\eta(z)^{\theta-1} - f(z,u_\eta(z)) + \mu u_\eta(z)^{p-1} & \text{if } x \leq u_\eta(z), \\ \lambda x^{\theta-1} - f(z,x) + \mu x^{p-1} & \text{if } u_\eta(z) < x. \end{cases} \tag{3.12}$$

We set $E_\lambda(z,x) = \int_0^x e_\lambda(z,s)ds$ and consider the C^1-functional $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \gamma(u) + \frac{\mu}{p}\|u\|_p^p - \int_\Omega E_\lambda(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

As in the proof of Proposition 3.2, via the direct method of the calculus of variations, we obtain $u_0 \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_0) = \inf[\psi_\lambda(u) : u \in W^{1,p}(\Omega)].$$

This implies that $u_0 \in K_{\psi_\lambda}$.

Then using (3.12) we conclude that

$$u_0 \in S_\lambda \subseteq D_+ \text{ and } u_\eta \le u_0. \tag{3.13}$$

Claim. $u_0 - u_\eta \in \operatorname{int} \widehat{C}_+$.

Let $\rho = \|u_0\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)$(iv). We have

$$\begin{aligned}
&- \Delta_p u_\eta - \Delta_q u_\eta + (\xi(z) + \widehat{\xi}_\rho)u_\eta^{p-1} \\
&= \eta u_\eta^{\theta-1} - f(z,u_\eta) + \widehat{\xi}_\rho u_\eta^{p-1} \\
&= \lambda u_\eta^{\theta-1} - f(z,u_\eta) + \widehat{\xi}_\rho u_\eta^{p-1} - (\lambda-\eta)u_\eta^{\theta-1} \\
&< \lambda u_0^{\theta-1} - f(z,u_0) + \widehat{\xi}_\rho u_0^{p-1} \quad \text{(see (3.13), hypothesis } H(f)\text{(iv) and recall } \eta<\lambda) \\
&\quad - \Delta_p u_0 - \Delta_q u_0 + (\xi(z) + \widehat{\xi}_\rho)u_0^{p-1}.
\end{aligned} \tag{3.14}$$

Since $u_\eta \in D_+$, we see that

$$0 < (\lambda-\eta)m_\eta^{\theta-1} \le (\lambda-\eta)u_\eta(z)^{\theta-1} \text{ with } m_\eta = \min_{\overline{\Omega}} u_\eta > 0.$$

So, from (3.14) and Proposition 2.4, we infer that $u_0 - u_\eta \in \operatorname{int}\widehat{C}_+$. This proves the Claim.

Recall that $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$ is the energy functional of problem (P_λ) (see the proof of Proposition 3.1). Let

$$[u_\eta) = \{u \in W^{1,p}(\Omega) : u_\eta(z) \le u(z) \text{ for a.a } z \in \Omega\}.$$

From (3.12) it is clear that

$$\varphi_\lambda\Big|_{[u_\eta)} = \psi_\lambda\Big|_{[u_\eta)} + \widehat{c}_\lambda \text{ with } \widehat{c}_\lambda \in \mathbb{R}. \tag{3.15}$$

From (3.15) and the Claim, it follows that u_0 is a local $C^1(\overline{\Omega})$-minimizer of φ_λ, so

$$u_0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda \text{ (see Proposition 2.3).} \tag{3.16}$$

Now let $\delta > 0$ be as postulated by hypothesis $H(f)$(iii). Then for $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \le \delta$ we have

$$\begin{aligned}\varphi_\lambda(u) &\ge c_7\|u^-\|^p + \frac{1}{p}[\gamma_p(u^+) + \eta_0\|u^+\|_p^p] - \frac{\lambda}{p}\|u^+\|_\theta^\theta \\ &\qquad\qquad\qquad\text{for some } c_7 > 0 \text{ (see hypothesis } H(f)\,\text{(iii))} \\ &\ge c_8\|u\|^p - \lambda c_9\|u\|^\theta \text{ for some } c_8, c_9 > 0 \text{ (since } \eta_0 > -\widehat{\lambda}_1(p)). \end{aligned} \tag{3.17}$$

From (3.17) it follows that by choosing $\delta \in (0,1)$ even smaller, we can have

$$\begin{aligned} &\varphi_\lambda(u) > 0 = \varphi_\lambda(0) \text{ for all } u \in C^1(\overline{\Omega}), \quad 0 < \|u\|_{C^1(\overline{\Omega})} \le \delta \\ \Rightarrow\quad & u = 0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda \\ \Rightarrow\quad & u = 0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda \text{ (see Proposition 2.3).} \end{aligned} \tag{3.18}$$

We assume that K_{φ_λ} is finite or otherwise we already have an infinity of positive solutions for problem (P_λ). Without any loss of generality we assume that

$$0 = \varphi_\lambda(0) \le \varphi_\lambda(u_0). \tag{3.19}$$

The argument is similar if the opposite inequality holds (using (3.18) instead of (3.16)).

On account of (3.16), we can find $\rho \in (0,1)$ small such that

$$\rho < \|u_0\|, \quad \varphi_\lambda(u_0) < \inf[\varphi_\lambda(u) : \|u - u_0\| = \rho] = \eta_\lambda \tag{3.20}$$

(see Aizicovici–Papageorgiou–Staicu [1, The proof of Proposition 29]).

The functional φ_λ is coercive (see the proof of Proposition 3.1). Therefore

$$\varphi_\lambda(\cdot) \text{ satisfies the } C\text{-condition (see Section 2).} \tag{3.21}$$

Then from (3.19), (3.20), (3.21) we see that we can apply Theorem 2.1 (the mountain pass theorem) and find $\widehat{u} \in W^{1,p}(\Omega)$ such that $\widehat{u} \in K_{\varphi_\lambda}$, $\widehat{u} \notin \{0, u_0\}$. Therefore $\widehat{u} \in S_\lambda \subseteq D_+$ and $\widehat{u} \ne u_0$. □

We show that the critical parameter value $\lambda_* > 0$ is admissible.

Proposition 3.5. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $\lambda_* \in \mathcal{L}$.*

Proof. Let $\{\lambda_n\}_{n\ge1} \subseteq \mathcal{L}$ such that $\lambda_n \to \lambda_*^+$. There exist $u_n = u_{\lambda_n} \in S_{\lambda_n} \subseteq D_+$ for all $n \in \mathbb{N}$ and we have

$$\begin{aligned} &\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma \\ &= \int_\Omega [\lambda_n u_n^{\theta-1} - f(z,u_n)] h dz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{3.22}$$

Hypotheses $H(f)$, (i), (ii) imply that given $\eta > \lambda_n$, we can find $c_\eta > 0$ such that

$$\eta x^{\theta-1} - c_\eta \le f(z,x) \text{ for a.a } z \in \Omega, \text{ all } x \ge 0. \tag{3.23}$$

In (3.22) we use $h = u_n \in W^{1,p}(\Omega)$ ($u_n \geq 0$ for all $n \geq 1$). So, (3.23) and hypothesis $H(\beta)$ imply

$$\begin{aligned}
&\|\nabla u_n\|_p^p + \int_\Omega \xi(z) u_n^p dz + (\eta - \lambda_n)\|u_n\|_\theta^\theta \leq c_{10} \\
&\qquad\qquad\qquad\qquad\text{for some } c_{10} > 0, \text{ all } n \in \mathbb{N} \\
\Rightarrow\quad &\|\nabla u_n\|_p^p + c_{11}\|u_n\|_p^\theta - \|\xi\|_\infty \|u_n\|_p^p \leq c_{10} \\
&\qquad\qquad\qquad\qquad\text{for some } c_{11} > 0, \text{ all } n \in \mathbb{N} \text{ (recall } \theta > p) \\
\Rightarrow\quad &\|\nabla u_n\|_p^p + [c_{11}\|u_n\|_p^{\theta-p} - \|\xi\|_\infty]\|u_n\|_p^p \leq c_{10} \text{ for all } n \in \mathbb{N} \\
\Rightarrow\quad &\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}
\end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.24}$$

In (3.22) we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.24). Then

$$\begin{aligned}
&\lim_{n\to+\infty} [\langle A_p(u_n), u_n - u_*\rangle + \langle A_q(u_n), u_n - u_*\rangle] = 0 \\
\Rightarrow\quad &\limsup_{n\to+\infty} [\langle A_p(u_n), u_n - u_*\rangle + \langle A_q(u_*), u_n - u_*\rangle] \leq 0, \\
&\qquad\qquad\qquad\text{(since } A_q(\cdot) \text{ is monotone, see Proposition 2.5)} \\
\Rightarrow\quad &\limsup_{n\to+\infty} \langle A_p(u_n), u_n - u_*\rangle \leq 0 \text{ (see (3.24))} \\
\Rightarrow\quad &u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2.5).} \qquad (3.25)
\end{aligned}$$

We pass to the limit as $n \to +\infty$ in (3.22) and use (3.25). We obtain

$$\begin{aligned}
&\langle A_p(u_*), h\rangle + \langle A_q(u_*), h\rangle + \int_\Omega \xi(z) u_*^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_*^{p-1} h d\sigma \\
&= \int_\Omega [\lambda_* u_*^{\theta-1} - f(z, u_*)] h dz \quad \text{for all } h \in W^{1,p}(\Omega).
\end{aligned}$$

Hence $u_* \in S_\lambda \cup \{0\}$.

We need to show that $u_* \neq 0$. Arguing by contradiction, suppose that $u_* = 0$. Then from (3.25) and since $q < p$, we have

$$u_n \to 0 \text{ in } W^{1,q}(\Omega). \tag{3.26}$$

From Papageorgiou–Rădulescu [21, Proposition 7], we know that we can find $c_{12} > 0$ such that

$$\|u_n\|_\infty \leq c_{12} \text{ for all } n \in \mathbb{N}.$$

Then the nonlinear regularity theory of Lieberman [15], implies that we can find $\alpha \in (0,1)$ and $c_{13} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}),\ \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_{13} \text{ for all } n \in \mathbb{N}. \tag{3.27}$$

From (3.25), (3.27) and the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we infer that

$$u_n \to 0 \text{ in } C^1(\overline{\Omega}) \text{ as } n \to +\infty.$$

Let $\|\cdot\|_{1,q}$ denote the $W^{1,q}(\Omega)$-norm and set $y_n = \dfrac{u_n}{\|u_n\|_{1,q}}$ $n \in \mathbb{N}$. Then $\|y_n\|_{1,q} = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,q}(\Omega),\ y_n \in C^1(\overline{\Omega}) \text{ for all } n \in \mathbb{N},\ y \geq 0. \tag{3.28}$$

From (3.22) we have

$$\begin{aligned} &\|u_n\|_{1,q}^{p-q}\langle A_p(y_n), h\rangle + \langle A_q(y_n), h\rangle \\ &\quad + \|u_n\|_{1,q}^{p-q}\left[\int_\Omega \xi(z)y_n^{p-1}hdz + \int_{\partial\Omega} \beta(z)y_n^{p-1}hd\sigma\right] \\ &= \int_\Omega \left[\lambda_n\|u_n\|_{1,q}^{\theta-q}y_n^{\theta-1} - \frac{f(z,u_n)}{\|u_n\|_{1,q}^q}\right] hdz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{3.29}$$

On account of hypothesis $H(f)$ (iii) and (3.26), we have

$$\frac{f(\cdot,u_n(\cdot))}{\|u_n\|_{1,q}^q} \xrightarrow{w} J(z)y^{q-1} \text{ in } L^{q'}(\Omega)\ \left(\frac{1}{q}+\frac{1}{q'}=1\right),\ 0<\eta_0\leq J(z)\leq\widetilde{\eta}_0 \text{ for a.a } z\in\Omega \tag{3.30}$$

(see Aizicovici–Papageorgiou–Staicu [1, The proof of Proposition 31]).

In (3.29), we set $h = y_n - y$, pass to the limit as $n \to +\infty$ and use (3.26), (3.30). Then

$$\lim_{n\to+\infty}\langle A_q(y_n), y_n - y\rangle = 0$$

which implies that

$$y_n \to y \text{ in } W^{1,q}(\Omega), \quad \|y\|_{1,q} = 1. \tag{3.31}$$

So, if in (3.29), we pass to the limit as $n \to +\infty$, then

$$\langle A_q(y), h\rangle = -\int_\Omega J(z)y^{q-1}hdz \text{ for all } h \in W^{1,p}(\Omega) \text{ (see (3.30))}. \tag{3.32}$$

In (3.32) we choose $h = y \in W^{1,p}(\Omega)$. Then

$$\|\nabla y\|_q^q = -\int_\Omega J(z)y^q dz \leq 0 \text{ (see (3.30))}. \tag{3.33}$$

Then

$$y = c^* \in (0, +\infty) \quad \text{(see (3.28) and (3.31))}.$$

But then from (3.33), we have

$$\|\nabla y\|_q^q = -(c^*)^q \int_\Omega J(z)dz < 0 \quad \text{(see (3.30))},$$

a contradiction. Therefore $u_* \neq 0$ and so $u_* \in S_\lambda \subseteq D_+$ and $\lambda_* \in \mathcal{L}$. □

Proposition 3.5 implies that $\mathcal{L} = [\lambda_*, +\infty)$.

Summarizing our findings in this section. we can state the following bifurcation-type result describing the set S_λ of positive solutions of problem (P_λ) as $\lambda > 0$ varies.

Theorem 3.6. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then there exists a critical parameter value $\lambda_* > 0$ such that*

(a) *for all $\lambda > \lambda_*$ problem (P_λ) has at least two positive solutions $u_0, \widehat{u} \in D_+$, $u_0 \neq \widehat{u}$;*
(b) *for $\lambda = \lambda_*$ problem (P_λ) has at least one positive solution $u_* \in D_+$;*
(c) *for all $\lambda \in (0, \lambda_*)$ problem (P_λ) has no positive solution.*

4. MINIMAL POSITIVE SOLUTIONS

In this section, we show that for every $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$ problem (P_λ) has a smallest positive solution.

From Papageorgiou–Rădulescu–Repovš [23, The proof of Proposition 7], we know that

$$S_\lambda \subseteq D_+ \text{ is downward directed,} \tag{4.1}$$

that is, if $u_1, u_2 \in S_\lambda$ then we can find $u \in S_\lambda$ such that $u \leq u_1$, $u \leq u_2$,

Proposition 4.1. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold and $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$, then problem (P_λ) admits a smallest positive solution $\overline{u}_\lambda \in S_\lambda \subseteq D_+$.*

Proof. Invoking Lemma 3.10, p. 178, of Hu–Papageorgiou [13], we can find a sequence $\{u_n\}_{n\geq 1} \subseteq S_\lambda \subseteq D_+$ decreasing (see (4.1)) such that

$$\inf S_\lambda = \inf_{n\geq 1} u_n.$$

We have

$$\begin{aligned} &\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma \\ &= \int_\Omega [\lambda u_n^{\theta-1} - f(z, u_n)] h dz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{4.2}$$

Choosing $h = u_n \in W^{1,p}(\Omega)$ in (4.2) and recalling that $0 \leq u_n \leq u_1$ for all $n \in \mathbb{N}$, we infer that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

Arguing as in the proof of Proposition 3.5 (see the part of the proof from (3.24) and after), we obtain that

$$u_n \to \overline{u}_\lambda \text{ in } W^{1,p}(\Omega), \quad \overline{u}_\lambda \neq 0.$$

Hence,

$$\overline{u}_\lambda \in S_\lambda \subseteq D_+ \text{ and } \overline{u}_\lambda = \inf S_\lambda. \qquad \square$$

Remark 4.2. It is an interesting open problem what are the monotonicity and the continuity properties of the map $\lambda \to \overline{u}_\lambda$ from $\mathcal{L} = [\lambda_*, +\infty)$ into $C^1(\overline{\Omega})$.

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